Abstract. We give sufficient conditions ensuring existence and regularity of a radial solution to the following equation

$$\det (\phi_{ij}) = F(|x|, \phi, |\nabla \phi|), \text{ in } \Omega$$

$$\phi|_{\partial \Omega} = c$$

when $\Omega$ is an exterior domain.

1. Introduction. In this work, we consider the Dirichlet problem for real Monge-Ampère equations in exterior domains. More precisely, let $B \subset \mathbb{R}^n$ be an open ball, centered at the origin, that can be supposed, without loss of generality, to be the unit ball. Our purpose is to establish the existence of radial, convex solution $u \in C^2(\mathbb{R}^n \setminus B)$ of radially symmetric Monge-Ampère equation

$$\begin{cases}
\det (\phi_{ij}) = F(|x|, \phi, |\nabla \phi|), & \text{in } \mathbb{R}^n \setminus B \\
\phi|_{\partial B} = c
\end{cases} \tag{1}$$

where $F$ is a nonnegative continuous function. As usual, $|x|$ denotes the Euclidean length of $x = (x_1, \ldots, x_n)$ and $n$ is (all over this paper) the dimension of our Euclidean space. Additional hypothesis on $F$ are described in §2.

When $\Omega$ is a strictly convex domain, this problem has received considerable study. Not many results are known about the solutions in unbounded domains. In the case when $F > 0$, F.Finster and O.C. Schnürer [2] proved the existence of smooth, strictly convex solution to (1) under some restrictions on $F$. We can also cite the work of T. Kusano and Ch.A. Swanson [3] related to radially symmetric two-dimensional elliptic Monge-Ampère equations.

Our attention will be directed toward the construction of radial solutions $u(x) = u(t) \text{ of } (1) \text{, } t = |x|$. Direct computation (see [1]), shows that solving the equation (1) in $C^2$ is equivalent to solving the ordinary differential equation

$$\begin{cases}
\left[(y')^n\right]' = nt^{n-1}F(t, y, y'), & \text{if } t > 1 \\
y(1) = c
\end{cases} \tag{2}$$

Without loss of generality, we can take $c = 0$.

If we take as initial condition $y'(1) = 0$, we can easily transform (2) into the following integro-differential equation

$$y(r) = \int_1^r \left[\int_1^p nt^{n-1}F(t, y(t), y'(t)) dt\right]^{\frac{1}{n}} dp \tag{3}$$

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Example. Let \( F(r) = (r - 1)^{n-1-\varepsilon} r^{1-n} \), with \( \varepsilon > 0 \) small enough. Then, 
\[ u'(r) = \left[ \frac{n}{n-\varepsilon} \right]^{1/2} (r - 1)^{1-\frac{n}{2}}. \]
In this case \( F \in C^0 \), but \( u \notin C^2 \).

This example shows that even when \( F \) depends only on \( r \), it may not yield a \( C^2 \) solution, if \( F \) is allowed to vanish in the domain. This implies that we should place some restrictions on \( F \).

Throughout this work, \( F \) satisfies some hypothesis be selected from the following list:

\((H_1)\):

e) \( F(t, y, z) \) is a nonincreasing function with respect to both \( y \) and \( z \) for each fixed \( (t, z) \) and \( (t, y) \), respectively.
ed) \( \int_1^{+\infty} t^{n-1} F(t, 0, 0) \, dt < +\infty \)

\((H_2)\): \( F(t, y, z) \leq C_0 t^{-\alpha} |y|^\beta |z|^\theta \), with \( \beta \geq 0 \), \( \alpha > \beta \), \( \theta \geq 0 \) and \( C_0 \leq \frac{2-\beta}{n} \) if \( \beta + \theta = n \).

\((H_3)\):

e) \( F(t, y, z) \) is a nondecreasing function with respect to both \( y \) and \( z \) for each fixed \( (t, z) \) and \( (t, y) \), respectively.
ed) There exists a constant \( a > 0 \) such that

\[ \int_1^{+\infty} nt^{n-1} F(t, (t - 1) a, a) \, dt \leq a^n \]

\((H_4)\): \( F(t, y, z) = (t - 1)^l \tilde{F}(t, y, z) \), with \( \tilde{F}(1, y, 0) \neq 0 \), for \( y \geq 0 \), \( l \geq n - 1 \), \( \tilde{F} \in C^0 \).

An example of a Monge-Ampère equation satisfying \((H_3)\) is the Gauss curvature equation

\[ \det (u_{ij}) = p(|x|) u^\gamma \left( 1 + |\nabla u|^2 \right)^\delta, \quad x \in \mathbb{R}^n \setminus B \]

with \( \gamma, \delta \geq 0 \), \( 2\delta + \gamma < n \) and \( p \) is a non-negative function satisfying:

\[ \int_1^{+\infty} t^{n+\gamma-1} p(t) \, dt < +\infty \]

In the following, \( \tilde{F} \) is used as introduced in \((H_4)\). We shall prove

**Theorem A.** If \((H_4)\) and either \((H_1)\), \((H_2)\) or \((H_3)\) holds, equation (1) has an infinitude of radial convex solutions \( u \in C^2 \) such that \( \frac{u(x)}{|x|} \) has a positive finite limit at \( \infty \).

**Theorem B.** If we suppose, in addition to the hypothesis of Theorem A, that

\[ \tilde{F} \in C^k \left( (\mathbb{R}^n \setminus B) \times \mathbb{R}^2 \right) \] (4)

and
either \( \frac{l+1}{n} \in \mathbb{N} \) or \( \frac{l+1}{n} \geq k + 1 \) \hspace{1cm} (5)

then the solutions given by theorem A are in \( C^{k+2} \)

2. Proof of theorem A. To prove the existence of a radially symmetric convex solution to the problem (1), we need to introduce the Frechet space \( C^1 \) of all continuously differentiable functions in \([1, +\infty[\), with the topology of uniform convergence of functions and their first derivatives on compact intervals. Consider now the closed convex subset \( K_R \) of \( C^1 \)

\[
K_R = \{ y \in C^1 \mid y(1) = 0, 0 \leq y'(t) \leq R \} \hspace{1cm} (6)
\]

and the operator \( T : K_R \rightarrow C^1 \) defined by

\[
T(y)(r) = \int_1^r \left[ \int_1^r nt^{n-1} F(t, y(t), y'(t)) \, dt \right] ^{\frac{1}{n}} \, d\rho, \quad r \geq 1 \hspace{1cm} (7)
\]

In order to prove that \( T \) has a fixed point \( y \in K_R \), we need to verify that \( T \) maps \( K_R \) continuously into a relatively compact subset of \( K_R \).

If \( y \in K_R \), (7) implies that \( T(y)(1) = 0 \) and

\[
0 \leq (Ty)'(r) = \left[ \int_1^r nt^{n-1} F(t, y(t), y'(t)) \, dt \right] ^{\frac{1}{n}}
\]

We shall need to verify that we can find a constant \( R > 0 \) such that

\[
\left[ \int_1^{+\infty} ns^{n-1} F(s, y(s), y'(s)) \, ds \right] ^{\frac{1}{n}} \leq R, \quad \forall y \in K_R \hspace{1cm} (8)
\]

* If \( F \) satisfies (\( H_1 \)), we can write using (\( H_1 \) (i)),

\[
(Ty)'(r) \leq \left[ \int_1^r ns^{n-1} F(s, 0, 0) \, ds \right] ^{\frac{1}{n}}
\]

by (\( H_1 \) (ii)), it suffices then to take

\[
R = \left[ n \int_1^{+\infty} s^{n-1} F(s, 0, 0) \, ds \right] ^{\frac{1}{n}}
\]

and we get

\[
(Ty)'(r) \leq R
\]

* When \( F \) satisfies (\( H_2 \)), then, since \( y(r) = \int_1^r y'(t) \, dt \), we get by (6),
\[ |y(r)| \leq (r - 1) R, \]

so,

\[
(Ty)'(r) \leq \left[ \int_1^r nC_0 s^{-\alpha - 1} (s - 1)^\beta R^{\beta + \theta} ds \right]^{\frac{1}{\theta}} \leq \left( \frac{n}{\alpha - \beta} C_0 \right)^{\frac{1}{\theta}} R^{\frac{\alpha - \beta}{\theta}}
\]

In order to get (8), it suffices to take \( R \) small enough when \( (\beta + \theta) > n \), big enough when \( (\beta + \theta) < n \). In the case when \( \beta + \theta = n \) and \( C_0 \leq \frac{\alpha - \beta}{n} \), any positive constant \( R \) lead to

\[
(Ty)'(r) \leq R
\]

* Finally, if \( F \) satisfies \((H_3)\), then, assumption \((H_3)\) (i) shows that

\[
(Ty)'(r) \leq \left[ \int_1^r n s^{n-1} F(s, (s-1) R, R) ds \right]^{\frac{1}{\theta}}
\]

it suffices then to take \( R = a \) to ensure by \((H_3)\) (ii) the inequality (8).

To establish the continuity of \( T \), let \((y_k)\) be a sequence in \( K_R \) with \( \lim_{k \to +\infty} y_k = y \in C^1 \) in the \( C^1 \)-topology. By the dominated convergence theorem, we have then

\[
\lim_{k \to +\infty} \int_1^r n s^{n-1} F(s, y_k(s), y_k'(s)) ds = \int_1^r n s^{n-1} F(s, y(s), y'(s)) ds
\]

uniformly on \([1, +\infty]\), from which \( Ty_k \) and \((Ty_k)'\) converge uniformly to \( Ty \) and \((Ty)'\), respectively, on compact intervals in \([1, +\infty]\). This means that \( Ty_k \) converges to \( Ty \) in the \( C^1 \)-topology.

The relative compactness of \( T(K_R) \) is a consequence of Ascoli’s Theorem; we need only verify the local uniform boundedness and local equicontinuity of the sets \( T(K_R) \) and \( T(K_R)' = \{(Ty)', y \in K_R\} \).

Let us denote \( G(t) = nF(t, u(t), u'(t)) \) and \( \tilde{G}(t) = n\tilde{F}(t, u(t), u'(t)) \).

For every \( y \in K_R \), \( 1 \leq t_1 \leq t_2 \), the inequality \( a^\frac{1}{\theta} - b^\frac{1}{\theta} \leq (a - b)^\frac{1}{\theta} \), true for \( a \geq b \geq 0 \), implies

\[
(Ty)'(t_2) - (Ty)'(t_1) = \left( \int_{t_1}^{t_2} t^n - 1 G(t) dt \right)^{\frac{1}{\theta}} - \left( \int_{t_1}^{t_2} t^n - 1 G(t) dt \right)^{\frac{1}{\theta}}
\]

\[
\leq \left( \int_{t_1}^{t_2} t^n - 1 G(t) dt \right)^{\frac{1}{\theta}}
\]

* If \( F \) satisfies \((H_1)\), then

\[
G(t) \leq nF(t, 0, 0)
\]

and

\[
(Ty)'(t_2) - (Ty)'(t_1) \leq \left( \int_{t_1}^{t_2} n t^{n-1} F(t, 0, 0) dt \right)^{\frac{1}{\theta}} \to 0, \text{ as } t_1, t_2 \to \infty
\]

* If \( F \) satisfies \((H_2)\), then,

\[
(Ty)'(t_2) - (Ty)'(t_1) \leq \left( \int_{t_1}^{t_2} n C_0 t^{n-1} (t - 1)^{\beta} R^{\beta + \theta} dt \right)^{\frac{1}{\theta}}
\]

\[
\leq C_1 \left( \int_{t_1}^{t_2} t^{\beta - \alpha - 1} dt \right)^{\frac{1}{\theta}} \to 0, \text{ as } t_1, t_2 \to \infty
\]

* Finally, when \( F \) satisfies \((H_3)\), then by (i), since \( R = a \),

\[
G(t) \leq nF(t, (t - 1)a, a)
\]

and
\[(Ty)' (t_2) \leq (f_{t_1}^{t_2} nt^{n-1} F (t, (t-1)a, a) dt)^{\frac{1}{n}} \to 0, \text{ as } t_1, t_2 \to \infty\]

Then, in all these cases, for any compact interval \(I\) in \[1, +\infty[\) and arbitrary \(\varepsilon > 0\), there is a corresponding \(\delta > 0\), independent of \(t_1, t_2\) and \(y \in \mathcal{K}_R\), such that
\[
| (Ty)' (t_2) - (Ty)' (t_1) | \leq \varepsilon
\]
for all \(t_1, t_2 \in I\) with \(|t_1 - t_2| < \delta\).

The local equicontinuity of \(T(\mathcal{K}_R)\) can be verified in the same way, and the local uniform boundedness is obvious.

Therefore the Schauder-Tychonoff fixed point theorem ([5]; lemma 1 and [6]; Theorem 4.5.1.) implies that \(T\) has a fixed point \(u \in \mathcal{K}_R\), satisfying the integro-differential equation (3) for any \(R\) such that (8) holds. It remains to prove that \(u' \in C^1\).

For \(t > 1\), we have

\[
u' (t) = \left( \int_1^t s^{n-1} (s-1)^l G (s) \, ds \right)^{\frac{1}{n}}
\]

Since \(G (1) \neq 0\), then \(u' \in C^1 [1, +\infty[\) and

\[
u'' (t) = t^{n-1} (t-1)^l G (t) \left( \int_1^t s^{n-1} (s-1)^l G (s) \, ds \right)^{\frac{1}{n}-1}
= t^{n-1} (t-1)^{l+1} G (t) \left( \int_1^t (t-1)^n \, s^{n-1} \, G ((t-1) s + 1) \, ds \right)^{\frac{1}{n}-1}
\]

which gives

\[
\lim_{t \to 1^+} \nu'' (t) = \begin{cases} 0, & \text{if } l > n-1 \\ \left[ \frac{G (1)}{t-1} \right]^{\frac{1}{n}-1} \, G (1) \frac{n}{n-1} & \text{if } l = n-1 \end{cases}
\]

Hence, \(u \in C^2 [1, +\infty[\). It is not to be noted that \(u\) is a solution of (1) satisfying \(u(1) = 0\) and \(u'(1) = 0\).

Furthermore, the relation (3) and the inequality (8) imply that the limit

\[
\lim_{t \to \infty} \frac{u(t)}{t} = \lim_{t \to \infty} u' (t) = \left[ \int_1^{\infty} n s^{n-1} F (s, u (s), u' (s)) \, ds \right]^{\frac{-1}{n}}
\]

is positive and finite, proving the asymptotic property in theorem A.

Since any non-negative constant \(b\) will serve as initial value \(y'(1) = b\), there exists an infinity of radial convex solutions to our problem.

3. Proof of theorem B. In this section, we study the regularity of the solution \(u\) given by theorem A. To prove the \(C^{k+2}\) regularity of \(u\), let us proceed by induction on \(k \in \mathbb{N}\). For \(k = 0\), we have established in section 2, that \(u \in C^2\). Suppose that \(F \in C^{k-1} \Rightarrow u \in C^{k+1}\) for some fixed \(k \geq 1\). Assume now that \(F \in C^k\). It follows in particular that \(u \in C^{k+1}\). Hence, from the integral formula (7) and the hypothesis \((H_4)\), we get \(u \in C^{k+2} [1, +\infty[\). It remains to check the regularity of \(u\) at the boundary \(t = 1\).

The following preliminary result will be needed

**Lemma** ([4] corollary 4.2). The \(k^{th}\) derivative of \(g^{\frac{k}{n}}\), can be written as a sum of terms of the form

\[
g^{\frac{k}{n}-\lambda} P_{\lambda} (g', g'', ..., g^{(k+1-\lambda)})
\]
where $P_\lambda$ is a monomial of degree $\lambda \leq k$ and of weighted degree $k$.

Now, using the notation
\[ H_y(t) = \int_0^1 [(t - 1)s + 1]^{n-1} s^l \tilde{G}_y((t - 1)s + 1) \, ds, \]
we can write
\[ u'(t) = (t - 1)^{\frac{k+1}{n}} H_u^\frac{k}{n}(t) \]
where, by the induction hypothesis, $H_u \in C^k$. Then,
\[ u^{(k+1)}(t) = \sum_{i=0}^{k} \binom{k}{i} \left[ (t - 1)^{\frac{k+1}{n}} \right]^{(i)} \left( H_u^\frac{k}{n} \right)^{(k-i)}(t) \]
furthermore, applying the above lemma, we get the following
\[ \left( H_u^\frac{k}{n} \right)^{(l)} = \sum_{i=2}^{l} c_i H_u^{\frac{k+1}{n}-1} P_i \left( H_u^{(i)}, ..., H_u^{(i+1-k)} \right) + \frac{1}{n} H_u^{\frac{k+1}{n}-1} H_u^{(l)}, \forall l \leq k \]
Since, $\forall j \leq k$,
\[ (H_u)^{(j)}(t) = \sum_{i=0}^{j} c_{i,j} \int_0^1 \left[ (t - 1)s + 1 \right]^{n-1} s^l \tilde{G}_u^{(i)}((t - 1)s + 1) \, ds \]
it suffices then to prove that
\[ f(t) = (t - 1)^{\frac{k+1}{n}} h_k \in C^1([1, +\infty]) \]
where $h_k(t) = \int_0^1 [(t - 1)s + 1]^{n-1} s^{k+1} \tilde{G}_u^{(k)}((t - 1)s + 1) \, ds$.
Differentiating $f$, yields
\[ \forall t > 1, f'(t) = (t - 1)^{\frac{k+1}{n}-1} \left[ t^{n-1} \tilde{G}_u^{(k)}(t) + c h_k(t) \right] \]
which implies
\[
\lim_{r \to 1^+} f'(t) = \begin{cases} 
0 & \text{if } l > n - 1 \\
-\tilde{G}_u^{(k)}(1) \left[ 1 + c \int_0^1 s^{l+k} \, ds \right] & \text{if } l = n - 1 
\end{cases}
\]
Consequently, $f \in C^1([1, +\infty])$, which completes the proof of theorem B.

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REFERENCES

