INFINITE INTERVAL PROBLEMS ARISING IN THE MODEL OF A
SLENDER DRY PATCH IN A LIQUID FILM DRAINING UNDER
GRAVITY DOWN AN INCLINED PLANE *

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Abstract. Existence results are established for a second order boundary value problem on the
half line motivated from the model of a slender dry patch in a liquid film draining under gravity
down an inclined plane.

1. Introduction. Consider a thin film of viscous liquid with constant density $\rho$ and viscosity $\mu$ flowing down a planer substrate inclined at an angle $\alpha$ $(0 < \alpha \leq \frac{\pi}{2})$ to the horizontal. We adopt Cartesian coordinates $(x, y, z)$ with the $x$-axis down the greatest slope and the $z$-axis normal to the plane. With the usual lubrication approximation the height of the free surface $z = h(x, y, z)$ satisfies [4]

\begin{equation}
3 \mu h_t = \nabla \cdot \left[ h^3 \nabla (\rho g h \cos \alpha - \sigma \nabla^2 h) \right] - \rho g \sin \alpha \left[ h^3 \right]_x,
\end{equation}

where $t$ denotes time, $g$ the magnitude of acceleration due to gravity and $\sigma$ the coefficient of surface tension. We are interested in solutions symmetric about $y = 0$, and we seek a steady state solution for a slender dry patch for which the length scale down the plane (i.e. in the $x$ direction) is much greater than in the transverse direction (i.e. in the $y$ direction), so the equation (1.1) is approximated by [4]

\begin{equation}
\left[ h^3 (\rho g h \cos \alpha - \sigma h_{yy}) \right]_y - \rho g \sin \alpha \left[ h^3 \right]_x = 0.
\end{equation}

The velocity component down the plane is $u(x, y, z) = \rho g \sin \alpha \left[ 2 h z - z^2 \right] / 2 \mu$ and so for a slender dry patch of semi-width $y_e = y_e(x)$ the average volume flux around the dry patch per unit width in the transverse direction down the plane (denoted by $Q(x)$) is approximately [4]

\begin{equation}
Q = \rho g \sin \alpha \lim_{y \to \infty} y^{-1} \int_y^{y_e(x)} h(x, w)^3 dw.
\end{equation}

We seek a similarity solution to equation (1.2) of the form $h = f(x) G(\eta)$ where $\eta = y / y_e(x)$. Note $G(1) = 0$ and (1.2) takes the form

\begin{equation}
\rho g \cos \alpha f^3 y_e^2 (G^3 G')' - \sigma f^2 (G^3 G'')' - 3 \rho g \sin \alpha y_e^2 G^2 (f' G y_e - f G' y_e) = 0
\end{equation}

with the corresponding expression for $Q$ being

\begin{equation}
Q = \rho g \sin \alpha \lim_{\eta \to \infty} \eta^{-1} \int_1^{\eta} G(w)^3 dw.
\end{equation}

For weak surface-tension effects the second term in (1.4) can be neglected and so the only relevant similarity solution is given (after a suitable choice of origin in $x$) by

\begin{equation}
f(x) = b (c x)^m \quad \text{and} \quad y_e(x) = (c x)^k
\end{equation}

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where the coefficients $b$ and $c$ and the exponents $m$ and $k$ are constants with $m = 2k - 1$. In this case $\alpha \neq \frac{\pi}{2}$ and so we may choose without loss of generality $b = c k \tan \alpha$ and so (1.4) becomes

\[(1.5) \quad ((G' + \eta')G^3)' - \left(7 - \frac{3}{k}\right) G^3 = 0.\]

The unknown exponent $k$ is determined by the requirement that the average volume flux per unit width around the dry patch, $Q$, is independent of $x$. This is possible only if $m = 0$ and $G \sim G_0 > 0$ (a constant) as $\eta \to \infty$. Thus

\[Q = \frac{pg \sin \alpha}{3\mu} (bG_0)^3 \text{ and so } m = 0, k = \frac{1}{2}.\]

Setting $k = \frac{1}{2}$ in (1.5) yields

\[(1.6) \quad (G^3 G')' + \eta (G^3)' = 0.\]

Also the solutions to (1.6) must satisfy the boundary condition $G(1) = 0$ and the far-field condition \[\lim_{\eta \to \infty} G(\eta) = G_0 > 0.\] As a result one is interested in the boundary value problem

\[(1.7) \quad \begin{cases} (G^3 G')' + \eta (G^3)' = 0, \quad 1 < \eta < \infty \\ G(1) = 0, \quad \lim_{\eta \to \infty} G(\eta) = G_0 > 0. \end{cases}\]

Keeping this problem in mind, in Section 2 we discuss the general boundary value problem

\[(1.8) \quad \begin{cases} (G'(y) + p(t) y^m)' + q(t) f(t, y) = p'(t) y^m, \quad a < t < n \\ y(a) = 0, \quad y(n) = b_0 > 0, \end{cases}\]

where $n > a$, $G(z) = \int_0^z g(x) dx$, $G'(y) = \frac{d}{dy} G(y(t))$ and

\[g(x) = \begin{cases} x^m, & x \geq 0 \\ -x^m, & x < 0 \end{cases}\]

with $m > 0$ odd. A very general existence theory will be presented for (1.8) in Section 2. Our theory relies on the following nonlinear alternative of Leray–Schauder type [1, 2].

**Theorem 1.1.** Let $U$ be an open subset of a Banach space $E$, $J : \overline{U} \to E$ a continuous compact map, $p^* \in U$ and let $N : \overline{U} \times [0, 1] \to E$ be a continuous compact map with $N_1 = J$ and $N_0 = p^*$ (here $N_\lambda(u) = N(u, \lambda)$). Also assume

\[(1.9) \quad u \neq N_\lambda (u) \text{ for } u \in \partial U \text{ and } \lambda \in (0, 1].\]

Then $J$ has a fixed point in $U$.

In Section 3 we discuss the following boundary value problem on the half line

\[\begin{cases} (G'(y) + p(t) y^m)' + q(t) f(t, y) = p'(t) y^m, \quad a < t < \infty \\ y(a) = 0, \quad y \text{ bounded on } [a, \infty), \end{cases}\]

and our existence theory will then be applied to (1.7).
2. Existence theory on finite intervals. In this section we first establish the existence of a solution to

\[
\begin{aligned}
& (G'(y) + p(t) y^m)' + q(t) f(t, y) = p'(t) y^m, \quad a < t < n \\
& y(a) = 0, \quad y(n) = b_0 > 0
\end{aligned}
\]

(2.1) where \( G(z) = \int_{a}^{z} g(x) \, dx \) and

\[
g(x) = \begin{cases}
  x^m, & x \geq 0 \\
  -x^m = |x|^m, & x < 0
\end{cases}
\]

and with \( m > 0 \) odd. Note \( G'(y) = \frac{d}{dt} G(y(t)) \) and

\[
G(z) = \begin{cases}
  \frac{z^{m+1}}{m+1}, & z \geq 0 \\
  -\frac{|z|^{m+1}}{m+1}, & z < 0
\end{cases}
\]

By a solution to (2.1) we mean a function \( y \in C[a, n] \), with \( G(y) \in C^1[a, n] \), \( G'(y) + p y^m \in AC[a, n] \cap C^1[a, n] \) which satisfies \( y(a) = 0 \), \( y(n) = b_0 \) and the differential equation in (2.1) on \((a, n)\).

**Theorem 2.1.** Suppose the following conditions are satisfied:

(2.2) \( f : [a, n] \times \mathbb{R} \to \mathbb{R} \) is continuous

(2.3) \( q \in C(a, n] \cap L^1[a, n] \) with \( q > 0 \) on \((a, n)\)

(2.4) \( p \in C^1[a, n] \) with \( p \geq 0 \) on \([a, n]\)

(2.5) \( f(t, 0) \geq 0 \) for \( t \in (a, n) \)

and

(2.6) \( f(t, b_0) \leq 0 \) for \( t \in (a, n) \).

Then (2.1) has a solution \( y \) with \( 0 \leq y(t) \leq b_0 \) for \( t \in [a, n] \).

**Proof.** Consider the boundary value problem

\[
\begin{aligned}
& (G'(y) + \lambda p y^m)' = \lambda f^*(t, y), \quad a < t < n \\
& y(a) = 0, \quad y(n) = b_0 > 0, \quad 0 < \lambda \leq 1
\end{aligned}
\]

(2.7) where

\[
f^*(t, y) = \begin{cases}
  -q(t) f(t, 0) + y, & y < 0 \\
  -q(t) f(t, y) + p'(t) y^m, & 0 \leq y \leq b_0 \\
  -q(t) f(t, b_0) + p'(t) b_0^m + y - b_0, & y > b_0.
\end{cases}
\]

Solving (2.7) is equivalent (see [2]) to finding a \( y \in C[a, n] \) which satisfies

\[
y(t) = G^{-1}(A(t - a) - \lambda \int_{a}^{t} p(s) y^m(s) \, ds + \lambda \int_{a}^{t} (t - x) \, f^*(x, y(x)) \, dx)
\]

(2.8)
where
\[(2.9)\]
\[A = G(b_0) + \lambda \int_a^t p(s) y^m(s) \, ds - \lambda \int_a^t (n - x) f^*(x, y(x)) \, dx.\]

Define the operator \(N_\lambda : C[a, n] \to C[a, n]\) by
\[N_\lambda y(t) = G^{-1}(A(t - a) - \lambda \int_a^t p(s) y^m(s) \, ds + \lambda \int_a^t (t - x) f^*(x, y(x)) \, dx).\]

The argument in [2] guarantees that \(N_\lambda : C[a, n] \to C[a, n]\) is continuous and completely continuous. We now show any solution \(y\) to (2.7) \((0 < \lambda \leq 1)\) satisfies
\[(2.10)\]
\[0 \leq G(y(t)) \leq G(b_0) \text{ for } t \in [a, n].\]

If (2.10) is true then
\[(2.11)\]
\[0 \leq y(t) \leq b_0 \text{ for } t \in [a, n].\]

Suppose \(G(y(t)) < 0\) for some \(t \in (a, n)\). Then \(G(y)\) has a negative minimum at say \(t_0 \in (a, n)\), so \(G'(y(t_0)) = 0\). Also there exists \(\delta_1 > 0, \delta_2 > 0\) with \((t_0 - \delta_1, t_0 + \delta_2) \subseteq [a, n]\) and with
\[(2.12)\]
\[\begin{cases} 
G(y(t)) < 0 & \text{for } t \in (t_0 - \delta_1, t_0 + \delta_2) \\
\text{and } G(y(t_0 - \delta_1)) = G(y(t_0 + \delta_2)) = 0.
\end{cases}\]

Now for \(t \in (t_0 - \delta_1, t_0 + \delta_2)\) we have
\[(G'(y(t)) + \lambda p(t) y^m(t))' = -\lambda q(t)f(t, 0) + \lambda y(t) < 0,\]
so integration from \(t_0\) to \(t_0 + \delta_2\) yields
\[G'(y(t_0 + \delta_2)) + \lambda p(t_0 + \delta_2) y^m(t_0 + \delta_2) < \lambda p(t_0) y^m(t_0).\]

Now \(y(t_0 + \delta_2) = 0\), so (note \(m\) is odd, \(p \geq 0\) and \(y(t_0) < 0\))
\[(2.13)\]
\[G'(y(t_0 + \delta_2)) < \lambda p(t_0) y^m(t_0) \leq 0.\]

Thus there exists \(\delta_3 > 0, \delta_3 < \delta_2\) with
\[(2.14)\]
\[G'(y(t)) < 0 \text{ for } t \in (t_0 + \delta_3, t_0 + \delta_2).\]

As a result
\[0 = G(y(t_0 + \delta_2)) < G(y(t_0 + \delta_3)),\]
and this contradicts (2.12). Thus \(0 \leq G(y(t))\) for \(t \in [a, n]\), so \(0 \leq y(t)\) for \(t \in [a, n]\).

Next suppose \(G(y(t)) > G(b_0)\) for some \(t \in (a, n)\). Then \(G(y)\) has a positive maximum at say \(t_1 \in (a, n)\), so \(G'(y(t_1)) = 0\). Also there exists \(\delta_4 > 0, \delta_5 > 0\) with \((t_1 - \delta_4, t_1 + \delta_5) \subseteq [a, n]\) and with
\[(2.15)\]
\[G(y(t)) > G(b_0) \text{ for } t \in (t_1 - \delta_4, t_1 + \delta_5)\]
and
\[(2.16)\]
\[G(y(t_1 - \delta_4)) = G(y(t_1 + \delta_5)) = G(b_0).\]
Also for \( t \in (t_1 - \delta_4, t_1 + \delta_5) \) we have
\[
(G'(y(t)) + \lambda p(t) y^m(t))' = -\lambda q(t)f(t, b_0) + \lambda p'(t) b_0^m + \lambda (y(t) - b_0) > \lambda p'(t) b_0^m,
\]
so integration from \( t_1 \) to \( t_1 + \delta_5 \) yields (note (2.16))
\[
G'(y(t_1 + \delta_5)) + \lambda p(t_1 + \delta_5) b_0^m > \lambda p(t_1) y^m(t_1) + \lambda b_0^m [p(t_1 + \delta_5) - p(t_1)].
\]
Thus
\[
G'(y(t_1 + \delta_5)) > \lambda p(t_1) [y^m(t_1) - b_0^m] \geq 0
\]
since \( p \geq 0 \). As a result there exists \( \delta_6 > 0, \delta_6 < \delta_5 \) with
\[
G'(y(t)) > 0 \text{ for } t \in (t_1 + \delta_6, t_1 + \delta_5),
\]
so
\[
G(b_0) = G(y(t_1 + \delta_5)) > G(y(t_1 + \delta_6)),
\]
and this contradicts (2.15). Thus \( G(y(t)) \leq G(b_0) \) for \( t \in [a, n] \), so (2.11) holds.

Now Theorem 1.1 applied to \( N_1 \) with \( E = C[a, n], U = \{u \in E : \sup_{[a, n]} |u(t)| < b_0 + 1\} \) and \( p^* = G^{-1}\left(\frac{G(b_0)(t-a)}{n-a}\right) \) guarantees that \( N_1 \) has a fixed point \( y \in U \). Thus \( y \) is a solution of (2.7) and the argument above guarantees that \( 0 \leq y(t) \leq b_0 \) for \( t \in [a, n] \). As a result \( y \) is a solution of (2.1). \( \square \)

**Remark 2.1.** It is possible to replace \( p \geq 0 \) on \([a, n]\) by \( p \leq 0 \) on \([a, n]\) and the result in Theorem 2.1 is again true; we leave the details to the reader.

Keeping our application in Section 1 in mind we now discuss the situation when our solution to (2.1) is positive on \([a, n]\). Suppose the following conditions hold:

\[
\begin{align*}
\exists \alpha \in C[a, n] & \quad \text{with } G(\alpha) \in C^1[a, n], G'(\alpha) + p \alpha^m \in AC[a, n] \\
\cap C^1[a, n] & \quad \text{with } b_0 \geq \alpha > 0 \text{ on } (a, n], \alpha(0) = 0, \alpha(n) \leq b_0 \\
& \quad \text{and } (G'(\alpha) + p \alpha^m)' + q(t) f(t, \alpha) \geq p'(t) \alpha^m(t) \text{ on } (a, n)
\end{align*}
\]

\( \qquad (2.17) \)

\[
\begin{align*}
\text{for each } t \in (a, n) & \quad \text{we have } q(t) [f(t, y) - f(t, \alpha(t))] \geq 0 \\
& \quad \text{for } 0 \leq y \leq \alpha(t)
\end{align*}
\]

and

\( \qquad (2.18) \)

\( p' > 0 \) on \((a, n)\).

\( \qquad (2.19) \)

Also in this case we discuss the boundary value problem

\[
\begin{align*}
(y^m y')' + p(y^m)' + q f(t, y) & = 0, \quad a < t < n \\
y(a) & = 0, \quad y(n) = b_0 > 0.
\end{align*}
\]

\( \qquad (2.20) \)

By a solution to (2.20) we mean a function \( y \in C[a, n] \cap C^1(a, n) \) with \( G(y) \in C^1[a, n], y^m y' \in C^1[a, n] \) which satisfies \( y(a) = 0, y(n) = b_0 \) and the differential equation in (2.20) on \((a, n)\).
THEOREM 2.2. Suppose (2.2)–(2.6), (2.17), (2.18) and (2.19) are satisfied. Then
(2.1) has a solution \( y \) with \( \alpha(t) \leq y(t) \leq b_0 \) for \( t \in [a, n] \). In addition \( y \in C^1(a, n) \)
with \( G'(y) = y^m y' \) on \( (a, n) \) and \( y \) is a solution of (2.20).

Proof. Theorem 2.1 guarantees that (2.1) has a solution \( y \) with \( 0 \leq y(t) \leq b_0 \)
for \( t \in [a, n] \). Next we claim that

\[
y(t) \geq \alpha(t) \quad \text{for} \quad t \in [a, n].
\]

Suppose \( G(\alpha(t)) > G(y(t)) \) for some \( t \in (a, n) \). Then \( G(y) - G(\alpha) \) has a negative
minimum at say \( t_0 \in (a, n) \), so \( G'(y(t_0)) = G'(\alpha(t_0)) \). Also there exists \( \delta_1 > 0, \delta_2 > 0 \)
with \((t_0 - \delta_1, t_0 + \delta_2) \subseteq [a, n]\) and with

\[
G(y(t_0)) < G(\alpha(t)) \quad \text{for} \quad t \in (t_0 - \delta_1, t_0 + \delta_2)
\]

and

\[
G(y(t_0 - \delta_1)) = G(\alpha(t_0 - \delta_1)) \quad \text{and} \quad G(y(t_0 + \delta_2)) = G(\alpha(t_0 + \delta_2)).
\]

Also for \( t \in (t_0 - \delta_1, t_0 + \delta_2) \) we have (note \( 0 \leq y \leq b_0 \) on \([a, n])

\[
(G'(y) + py^m)'(t) - (G'(\alpha) + p\alpha^m)'(t) \leq q(t)[f(t, \alpha(t)) - f(t, y(t))]
\]

\[
+ p'(t)[y^m(t) - \alpha^m(t)]
\]

\[
< 0,
\]

since \( p' > 0 \) on \((a, n)\). Integrate from \( t_0 \) to \( t_0 + \delta_2 \) to obtain

\[
G'(y(t_0 + \delta_2)) + p(t_0 + \delta_2) y^m(t_0 + \delta_2) - G'(y(t_0)) - p(t_0) y^m(t_0)
\]

\[
< G'(\alpha(t_0 + \delta_2)) + p(t_0 + \delta_2) \alpha^m(t_0 + \delta_2) - G'(\alpha(t_0)) - p(t_0) \alpha^m(t_0),
\]

so (note (2.23))

\[
G'(y(t_0 + \delta_2)) - G'(\alpha(t_0 + \delta_2)) < p(t_0)[y^m(t_0) - \alpha^m(t_0)] \leq 0,
\]

since \( p \geq 0 \) on \([a, n]\). Thus there exists \( \delta_3 > 0, \delta_3 < \delta_2 \) with

\[
G'(y(t)) - G'(\alpha(t)) < 0 \quad \text{for} \quad t \in (t_0 + \delta_3, t_0 + \delta_2).
\]

As a result

\[
0 = G(y(t_0 + \delta_2)) - G(\alpha(t_0 + \delta_2)) < G(y(t_0 + \delta_3)) - G(\alpha(t_0 + \delta_3)),
\]

i.e.

\[
G(\alpha(t_0 + \delta_3)) < G(y(t_0 + \delta_3)),
\]

and this contradicts (2.22). Thus \( G(\alpha(t)) \leq G(y(t)) \) for \( t \in [a, n] \), so \( \alpha(t) \leq y(t) \) for
\( t \in [a, n] \) i.e (2.21) is true.

In particular note \( y(t) > 0 \) for \( t \in (a, n) \). Also

\[
\frac{y^{m+1}(t)}{m+1} = A(t-a) - \int_a^t p(s) y^m(s) \, ds
\]

\[
+ \int_a^t (t-x)[-q(x)f(x, y(x)) + p'(x) y^m(x)] \, dx
\]
where $A$ is given in (2.9) with $\lambda = 1$ and $f^*(x, y(x)) = -q(x) f(x, y(x)) + p'(x) y^m(x)$. Since $y > 0$ on $(a, n]$ we have $y' \in C[a, n]$. Then the change of variables theorem [3 pp. 181] guarantees that $G'(y) = g(y) y' = y^m y'$ on $(a, n)$. Also for $t \in (a, n)$ we have

$$g(y) y' = A - p y^m + \int_a^t [-q(x) f(x, y(x)) + p'(x) y^m(x)] dx,$$

so $g(y) y' \in C^1(a, n)$. Thus for $t \in (a, n)$ we have

$$-q f(t, y) + p' y^m = (g(y) y' + p y^m)' = (g(y) y')' + (p y^m)',$$

so $y$ is a solution of (2.20). □

Suppose the following condition is satisfied:

$$\begin{cases}
\exists \alpha \in C[a, n] \cap C^1(a, n) \text{ with } G(\alpha) \in C^1[a, n], \\
\alpha'^m \alpha' \in C^1[a, n], \ b_0 \geq \alpha > 0 \text{ on } (a, n], \ \alpha(a) = 0, \ \alpha(n) \leq b_0 \\
\text{and } (\alpha'^m \alpha')' + p(\alpha'^m)' + q(f(t, \alpha) \geq 0 \text{ on } (a, n).
\end{cases}
$$

Then we have the following theorem.

**Theorem 2.3.** Suppose (2.2)–(2.6), (2.18), (2.19) and (2.24) are satisfied. Then (2.20) has a solution $y$ with $\alpha(t) \leq y(t) \leq b_0$ for $t \in [a, n]$.

**Proof.** Now the change of variables theorem [3 pp. 181] guarantees that $G'(\alpha) = g(\alpha) \alpha' = \alpha'^m \alpha'$ on $(a, n)$, so for $t \in (a, n)$ we have

$$(G'(\alpha) + p \alpha'^m)' + q f(t, \alpha) = (\alpha'^m \alpha' + p \alpha'^m)' + q f(t, \alpha)$$

$$= (\alpha'^m \alpha')' + (p \alpha'^m)' + q f(t, \alpha)$$

$$\geq (p \alpha'^m)' - p(\alpha'^m)' = p' \alpha'^m.$$

Thus (2.17) holds and the result follows from Theorem 2.2. □

**3. Existence theory on infinite intervals.** In this section we first establish the existence of a solution to

$$\begin{cases}
(G'(y) + p(t) y^m)' + q(t) f(t, y) = p'(t) y^m, \ a < t < \infty \\
y(a) = 0, \ y \text{ bounded on } [a, \infty)
\end{cases}
$$

where $g$ and $G$ are as in Section 2 and $m > 0$ is odd. By a solution to (3.1) we mean a function $y \in BC[a, \infty) \text{ (bounded continuous functions on } [0, \infty))$ with $G(y) \in C^1[a, \infty)$, $G'(y) + p y^m \in AC_{loc}[a, \infty) \cap C^1(a, \infty)$ which satisfies $y(a) = 0$ and the differential equation in (3.1) on $(a, \infty)$.

**Theorem 3.1.** Suppose the following conditions are satisfied:

(3.2) $f : [a, \infty) \times R \rightarrow R$ is continuous

(3.3) $q \in C(a, \infty) \cap L^1_{loc}[a, \infty)$ with $q > 0$ on $(a, \infty)$

(3.4) $p \in C^1(a, \infty)$ with $p \geq 0$ on $(a, \infty)$
Theorem 3.1 guarantees that there exists a solution to (3.5) with \( f(t, 0) \geq 0 \) for \( t \in (a, \infty) \) and \( f(t, b_0) \leq 0 \) for \( t \in (a, \infty) \).

\[
\exists b_0 > 0 \text{ with } f(t, b_0) \leq 0 \text{ for } t \in (a, \infty)
\]

and

\[
\begin{align*}
\exists \mu & \in L^1_{loc}[a, \infty) \text{ with } |f(t, u)| \leq \mu(t) \\
& \text{for a.e. } t \in [a, \infty) \text{ and } u \in [0, b_0].
\end{align*}
\]

Then (3.1) has a solution \( y \) with \( 0 \leq y(t) \leq b_0 \) for \( t \in [a, \infty) \).

**Proof.** Fix \( n \in N = \{1, 2, \ldots\} \) with \( n \geq a + 1 \) and consider the boundary value problem

\[
\begin{align*}
& G'(y) + p(t) y^m' + q(t) f(t, y) = p'(t) y^m, \quad a < t < n \\
& y(a) = 0, \quad y(n) = b_0 > 0.
\end{align*}
\]

Theorem 3.1 guarantees that there exists a solution \( y_n \) to (3.8) (i.e. \( y_n \in C[a, n] \), with \( G(y_n) \in C^1[a, n] \), \( G'(y_n) + p y_n^m \in AC[a, n] \cap C^1(a, n] \)) with \( 0 \leq y_n(t) \leq b_0 \) for \( t \in [a, n] \). We now claim that there exist constants \( A_1 \) and \( A_2 \) (independent of \( n \)) with

\[
|G'(y_n(t))| \leq A_1 + A_2 \int_a^t |p'(s)| ds + \int_a^t \mu(s) ds \text{ for } t \in [a, n].
\]

The mean value theorem guarantees that there exists \( \xi \in (a, a + 1) \) with \( G'(y_n(\xi)) = G(y_n(a + 1)) - G(0) \), and so

\[
|G'(y_n(\xi))| \leq G(b_0) \equiv K_0.
\]

To prove (3.9) we consider first the case when \( t \in [a, n] \) and \( t > \xi \). Integrate (3.8) from \( \xi \) to \( t \) to obtain (note (3.7)),

\[
|G'(y_n(t))| \leq |G'(y_n(\xi))| + |p(t) y^m(t) - p(\xi) y^m(\xi)|
\]

\[
+ b_0^m \int_{\xi}^t |p'(s)| ds + \int_{\xi}^t \mu(s) ds
\]

\[
\leq K_0 + |p(t) y^m(t) - y^m(\xi)| + |p(t) - p(\xi)| y^m(t)
\]

\[
+ b_0^m \int_{a}^t |p'(s)| ds + \int_{a}^t \mu(s) ds
\]

\[
\leq K_0 + 2 b_0^m \sup_{s \in [a, a + 1]} p(s) + b_0^m \left| \int_{\xi}^t p'(s) ds \right|
\]

\[
+ b_0^m \int_{a}^t |p'(s)| ds + \int_{a}^t \mu(s) ds
\]

\[
\leq K_0 + 2 b_0^m \sup_{s \in [a, a + 1]} p(s) + 2 b_0^m \int_{a}^t |p'(s)| ds
\]

\[
+ \int_{a}^t \mu(s) ds,
\]
so (3.9) is true in this case. Next consider the case when \( t < \xi \). Note in particular that \( t < a + 1 \). Integrate the differential equation in (3.8) from \( t \) to \( \xi \) to obtain

\[
|G'(y_n(t))| \leq K_0 + |p(\xi)||y^m(t) - y^m(\xi)| + |p(t) - p(\xi)||y^m(t) + b_0^m \int_t^\xi |p'(s)|\,ds + \int_t^\xi \mu(s)\,ds \\
\leq K_0 + 2b_0^m \sup_{s \in [a,a+1]} p(s) + 2b_0^m \int_a^{a+1} |p'(s)|\,ds + \int_a^{a+1} \mu(s)\,ds,
\]

so (3.9) is again true.

Thus (3.9) is true in all cases, so for \( t, s \in [a,n] \) with \( s < t \) we have

\[
|G(y_n(s)) - G(y_n(t))| = \left| \int_s^t G'(y_n(x))\,dx \right| \leq A_1 |t - s| \\
+ A_2 \int_s^t \int_a^x |p'(z)|\,dz\,dx + \int_s^t \int_a^x \mu(x)\,dz\,dx.
\]

We can do this argument for each \( k \in N \) with \( k \geq n \). Define for \( k \geq n \) an integer

\[
u_k(x) = \begin{cases} y_k(x), & x \in [a,k] \\ b_0, & x \in [k,\infty), \end{cases}
\]

so

\[
G(u_k(x)) = \begin{cases} G(y_k(x)), & x \in [a,k] \\ G(b_0), & x \in [k,\infty). \end{cases}
\]

It is easy to see that

\[
|G(u_k(s)) - G(u_k(t))| \leq A_1 |t - s| + A_2 \left| \int_s^t \int_a^x |p'(z)|\,dz\,dx \right| \\
+ \int_s^t \int_a^x \mu(x)\,dz\,dx \quad \text{for} \ t, s \in [a,\infty).
\]

Consider \( \{u_k\}_{k=n}^\infty \). The Arzela–Ascoli theorem guarantees that there is a subsequence \( N^*_n \) of \( \{n,n+1,\ldots\} \) and a function \( G(z_n) \in C[a,n] \) with \( G(u_k) \) converging uniformly on \( [a,n] \) to \( G(z_n) \) as \( k \to \infty \) through \( N^*_n \). This together with the fact that \( G^{-1} \) is continuous and \( G(u_k(t)) \in [0,b_0] \) for \( t \in [a,n] \) implies \( u_k \) converges uniformly on \( [a,n] \) to \( z_n \) as \( k \to \infty \) through \( N^*_n \). Note \( 0 \leq z_n(t) \leq b_0 \) for \( t \in [a,n] \), let \( N_n = N^*_n \setminus \{n\} \). Also the Arzela–Ascoli theorem guarantees the existence of a subsequence \( N^*_{n+1} \) of \( N_n \) and a function \( G(z_{n+1}) \in C[a,n+1] \) with \( G(u_k) \) converging uniformly on \( [a,n+1] \) to \( G(z_{n+1}) \) as \( k \to \infty \) through \( N^*_{n+1} \), and so \( u_k \) converges uniformly on \( [a,n+1] \) to \( z_{n+1} \) as \( k \to \infty \) through \( N^*_{n+1} \). Note \( 0 \leq z_{n+1}(t) \leq b_0 \) for \( t \in [a,n+1] \) and \( z_{n+1} = z_n \) on \( [a,n] \) since \( N^*_{n+1} \subseteq N_n \). Let \( N_{n+1} = N^*_{n+1} \setminus \{n+1\} \). Proceed inductively to obtain for \( m \in \{n+2,n+3,\ldots\} \) a subsequence \( N_m \) of \( N_{m-1} \) and a function \( z_m \in C[a,m] \) with \( u_k \) converges uniformly on \( [a,m] \) to \( z_m \) as \( k \to \infty \) through \( N^*_m \). Note \( 0 \leq z_m(t) \leq b_0 \) for \( t \in [a,m] \) and \( z_m = z_{m-1} \) on \( [a,m-1] \). Let \( N_m = N^*_m \setminus \{m\} \).
Define a function $y$ as follows. Fix $x \in (a, \infty)$ and let $l \in \{n, n + 1, \ldots\}$ with $x \leq l$. Then define $y(x) = z_l(x)$ so $y \in C[a, \infty)$ and $0 \leq y(t) \leq b_0$ on $[a, \infty)$. Also for $n \in \mathbb{N}$ we have

$$G(u_n(x)) = A_l(x - a) - \int_a^x p(s) u_n^m(s) \, ds$$

$$+ \int_a^x (x - s) [-q(s) f(s, u_n(s)) + p'(s) u_n^m(s)] \, ds$$

where

$$A_l(l - a) = G(u_n(l)) + \int_a^l p(s) u_n^m(s) \, ds$$

$$- \int_a^l (l - s) [-q(s) f(s, u_n(s)) + p'(s) u_n^m(s)] \, ds.$$ Let $n \to \infty$ through $N_l$ to obtain

$$G(z_l(x)) = A_l^*(x - a) - \int_a^x p(s) z_l^m(s) \, ds$$

$$+ \int_a^x (x - s) [-q(s) f(s, z_l(s)) + p'(s) z_l^m(s)] \, ds$$

where

$$A_l^*(l - a) = G(z_l(l)) + \int_a^l p(s) z_l^m(s) \, ds$$

$$- \int_a^l (l - s) [-q(s) f(s, z_l(s)) + p'(s) z_l^m(s)] \, ds.$$ Thus

$$G(y(x)) = A_l^*(x - a) - \int_a^x p(s) y^m(s) \, ds$$

$$+ \int_a^x (x - s) [-q(s) f(s, y(s)) + p'(s) y^m(s)] \, ds$$

where

$$A_l^*(l - a) = G(y(l)) + \int_a^l p(s) y^m(s) \, ds$$

$$- \int_a^l (l - s) [-q(s) f(s, y(s)) + p'(s) y^m(s)] \, ds.$$ We can do this for each $x > a$ and so the above integral equation yields for each $l \in \mathbb{N}$ and $t \in [a, l]$ that

$$G'(y(t)) = -p(t) y^m(t) + A_l^* + \int_a^l [-q(s) f(s, y(s)) + p'(s) y^m(s)] \, ds,$$ so $G' \in C^1[a, l]$, $G'(y) + p y^m \in AC[a, l] \cap C^1(a, l]$ and

$$(G'(y) + p y^m)'(t) = q(t) f(t, y(t)) + p'(t) y^m(t) \quad \text{for} \quad t \in [a, l].$$
Keeping the application in section 1 in mind it is important to discuss the situation when our solution to (3.1) is positive on \((a, \infty)\). Suppose the following conditions hold:

\[
\begin{aligned}
\exists \alpha &\in BC[a, \infty) \text{ with } G(\alpha) \in C^1[a, \infty), \quad \alpha'(\alpha) + p \alpha^m \\
\in AC_{\text{loc}}[a, \infty) \cap C^1(a, \infty) \text{ with } b_0 \geq \alpha > 0 \text{ on } (a, \infty), \\
\alpha(a) &= 0 \text{ and } (\alpha'(\alpha) + p \alpha^m)(t) + q(t) f(t, \alpha) \geq p'(t) \alpha^m(t) \\
\text{on } (a, \infty)
\end{aligned}
\]

(3.10)

Also in this case we discuss the boundary value problem

\[
\begin{aligned}
\{ &\text{for each } t \in [a, \infty) \text{ we have } q(t) [f(t, y) - f(t, \alpha(t))] \geq 0 \\
&\text{for } 0 \leq y \leq \alpha(t)
\end{aligned}
\]

(3.11)

and

\(p' > 0 \text{ on } (a, \infty)\).

Also in this case we discuss the boundary value problem

\[
\begin{aligned}
\{ &\text{(g(y))y' + p(y^m)y' + q f(t, y) = 0, } a < t < \infty \\
y(\alpha) = 0, y \text{ bounded on } [a, \infty).
\end{aligned}
\]

(3.13)

By a solution to (3.13) we mean a function \(y \in BC[a, \infty) \cap C^1(a, \infty)\) with \(y' \in C^1(a, \infty)\) which satisfies \(y(\alpha) = 0\) and the differential equation in (3.13) on \((a, \infty)\).

**Theorem 3.2.** Suppose (3.2)–(3.7), (3.10), (3.11) and (3.12) hold. Then (3.1) has a solution \(y\) with \(0 \leq y(t) \leq b_0\) for \(t \in [a, \infty)\). In addition \(y \in C^1(a, \infty)\) with \(G'(y) = y' \text{ on } (a, \infty)\) and \(y\) is a solution of (3.13).

**Proof.** Fix \(n \in N = \{1, 2, \ldots\}\) with \(n \geq a + 1\) and consider (3.8). Theorem 2.2 guarantees that there exists a solution \(y_n\) to (3.8) with \(\alpha(t) \leq y_n(t) \leq b_0\) for \(t \in [a, n]\). Essentially the same reasoning as in Theorem 3.1 guarantees that (3.1) has a solution \(y \in BC[a, \infty)\) with \(G(y) \in C^1[a, \infty)\), \(G'(y) + py^m \in AC_{\text{loc}}[a, \infty) \cap C^1(a, \infty)\) and with \(\alpha(t) \leq y(t) \leq b_0\) for \(t \in [a, \infty)\). In particular note \(y > 0\) on \((a, \infty)\). Fix \(l \in \{n, n+1, \ldots\}\) and consider \(t \in [a, l]\). We know (see Theorem 3.1) that

\[
\frac{y^{m+1}(t)}{m+1} = A^*_t (l-a) - \int_a^t p(s) y^m(s) \, ds \\
+ \int_a^t (t-s) [-q(s) f(s, y(s)) + p'(s) y^m(s)] \, ds
\]

where

\[
A^*_t (l-a) = G(y(l)) + \int_a^t p(s) y^m(s) \, ds \\
- \int_a^t (l-s) [-q(s) f(s, y(s)) + p'(s) y^m(s)] \, ds,
\]

and since \(y > 0\) on \((a, l)\) we have \(y' \in C^1(a, l)\). Then [3 pp. 181] guarantees that \(G'(y) = g(y) y' = y^m y' \text{ on } (a, l)\). Also for \(t \in (a, l)\) we have

\[
g(y) y' = A^*_t - py^m + \int_a^t [-q(s) f(s, y(s)) + p'(s) y^m(s)] \, ds,
\]
so \( g(y)y' \in C^1(a,l) \). In addition for \( t \in (a,l) \) we have
\[
-qf(t,y) + p'(y')m = (g(y)y' + py'm)' = (g(y)y')' + (pym)'.
\]
We can do this for each \( l \in N \), so \( y \) is a solution of (3.13). \( \square \)

**Remark 3.1.** If \( \lim_{t \to \infty} \alpha(t) = b_0 \) (here \( b_0 \) is as in (3.6)) then the solution \( y \) to (3.1) (guaranteed from Theorem 3.2) is a solution of the boundary value problem
\[
\begin{cases}
(g(y)y')' + p(y')m = 0, & 0 < t < \infty \\
y(0) = 0, & \lim_{t \to \infty} y(t) = b_0.
\end{cases}
\]
(3.14)

Suppose the following condition is satisfied:
\[
\begin{cases}
\exists \alpha \in BC([a,\infty)) \cap C^1(a,\infty) \text{ with } G(\alpha) \in C^1[a,\infty), \\
\alpha^m \alpha' \in C^1(a,\infty), \quad b_0 \geq \alpha > 0 \text{ on } (a,\infty), \quad \alpha(a) = 0 \\
\text{and } (\alpha^m \alpha')' + p(\alpha^m)' + q(t)f(t,\alpha) \geq 0 \text{ on } (a,\infty).
\end{cases}
\]
(3.15)

Then we have the following theorem.

**Theorem 3.3.** Suppose (3.2)–(3.7), (3.11), (3.12) and (3.15) hold. Then (3.13) has a solution \( y \) with \( \alpha(t) \leq y(t) \leq b_0 \) for \( t \in [a,\infty) \).

**Proof.** Now [3 pp. 181] guarantees that \( G'(\alpha) = g(\alpha) \alpha' = \alpha_m \alpha' \) on \( (a,l) \) for each \( l \in N \), so for \( t \in (a,l) \) we have
\[
(G'(\alpha) + p(\alpha^m)')' + q(t,\alpha) = (\alpha_m \alpha')' + q(t,\alpha)
\]
\[
= (\alpha_m \alpha')' + q(t,\alpha)
\]
\[
\geq (p(\alpha^m)') - p(\alpha^m)' = p'(\alpha^m).
\]
Thus (3.10) holds and the result follows from Theorem 3.2. \( \square \)

**Remark 3.2.** If \( \lim_{t \to \infty} \alpha(t) = b_0 \) (here \( b_0 \) is as in (3.6)) then the solution \( y \) to (3.13) (guaranteed from Theorem 3.3) is a solution of (3.14).

**Example.** (Slender dry patch in a liquid film).

From Section 1 consider the boundary value problem
\[
\begin{cases}
(y^3 y')' + t(y^3)' = 0, & 1 < t < \infty \\
y(1) = 0, & \lim_{t \to \infty} y(t) = G_0 > 0.
\end{cases}
\]
(3.16)

We will now use Theorem 3.3 (with Remark 3.2) to show that (3.16) has a solution. To see this consider
\[
\begin{cases}
(y^3 y' + t y^3)' = y^3, & 1 < t < \infty \\
y(1) = 0, & y \text{ bounded on } [1,\infty).
\end{cases}
\]
(3.17)

**Remark 3.3.** Notice \( y \equiv 0 \) is a solution of (3.17).

Let \( m = 3, a = 1, p = t, q \equiv 0, f(t,y) \equiv 0, b_0 = G_0 \) and
\[
g(z) = \begin{cases}
z^3, & z \geq 0 \\
-z^3 = |z|^3, & z < 0.
\end{cases}
\]
Clearly (3.2)–(3.7), (3.11) and (3.12) hold. Let

\[ \alpha(t) = A \int_1^t \exp \left( -\frac{3s^2}{2G_0} \right) \, ds \]

where

\[ A = \frac{G_0}{\int_1^\infty \exp \left( -\frac{3s^2}{2G_0} \right) \, ds} . \]

Note \( \alpha(1) = 0 \) and \( \alpha' = A \exp \left( -\frac{3t^2}{2G_0} \right) \). Also for \( t \in (1, \infty) \) we have

\[
(\alpha^3 \alpha')' + t (\alpha^3)' = A^4 \left( \int_1^t \exp \left( -\frac{3s^2}{2G_0} \right) \, ds \right)^2 \left[ 3 \exp \left( -\frac{3t^2}{G_0} \right) \right] \\
- \frac{3t}{G_0} \exp \left( -\frac{3t^2}{2G_0} \right) \left( \int_1^t \exp \left( -\frac{3s^2}{2G_0} \right) \, ds \right) \\
+ 3t A^3 \left( \int_1^t \exp \left( -\frac{3s^2}{2G_0} \right) \, ds \right)^2 \exp \left( -\frac{3t^2}{2G_0} \right) \\
= 3t A^3 \left( \int_1^t \exp \left( -\frac{3s^2}{2G_0} \right) \, ds \right)^2 \exp \left( -\frac{3t^2}{2G_0} \right) \\
\times \left[ 1 - \frac{A}{G_0} \int_1^t \exp \left( -\frac{3s^2}{2G_0} \right) \, ds \right] \\
+ 3 A^4 \left( \int_1^t \exp \left( -\frac{3s^2}{2G_0} \right) \, ds \right)^2 \exp \left( -\frac{3t^2}{G_0} \right) \\
\geq 0,
\]

since

\[ \frac{A}{G_0} \int_1^t \exp \left( -\frac{3s^2}{2G_0} \right) \, ds = \frac{\int_1^t \exp \left( -\frac{3s^2}{2G_0} \right) \, ds}{\int_1^\infty \exp \left( -\frac{3s^2}{2G_0} \right) \, ds} \leq 1. \]

Thus (3.15) holds so Theorem 3.3 guarantees that (3.17) has a solution \( y \) with \( \alpha(t) \leq y(t) \leq G_0 \) for \( t \in [1, \infty) \). Also since \( \lim_{t \to \infty} \alpha(t) = G_0 \) then \( y \) is a solution of (3.16).

REFERENCES
