PARTIAL REGULARITY OF WEAK SOLUTIONS TO MAXWELL’S EQUATIONS IN A QUASI-STATIC ELECTROMAGNETIC FIELD

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Dedicated to Professor Neil Trudinger on the occasion of his 65th birthday

Abstract. We study Maxwell’s equations in a quasi-static electromagnetic field, where the electrical conductivity of the material depends on the temperature. By establishing the reverse Hölder inequality, we prove partial regularity of weak solutions to the non-linear elliptic system and the non-linear parabolic system in a quasi-static electromagnetic field.

Key words. Partial regularity, elliptic systems, parabolic systems.

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1. Introduction. In this paper, let \( \Omega \) be a domain in \( \mathbb{R}^n \) with \( n \geq 3 \), and let \( u(x) \) and \( H^i(x) \) for \( i = 1, \ldots, n \) be scalar functions defined on \( \Omega \). For any positive integer \( k \), let \( \Lambda_k(\Omega) \) denote the space of \( k \)-forms on \( \Omega \). We have the usual exterior derivative \( d \) of forms with \( d : \Lambda_k(\Omega) \to \Lambda_{k+1}(\Omega) \). Consider a 1-form \( H = \sum_{i=1}^n H^i(x) dx_i \), which may be regarded as a connection in differential geometry. We define the curvature \( F \) of the connection \( H \) by

\[
F = dH = \sum_{i<j} F^{ij} dx_i \wedge dx_j,
\]

where \( F^{ij} = \frac{\partial H^j}{\partial x_i} - \frac{\partial H^i}{\partial x_j} \) (e.g. [9]).

Let \( \ast \) be the Hodge star linear operator which assigns to each \( k \)-form on \( \Omega \) an \((n-k)\)-form and which satisfies

\[
\ast \ast = (-1)^{k(n-k)}.
\]

We have a product \( \langle \cdot, \cdot \rangle \) in the \( k \)-form space \( \Lambda_k(\Omega) \)

\[
\langle a, b \rangle dx_1 \wedge \ldots \wedge dx_n = a \wedge \ast b, \quad |a|^2 = \langle a, a \rangle
\]

for all \( a, b \in \Lambda_k(\Omega) \) (e.g. [15]).

By definition, we have

\[
|H|^2 = \langle H, H \rangle = \sum_{i=1}^n (H^i)^2, \quad |dH|^2 = \langle dH, dH \rangle = \frac{1}{2} \sum_{i,j=1}^n (F^{ij})^2.
\]

Let \( d^\ast \) be the adjoint operator of \( d \) with \( d^\ast = (-1)^{n+k+1} \ast d^\ast : \Lambda_k(\Omega) \to \Lambda_{k-1}(\Omega) \) and

\[
\int_{\Omega} \langle da, b \rangle dx = \int_{\Omega} \langle a, d^\ast b \rangle dx
\]

for \( a \in \Lambda_k(\Omega), b \in \Lambda_{k+1}(\Omega) \), where \( b \) or \( a \) has compact support inside of \( \Omega \).

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We consider the following system
\begin{align}
(1.1) \quad d^*\left[\sigma(u)dH\right] &= 0 \quad \text{in } \Omega \\
(1.2) \quad -\Delta u &= \sigma(u)|dH|^2 \quad \text{in } \Omega
\end{align}
where \(\sigma\) is a positive function defined on \(\mathbb{R}\).

We say that a pair \((u, H)\) is a weak solution to the system (1.1)-(1.2) if \(u \in W^{1,q}(\Omega)\) for some \(q \in (1, \frac{n}{n-1})\) and \(H \in W^{1,2}(\Omega; \mathbb{R}^n)\), and the pair \((u, H)\) satisfies the following:
\[
\int_{\Omega} \left(\sigma(u)dH, d\phi\right) dx = 0,
\]
\[
\int_{\Omega} \nabla u \cdot \nabla \psi dx = \int_{\Omega} \sigma(u)|dH|^2 \psi dx
\]
for all \(\phi := \sum_{i=1}^{n} \phi^i(x)dx_1\) for \(i = 1, \ldots, n\), where \(\phi^i \in C^0(\Omega; \mathbb{R})\) and \(\psi \in C^0(\Omega; \mathbb{R})\).

**Assumption (S).** \(\sigma(u)\) is uniformly Hölder continuous in \(\mathbb{R}\) and there exist two constants \(\sigma_1 \) and \(\sigma_2\) such that
\[0 < \sigma_1 \leq \sigma(u) \leq \sigma_2.\]

Uniform Hölder continuity above can be replaced by the assumption of Hölder continuity of \(\sigma(u)\) (see [1]). Without loss of generality, we assume that Assumption (S) holds throughout this paper.

In this paper, we prove the partial regularity of the above weak solution to the system (1.1)-(1.2) in the following:

**Theorem A.** Let a pair \((u, H)\) be a weak solution to the system (1.1)-(1.2) with \(u \in W^{1,q}(\Omega, \mathbb{R})\) for some \(q \in (1, \frac{n}{n-1})\), \(H \in W^{1,2}(\Omega; \mathbb{R}^n)\), and \(d^*H(x) = 0\) for a.e. \(x \in \Omega\). Then there exists an open subset \(\Omega_0\) of \(\Omega\) such that the solution \((u, H)\) is \(C^{1,\alpha}\) locally in \(\Omega_0\), and \(H^{n-q_1}(\Omega \setminus \Omega_0) = 0\) for some \(q_1 > \frac{n}{n-1}\), where \(H^{n-q_1}\) denotes the \((n-q_1)\)-dimensional Hausdorff measure.

The system (1.1)-(1.2) is not elliptic since it is invariant under the gauge transformation \((u, H) \rightarrow (u, H + \nabla \xi)\) for all \(\xi \in W^{2,2}(\Omega)\). By a gauge transformation, one can fix a gauge satisfying
\[d^*H = \operatorname{div} H = \sum_i \frac{\partial H^i}{\partial x_i} = 0.\]

The system (1.1)-(1.2) with \(d^*H = 0\) on \(\Omega\) is a quasi-linear elliptic system which has a natural growth structure. When \(n = 3\), Yin in [13], [14] proved the existence of weak solutions \((u, H)\) to (1.1)-(1.2) with \(u \in W^{1,q}(\Omega, \mathbb{R})\), \(q \in (1, \frac{n}{n-1})\), \(H \in W^{1,2}(\Omega; \mathbb{R}^3)\), and \(\operatorname{div} H = 0\) in \(\Omega\). Moreover, he also proved the regularity of continuous weak solutions to (1.1)-(1.2). However, he also pointed out that the continuity of the weak solution is unknown. For \(n > 3\), we have a similar existence result for weak solutions to the system (1.1)-(1.2) using the same proof as in [13] and [14]. Generally, weak solutions of non-linear elliptic systems may have singularities by De Giorgi’s example and Giusti-Miranda’s example (see [8]). Partial regularity theory for weak solutions of non-linear elliptic systems began around 1968 by Morrey, Giusti-Miranda (e.g. see [1] or [2]). The reader may refer to an excellent book [1] on the further development of the general theory of partial regularity. For many cases of quasi-linear elliptic systems.
which have natural growth, e.g. harmonic map equations, one usually assumes that weak solutions to (1.1)-(1.2) are in the space $W^{1,2} \cap L^\infty(\Omega)$. From the existence result for weak solutions, we only know $u \in W^{1,q}(\Omega)$ with $q \in (1, \frac{n}{n-1})$, we do not know if $u$ in $W^{1,2} \cap L^\infty(\Omega)$, so the general theory of non-linear elliptic systems in [1] does not apply to our system (1.1)-(1.2). Recently, the partial regularity of non-linear elliptic systems involving forms and maps was studied in [4].

When $n = 3$, the system (1.1)-(1.2) arises from approximating Maxwell’s equations in a quasi-stationary electromagnetic field with non-ferromagnetic bodies (e.g. [11]). In the study of the penetration of a magnetic field in a medium, the electrical resistance strongly depends on the temperature. By taking the temperature effect into consideration, the classical Maxwell system in the quasi-static electromagnetic field can be reduced to the following system (see [11], [13] and [14]):

\begin{align}
\partial_t H + \nabla \times \sigma(u) \nabla \times H &= 0; \quad (x, t) \in \Omega \times (0, T) \\
\partial_t u - \Delta u &= \sigma(u)|\nabla \times H|^2; \quad (x, t) \in \Omega \times (0, T) \\
\text{div} H &= 0; \quad (x, t) \in \Omega \times (0, T),
\end{align}

where $H = (H^1(x,t), H^2(x,t), H^3(x,t))$ and $u(x,t)$ represent the strength of the magnetic field and the temperature respectively, and $\sigma^{-1}(u)$ denotes the electrical conductivity of the material. By changing the notation from vector functions to forms, we can consider the vector function $H$ and its ‘curl’ $\nabla \times H$ as a 1-form $H(x)$ and its curvature $dH$ respectively.

Now we generalize the Maxwell systems (1.3)-(1.5) to higher dimensional cases; i.e. $n > 3$. Let $u = u(x,t)$ and $H = \sum_i H^i(x,t)dx_i$ be a function and a 1-form on $Q_T = \Omega \times [0,T]$ respectively. Then we consider the following system

\begin{align}
\partial_t H &= -d^*[\sigma(u)dH]; \quad \text{in } Q_T \\
\partial_t u &= \Delta u + |\sigma(u)|dH|^2; \quad \text{in } Q_T,
\end{align}

with $d^*H(x,t) = 0$ for a.e. $(x,t) \in Q_T$, where $\sigma$ is a positive function satisfying Assumption (S). The weak solution in $V^{1,0}_{1,q}(Q_T)$ to system (1.6)-(1.7) is defined in Section 4.

The second main result of this paper is the following:

**Theorem B.** Let $(u, H)$ be a weak solution to equations (1.6) and (1.7) with $u \in V^{1,0}_{1,q}(Q_T)$ for some $q \in (1, \frac{n+2}{n-1})$, $H^i \in V^{1,0}_{2}(Q_T; \mathbb{R}^n)$ for $i = 1, \ldots, n$ and $d^*H = 0$ for a.e. $(x,t) \in Q_T$. Then when $n \geq 3$, there exists an open subset $\hat{Q}$ of $Q_T$ such that the solution $(u, H)$ is $C^{1,\alpha}$ in $\hat{Q}$, and $\mathcal{H}^{n+2-q_3}(Q_T \setminus \hat{Q}) = 0$ with $q_3 = \frac{(n+2)p}{n+2-2p}$ for some $p > 2$, where $\mathcal{H}^{n+2-q_3}$ denotes the Hausdorff measure.

The paper is organized as follows. In Section 2, we prove Caccioppoli’s inequality for $H$ (Lemma 1) and then obtain $L^p$-estimates (Theorem 3) by applying the reverse Hölder inequality. In Section 3, we prove partial regularity for system (1.1)-(1.2) by applying Theorem 3. Finally, in Section 4, we establish partial regularity of weak solutions for the parabolic problem (1.6)-(1.7) using the analogous techniques as in the elliptic case.
2. Reverse Hölder inequalities and $L^p$-estimates. In this section, we establish the Caccioppoli inequality for $H$ and the $L^p$-estimate.

Let $x_0$ be a point in $\Omega$ with $B_R(x_0) \subset \Omega$. For any function $f$, any 1-form $H$ and any measurable set $A$, denote

\[ \int f \ dx = \frac{1}{|A|} \int_A f \ dx, \quad f_{x_0,R} = \int_{B_R(x_0)} f \ dx, \quad (H)_{x_0,R} = H^i_{x_0,R} dx_i. \]

**Lemma 1.** (Caccioppoli’s inequality for $H$) Assume that $(u,H)$ is a weak solution of (1.1)-(1.2) with $u \in W^{1,q}$, $H \in W^{1,2}$ and $d^*H(x) = 0$ for a.e. $x \in \Omega$. Then there exists a constant $C$ such that for any $x_0 \in \Omega$ and $\rho, R$ with $\rho < R$ with $B_R(x_0) \subset \Omega$,

\[ \int_{B_\rho(x_0)} |\nabla H|^2 \ dx \leq \frac{C}{(R - \rho)^2} \int_{B_R(x_0)} |H - (H)_{x_0,R}|^2 \ dx. \]

**Proof.** Without loss of generality, we assume $x_0 = 0$. Let $\phi$ be a smooth cut-off function with $\phi = 1$ on $B_\rho$, $\phi = 0$ outside $B_R$, $|\phi| \leq 1$ on $B_R \setminus B_\rho$, and $|\nabla \phi| \leq \frac{C}{R - \rho}$ on $B_R \setminus B_\rho$. Choosing $\phi^2(H - H_{0,R})$ as a test function in (1.1), we have

\[ \int_{B_R} \langle d^*\sigma(u)dH, \phi^2(H - H_{0,R}) \rangle \ dx = 0. \]

By Stokes’ formula, we obtain

\begin{align*}
\int_{B_R} \sigma(u)|dH|^2 \phi^2 \ dx = & -2 \int_{B_R} \langle \sigma(u)dH, \phi d\phi \wedge (H - H_{0,R}) \rangle \ dx \\
\leq & \varepsilon \int_{B_R} |dH|^2 \phi^2 \ dx + \frac{C}{(R - \rho)^2} \int_{B_R} |H - H_{0,R}|^2 \ dx.
\end{align*}

Choosing $\varepsilon$ to be sufficiently small, we have

\[ \int_{B_R} |dH|^2 \phi^2 \ dx \leq \frac{C}{(R - \rho)^2} \int_{B_R} |H - H_{0,R}|^2 \ dx. \]  

(2.1)

We note

\[ |dH|^2 = \frac{1}{2} \sum_{i,j=1}^n \left( \frac{\partial H^i}{\partial x_j} - \frac{\partial H^j}{\partial x_i} \right)^2 \]

\[ = |\nabla H|^2 - \sum_{i,j=1}^n \frac{\partial H^i}{\partial x_j} \frac{\partial H^j}{\partial x_i}. \]

Since $H \in W^{1,2}$, we can approximate it by smooth functions $H_k$ in $W^{1,2}$ for
Therefore, this proves our claim.

Now it follows from (2.1) that

\[
\int_{B_R} |\nabla H|^2 \phi^2 \, dx = \int_{B_R} |\nabla H_k|^2 \phi^2 \, dx + 2 \sum_{i,j=1}^n \int_{B_R} \frac{\partial H_{k}^i}{\partial x_j} \frac{\partial \phi}{\partial x_i} [H^i_k - (H^i_k)_{0,R}] \, dx
\]

where we note \( \frac{\partial^2 H^i_k}{\partial x_j \partial x_i} = \frac{\partial^2 H^i_k}{\partial x_i \partial x_j} \). As \( k \to \infty \), it follows from using \( \sum \frac{\partial H^i_k}{\partial x_i} = 0 \) that

\[
\int_{B_R} |dH|^2 \phi^2 \, dx = \int_{B_R} |\nabla H|^2 \phi^2 \, dx + 2 \sum_{i,j=1}^n \int_{B_R} \frac{\partial H^i_k}{\partial x_j} \frac{\partial \phi}{\partial x_i} [H^i_k - (H^i_k)_{0,R}] \, dx
\]

Therefore

\[
\int_{B_R} |\nabla H|^2 \phi^2 \, dx \leq \int_{B_R} |dH|^2 \phi^2 \, dx + \frac{1}{2} \int_{B_R} |\nabla H|^2 \phi^2 \, dx
\]

\[
+ \frac{C}{(R-\rho)^2} \int_{B_R} |H - (H)_{0,R}|^2 \, dx.
\]

Now it follows from (2.1) that

\[
\int_{B_R} |\nabla H|^2 \phi^2 \, dx \leq \frac{C}{(R-\rho)^2} \int_{B_R} |H - (H)_{0,R}|^2 \, dx.
\]

This proves our claim.

By the Proposition in [1; Chapter V. Proposition 1.1, page 122-123], we have

PROPOSITION 2. (Reverse Hölder inequalities) Let \( \Omega \) be an open domain and let \( f \) and \( g \) be positive functions. Suppose

\[
\int_{B_{2R}(x_0)} g^q \, dx \leq b \left( \int_{B_{2R}(x_0)} g \, dx \right)^q + \int_{B_{2R}(x_0)} f^q \, dx + \theta \int_{B_{2R}(x_0)} g^q \, dx
\]

for each \( x_0 \in \Omega \) and each \( R < \frac{1}{2} \text{dist} (x_0, \partial \Omega) \cap R_0 \), where \( R_0, b, \theta \) are constants with \( b > 1, R_0 > 0, 0 \leq \theta < 1 \). Then \( g \in L^p_{loc}(\Omega) \) for \( p \in [q, q + \varepsilon] \) and

\[
\left( \int_{B_{2R}(x_0)} g^p \, dx \right)^{1/p} \leq c \left( \int_{B_{2R}(x_0)} g^q \, dx \right)^{1/q} + c \left( \int_{B_{2R}(x_0)} f^p \, dx \right)^{1/p}
\]

for \( B_{2R} \subset \Omega, R < R_0 \), where \( c \) and \( \varepsilon \) are positive constants depending on \( b, \theta, n \).
Theorem 3. \((L^p\)-estimates\) Let \((u, H)\) be a weak solution of (1.1)-(1.2) with \(u \in W^{1,p}(\Omega, \mathbb{R}),\) \(H \in W^{1,1}(\Omega, \mathbb{R}^n)\) and \(d^p H(x) = 0\) for a.e. \(x \in \Omega.\) Then there exists a small positive constant \(\varepsilon\) such that \(H \in W^{1,p}_{loc}(\Omega, \mathbb{R}^n)\) for some \(p \in (2, 2 + \varepsilon).\) More precisely,

\[
(2.2) \quad \left( \frac{1}{p} \int_{B_R(x_0)} |\nabla H|^p \, dx \right)^{1/p} \leq c \left( \int_{B_{2R}(x_0)} |\nabla H|^2 \, dx \right)^{1/2}
\]

for all \(x_0 \in \Omega\) and all \(R\) with \(2R < R_0\) with \(B_{R_0}(x_0) \subset \Omega\) for some \(R_0 > 0.\) Moreover \(u \in W^{1,q_1}_{loc}(\Omega)\) with \(q_1 = \frac{np}{(2n-p)} > \frac{n}{n-1}\) where \(p > 2\) is fixed above.

Proof. By the Sobolev-Poincaré inequality, we have

\[
\int_{B_R} |H - (H)_{x_0,R}|^2 \, dx \leq CR^{2+1 - \frac{2n}{m}} \left( \int_{B_R} |\nabla H|^{q_2} \, dx \right)^{2/q_2}
\]

for \(q_2 = \frac{2n}{n+2} < 2.\)

Letting \(\rho = R/2\) in Lemma 1, we have

\[
\left( \int_{B_{R/2}(x_0)} |\nabla H|^2 \, dx \right)^{1/2} \leq C \left( \int_{B_R(x_0)} |\nabla H|^{q_2} \, dx \right)^{1/q_2}
\]

Applying Proposition 2, there exists a \(p > 2\) such that \(H \in W^{1,p}(\Omega; \mathbb{R}^n)\) and (2.2) holds. Applying the standard \(L^p\)-theory for equation (1.2), we get \(u \in W^{2,p/2}_{loc}(\Omega; \mathbb{R}).\)

By Sobolev’s inequality again, we have \(u \in W^{1,\frac{m-1}{m}\frac{np}{n-1}}_{loc}.\)

3. Proof of Theorem A. In this section, we give a proof of Theorem A.

Let \(\Omega(x, \rho) = \Omega \cap B_\rho(x)\) and let \(p \geq 1\) and \(\lambda \geq 0.\) At first, let us define the Morrey space \(L^{p,\lambda}(\Omega)\) in the following

Definition A. (Morrey spaces) We say that \(u\) belongs to \(L^{p,\lambda}(\Omega)\) if \(u \in L^p(\Omega)\) satisfies

\[
\|u\|_{L^{p,\lambda}(\Omega)} = \left\{ \sup_{x_0 \in \Omega, 0 < \rho < \text{diam } \Omega} \rho^{-\lambda} \int_{\Omega(x_0, \rho)} |u|^p \, dx \right\}^{1/p} < +\infty
\]

and the Campanato space \(L^{p,\lambda}(\Omega)\)

Definition B. (Campanato space) We say that \(u\) belongs to \(L^{p,\lambda}(\Omega)\) if \(u \in L^p(\Omega)\) satisfies

\[
|u|_{p,\lambda} = \left\{ \sup_{x_0 \in \Omega, 0 < \rho < \text{diam } \Omega} \rho^{-\lambda} \int_{\Omega(x_0, \rho)} |u - u_{x_0, \rho}|^p \, dx \right\}^{1/p} < +\infty,
\]

where \(u_{x_0, \rho} = \frac{1}{|\Omega(x, \rho)|} \int_{\Omega(x, \rho)} u(x) \, dx.\)

Let us recall some results about Morrey and Campanato spaces from [1] and [2]. If there exists a constant \(A\) such that \(|\Omega(x, \rho)| \geq A \rho^n\) for all \(\Omega(x, \rho),\) the Campanato space \(L^{p,\lambda}(\Omega)\) is isomorphic to the Morrey space \(L^{p,\lambda}(\Omega)\) when \(0 \leq \lambda < n,\) and moreover, when \(n < \lambda \leq n + p,\) \(L^{p,\lambda}(\Omega)\) is isomorphic to the Hölder space \(C^{0,\alpha}\) with \(\alpha = \frac{\lambda - n}{p}.\)
Lemma 4. Let \((u, H)\) be a weak solution to (1.1)-(1.2). Then \(u\) is also a weak solution to the following equation
\[
\Delta u = d^* [\sigma(u) \langle dH, H \rangle],
\]
where
\[
(dH, H) := \sum_{i,j=1}^n F^{ij} H^j dx_i.
\]

Proof. Taking \(\phi H\) as a test function in (1.1), we obtain
\[
\int_{\Omega} (\sigma(u) dH, d(\phi H)) \, dx = 0,
\]
where \(\phi\) is a function with \(\phi \in C^2_0(\Omega; \mathbb{R})\). Then by the definition in Section 1, we get
\[
\int_{\Omega} \phi \sigma(u) |dH|^2 \, dx = - \int_{\Omega} (\sigma(u) dH, d\phi \wedge H) \, dx
\]
\[
= - \int_{\Omega} \sigma(u) \left( \sum_{i,j} F^{ij} H^j dx_i \right) \, dx
\]
\[
= - \int_{\Omega} \phi d^* [\sigma(u) \langle dH, H \rangle] \, dx
\]
for all \(\phi \in C^2_0(\Omega; \mathbb{R})\), where \(\langle dH, H \rangle\) is defined in (3.2). This proves our claim.

Now we prove partial regularity of the weak solutions \((u, H)\) to the system (1.1)-(1.2).

Proof of Theorem A. Under the gauge condition \(d^* H = 0\), we know from the Hodge theory that
\[- \Delta H = d^* dH + dd^* H = d^* dH.\]
Let \(x_0 \in \Omega\) with \(B_{R_0}(x_0) \subset \subset \Omega\) for some \(R_0 > 0\). Let a 1-form \(H_1 \in W^{1,2}(B_R(x_0))\) be a weak solution of the following Dirichlet problem
\[
\sigma(u_{x_0,R}) \Delta H_1 = 0, \forall x \in B_R(x_0),
\]
\[
H_1 - H \in W^{1,2}_0(B_R(x_0), \mathbb{R}^n).
\]
Then for all \(\rho < R \leq R_0\), we have
\[
\int_{B_{\rho}(x_0)} |\nabla H_1|^2 \, dx \leq C \left( \frac{\rho}{R} \right)^n \int_{B_R(x_0)} |\nabla H_1|^2 \, dx.
\]
and therefore for all \(\rho < R \leq R_0\) with some \(R_0 > 0\)
\[
\int_{B_{\rho}(x_0)} |\nabla H|^2 \, dx \leq C \left( \frac{\rho}{R} \right)^n \int_{B_R(x_0)} |\nabla H|^2 \, dx + C \int_{B_R(x_0)} |\nabla (H - H_1)|^2 \, dx.
\]
Let \(W = H - H_1\). Using equations (1.1) and (3.3), \(W\) is the weak solution of the following
\[
\sigma(u_{x_0,R}) \Delta W = d^* \{ [\sigma(u) - \sigma(u_{x_0,R})] dH \}.
\]
with boundary condition $W = 0$ on $\partial B_R(x_0)$. Using $W$ as a test function in the above equation, we get

$$
(3.5) \quad \sigma(u_{x_0,R}) \int_{B_R} |\nabla W|^2 \, dx = - \int_{B_R} \langle [\sigma(u) - \sigma(u_{x_0,R})]dH, dW \rangle \, dx.
$$

By the assumption on $\sigma(u)$, there exists a non-negative, bounded function $\omega(t)$ increasing in $t$, concave, continuous with $\omega(0) = 0$, such that for $u, v \in \mathbb{R}$,

$$
(3.6) \quad |\sigma(u) - \sigma(v)| \leq \omega(|u - v|^{q_1}),
$$

where $q_1 = \frac{np}{2n-p}$ and $p$ is a fixed exponent in $(2, 2 + \varepsilon)$ from Theorem 3. Hence we get from (3.5)-(3.6)

$$
\int_{B_R(x_0)} |\nabla W|^2 \, dx \leq C \int_{B_R(x_0)} \omega^2(|u - u_{x_0,R}|^{q_1})|\nabla H|^2 \, dx.
$$

By the Sobolev-Poincare inequality, we obtain

$$
\int_{B_R} |u - u_{x_0,R}|^{q_1} \, dx \leq CR^{qn} \int_{B_R} |\nabla u|^{q_1} \, dx.
$$

Using the $L^p$-estimate (Theorem 3) and the boundedness and concavity of $\omega$, we have

$$
\int_{B_R(x_0)} \omega^2(|u - u_{x_0,R}|^{q_1})|\nabla H|^2 \, dx
\leq C \left( \int_{B_R(x_0)} |\nabla H|^p \, dx \right)^{2/p} \left( \int_{B_R(x_0)} \omega^{\frac{2n}{2n-p}}(|u - u_{x_0,R}|^{q_1}) \, dx \right)^{\frac{p-2}{p}}
\leq C \left( \int_{B_{2R}(x_0)} |\nabla H|^2 \, dx \right) \left( |B_R(x_0)|^{-1} \int_{B_R(x_0)} \omega(|u - u_{x_0,R}|^{q_1}) \, dx \right)^{\frac{p-2}{p}}
\leq C \omega^{\frac{p-2}{p}} \left( CR^{n-1} \int_{B_R(x_0)} |\nabla u|^{q_1} \, dx \right) \left( \int_{B_{2R}} |\nabla H|^2 \, dx \right),
$$

where last inequality comes from the concavity of $\omega$ using the Jensen inequality and the Poincare inequality.

Therefore for all $\rho < R < 2R \leq R_0$ we have

$$
(3.7) \quad \int_{B_{\rho}(x_0)} |\nabla H|^2 \, dx \leq C \left( \frac{\rho}{R} \right)^n \int_{B_{2R}(x_0)} |\nabla H|^2 \, dx +
C \omega^{\frac{p-2}{p}} \left( CR^{n-1} \int_{B_{2R}(x_0)} |\nabla u|^{q_1} \, dx \right) \int_{B_{2R}(x_0)} |\nabla H|^2 \, dx
$$

By Theorem 3, $u$ belongs to $W^{2,p/2}(\Omega)$. Let $v \in W^{2,p/2}(B_R(x_0))$ be a weak solution of the following Dirichlet problem:

$$
- \Delta v = 0, \quad \text{in } B_R(x_0),
$$

$$
v|_{\partial B_R} = u|_{\partial B_R}, \quad x \in \partial B_R(x_0).
$$
For the harmonic function $v$, it is easy to see that for $\rho \leq R < 2R \leq R_0$, we obtain
\[
\int_{B_{\rho}(x_0)} |\nabla v|^{q_1} \, dx \leq C\left(\frac{\rho}{R}\right)^n \int_{B_R(x_0)} |\nabla v|^{q_1} \, dx.
\]

Let $w = u - v$. Then $w \in W^{2,p/2}(B_R(x_0); \mathbb{R})$ satisfies
\[
- \Delta w = \sigma(u)|dH|^2, \quad \text{in } B_R(x_0),
\]
\[
w = 0 \quad \text{on } \partial B_R(x_0).
\]

Then
\[
\int_{B_{\rho}(x_0)} |\nabla u|^{q_1} \, dx \leq C\left(\frac{\rho}{R}\right)^n \int_{B_R(x_0)} |\nabla u|^{q_1} \, dx + C \int_{B_R} |\nabla w|^{q_1} \, dx.
\]

We rescale
\[
\tilde{u}(x) = u(x_0 + Rx), \quad \tilde{w}(x) = w(x_0 + Rx), \quad \tilde{H}(x) = H(x_0 + Rx) = H^i(x_0 + Rx)dx_i.
\]

Then
\[
- \Delta \tilde{w} = \sigma(u)|d\tilde{H}|^2, \quad \text{in } B_1,
\]
\[
\tilde{w} = 0; \quad \text{on } \partial B_1,
\]
where $B_1 = B(0,1)$ is the unit ball in $\mathbb{R}^n$. Applying the standard elliptic $L^p$-theory (see [7]) to (3.8)-(3.9), we obtain
\[
\left(\frac{1}{|B_1|} \int_{B_1} |\nabla^2 \tilde{w}|^{p/2} \, dx\right)^{2/p} \leq C \left(\frac{1}{|B_1|} \int_{B_1} |\nabla \tilde{H}|^p \, dx\right)^{2/p},
\]
where $C$ is a constant independent of $R$.

Rescaling back, we have
\[
\left(\frac{1}{|B_R(x_0)|} \int_{B_{\rho}(x_0)} |\nabla^2 w|^{p/2} \, dx\right)^{2/p} \leq C \left(\frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} |\nabla H|^p \, dx\right)^{2/p},
\]
where $C$ is a constant independent of $R$. By the Sobolev inequality and using $L^p$-estimates, we see
\[
\left(\int_{B_R(x_0)} |\nabla w|^{q_1} \, dx\right)^{\frac{1}{q_1}} \leq CR \left(\int_{B_R(x_0)} |\nabla^2 w|^{p/2} \, dx\right)^{2/p}
\]
\[
\leq CR^{1-n} \int_{B_{2R}(x_0)} |\nabla H|^2 \, dx.
\]

Therefore for all $\rho < R < 2R \leq R_0$, we have
\[
\int_{B_\rho(x_0)} |\nabla u|^{q_1} \, dx \leq C\left(\frac{\rho}{R}\right)^n \int_{B_R(x_0)} |\nabla u|^{q_1} \, dx
\]
\[
+ CR^{n+q_1(1-n)} \left(\int_{B_{2R}(x_0)} |\nabla H|^2 \, dx\right)^{q_1}.
\]

(3.10)
For any \( x_0 \in \Omega \) and \( r \) with \( R_0 \geq r > 0 \), we denote
\[
\Phi(x_0, r) = r^{2-n} \int_{B_r(x_0)} |\nabla H|^2 \, dx, \quad \xi(x_0, r) = r^{q_1-n} \int_{B_r(x_0)} |\nabla u|^{q_1} \, dx,
\]
Note that (3.7) and (3.10) also hold for \( R < \rho < 2R \leq R_0 \). Then for all \( \tau < 1 \), we have
\[
\Phi(x_0, \tau R) \leq C_1 [1 + \omega^{\frac{n-2}{2}} (C_2 \xi(x_0, R))]^{\tau-n} \tau^2 \Phi(x_0, R)
\]
and
\[
\xi(x_0, \tau R) \leq C_1 \tau^n \xi(x_0, R) + \tau^{q_1-n} \Phi^{q_1}(x_0, R)
\]
by using \( R \) instead of \( 2R \) in (3.7) and (3.10). For any \( \alpha < 1 \), choose \( \tau < 1 \) such that
\[
2C_1 \tau^{q_1-\alpha} = 1.
\]
There exists a small constant \( \varepsilon_0 > 0 \) such that if
\[
\xi(x_0, R) + \Phi(x_0, R) < \varepsilon_0
\]
for some \( R < R_0 \), then we have
\[
\Phi^{q_1-1}(x_0, R) < \tau^n, \quad \omega^{\frac{n-2}{2}} (C_2 \xi(x_0, R)) < \tau^n
\]
provided that \( R \) is less than some \( R_0 \). Hence
\[
\xi(x_0, \tau R) + \Phi(x_0, \tau R) \leq \tau^{2\alpha} \xi(x_0, R) + \Phi(x_0, R).
\]
Therefore by iteration we obtain
\[
\xi(x_0, \tau^k R) + \Phi(x_0, \tau^k R) \leq \tau^{2\alpha} \xi(x_0, R) + \Phi(x_0, R) < \varepsilon_0
\]
In conclusion, if \( \xi(x_0, R) + \Phi(x_0, 2R) < \varepsilon_0 \) for some \( R < R_0 \), then
\[
\xi(x_0, \tau^k R) + \Phi(x_0, \tau^k R) \leq \tau^{2\alpha} \varepsilon_0.
\]
Hence for any \( \rho < R_0 \), we have
\[
(3.11) \quad \xi(x_0, \rho) + \Phi(x_0, \rho) \leq C \left( \frac{\rho}{R} \right)^{2\alpha},
\]
where \( C \) is a constant independent of \( \rho \) and \( R \).

Note that \( \xi(x_0, R) \) and \( \Phi(x_0, R) \) are continuous functions of \( x_0 \). There exits an open \( \Omega_0 \subset \Omega \) such that \( u \) and \( H \) are in \( C^{0,\alpha}_{loc}(\Omega_0) \) for every \( \alpha < 1 \). Moreover, \( \Omega \setminus \Omega_0 \subset \Sigma_1 \cup \Sigma_2 \), where
\[
\Sigma_1 = \{ x \in \Omega : \liminf_{R \to 0^+} \frac{1}{R^2-n} \int_{B_R(x)} |\nabla H|^2 \, dx > 0 \},
\]
\[
\Sigma_2 = \{ x \in \Omega : \liminf_{R \to 0^+} \frac{1}{R^{q_1-n}} \int_{B_R(x)} |\nabla u|^{q_1} \, dx > 0 \}.
\]
Moreover, since \( H \in W^{1,2}(\Omega, \mathbb{R}^n) \) and \( u \in W^{1,q_1}_{loc}(\Omega, \mathbb{R}^n) \) with \( q_1 = \frac{np}{2n-p} \) for some \( p > 2 \), we have
\[
\mathcal{H}^{n-q_1}(\Omega \setminus \Omega_0) = 0
\]
where \( \mathcal{H}^{n-q_1} \) denote \( (n-q_1) \)-Hausdorff measure.
Next we prove $C^{1,\alpha}$-regularity inside $\Omega_0$. We assume that $x_0 \in \Omega$ with $B_{2R}(x_0) \subset \Omega_0$. From the above results, we know that $u$ and $H$ are $C^{0,\alpha}(\Omega_0)$ for every $\alpha < 1$ and
\[
R^{n-n} \int_{B_R(x_0)} |\nabla u|^n \, dx \leq CR^{2\alpha}, \quad R^{2-n} \int_{B_R(x_0)} |\nabla H|^2 \, dx \leq CR^{2\alpha},
\]
where $C$ is a constant independent of $R$. Note that $H_1$ is the solution to equations (3.3)-(3.4). For any $\rho$ and $R$ with $\rho < R \leq R_0$, we have
\[
\int_{B_\rho(x_0)} |\nabla H_1 - (\nabla H_1)_{x_0,\rho}|^2 \, dx \leq C \left( \frac{\rho}{R} \right)^{n+2} \int_{B_R(x_0)} |\nabla H_1 - (\nabla H_1)_{x_0,\rho}|^2 \, dx.
\]
Repeating the same proof as before (3.7), we get
\[
\int_{B_\rho(x_0)} |\nabla W|^2 \, dx \leq C\omega^{\frac{p-2}{2}} \left( CR^{n-n} \int_{B_\rho(x_0)} |\nabla u|^2 \, dx \right) \int_{B_R(x_0)} |\nabla H|^2 \, dx
\]
for some $p > 2$.

Since $\omega$ is uniformly Hölder continuous, there exist constants $\beta$ and $C$ with $0 < \beta < 1$ such that $\omega(t) \leq Ct^\beta$. Therefore
\[
\int_{B_\rho(x_0)} |\nabla H - (\nabla H)_{x_0,\rho}|^2 \, dx \leq C \left( \frac{\rho}{R} \right)^{n+2} \int_{B_{2R}(x_0)} |\nabla H - (\nabla H)_{x_0,R}|^2 \, dx \]
\[+ CR^{n-2+\alpha[2+\beta\frac{p-2}{p}]},\]
where $\alpha[2 + \beta\frac{p-2}{p}] > 2$ by letting $\alpha$ be closing to 1. Then the standard procedure yields that $\nabla H$ is $C^{0,\gamma}$ for some $0 < \gamma < 1$. By applying standard PDE theory to equation (1.2), it is easy to see that $\nabla u$ is also locally in $C^{0,\gamma_1}(\Omega_0)$ for some $\gamma_1 > 0$. This proves our claim. □

4. Partial regularity for the parabolic system. In this section, we prove the partial regularity of the weak solutions to system (1.5)-(1.6).

Denote $Q_T = \Omega \times (0, T)$ and let $z = (x, t)$ for $x \in \Omega$ and $t \in (0, T)$. We recall some definitions from [9]. $L_{p,r}(Q_T)$ is the Banach space consisting of all measurable functions on $Q_T$ with a finite norm
\[
\|u\|_{p,r,Q_T} = \left( \int_0^T \left( \int_{\Omega} |u(x, t)|^p \, dx \right)^{r/p} \, dt \right)^{1/r}.
\]
We denote $\|u\|_{p,r,Q_T} = \|u\|_{p,p,Q_T}$. The space $V_p^{1,0}(Q_T)$ is the completion of $C^1(Q_T)$ with respect to the norm
\[
|u|_{p,Q_T} = \left\{ \int_{Q_T} (|u|^p + |\nabla u|^p) \, dz \right\}^{1/p}.
\]

The space $W^{2,1}_p(Q_T)$ with $p \geq 1$ is the Banach space consisting of the elements of $L_p(Q_T)$ having generalized derivatives of the form $D_t^r D_x^s$ with any $r$ and $s$ satisfying the inequality $2r + s \leq 2$. The norm is defined by
\[
\|u\|_{q,Q_T}^{(2)} = \sum_{j=0}^2 (\langle u \rangle_{q,Q_T}^{(j)})^2.
\]
with
\[ \langle \langle u \rangle \rangle_{p,Q_T}^{(j)} = \sum_{2r+s=j} \| D_{2r}^{s} u \|_{q,Q_T}. \]

We say that a pair \((u, H)\) is a weak solution to equations (1.6)-(1.7) if \(u \in V^{1,0}_{q}(Q_T)\) for some \(q \in (1, \frac{n}{n-1})\) and \(H^{i} \in V^{1,0}_{2}(Q_{T};\mathbb{R}^{n})\), and the pair \((u, H)\) satisfies the following:

(4.1) \[
\int_{Q_T} [\langle H, \partial_{t} \phi \rangle + \langle \sigma(u)dH, d\phi \rangle] \, dz = 0,
\]

(4.2) \[
\int_{Q_T} [-u\psi + \nabla u \cdot \nabla \psi] \, dz = \int_{Q_T} \sigma(u)|dH|^2 \psi \, dz
\]

for all \(\phi := \sum_{i=1}^{n} \phi^{i}(x,t) dx_{i}\) for \(i = 1, ..., n\) with \(d^{*}H = 0\) in \(Q_T\) in the weak sense, where \(\phi^{i}(x,t) \in C^{2}_{0}(Q_{T};\mathbb{R})\) and \(\psi(x,t) \in C^{2}_{0}(Q_{T};\mathbb{R})\). The existence of weak solutions of (4.1)-(4.2) with \(d^{*}H = 0\) in \(Q_T\) was obtained by Yin in \([13]\) and \([14]\).

For any \(R > 0\), denote \(Q_{R}(z_{0}) = B_{R}(x_{0}) \times (t_{0} - R^2, t_{0} + R^2)\) with \(z_{0} = (x_{0},t_{0})\). We denote for any function \(u(x,t)\)

\[ u_{z_{0},R} = \int_{Q_{R}(z_{0})} u(z) \, dz. \]

Next, we prove partial regularity of weak solutions to the system (4.1)-(4.2) by modifying the method for elliptic case of Sections 2-3. The first step towards the proof of Theorem B is to establish a Caccioppoli’s inequality and \(L^{p}\)-estimates for weak solutions to the parabolic system (4.1)-(4.2) by applying the proof of \([3]\) and \([6]\).

More precisely, we have

**Lemma 7.** (Caccioppoli’s inequality for parabolic problems) Assume that \((u, H)\) is a weak solution of (4.1)-(4.2) with the assumptions of Theorem B. Then there exists a constant \(C\) such that for any \(x_{0} \in Q_{T}\) and any \(R \) with \(2R \leq R_{0}\) with \(Q_{R_{0}}(z_{0}) \subset Q_{T}\) for some \(R_{0} > 0\),

\[ \int_{Q_{R}(z_{0})} |\nabla H|^2 \, dz \leq \frac{C}{R^{2}} \int_{Q_{2R}(z_{0})} |H - \tilde{H}_{x_{0},2R}(t)|^2 \, dz. \]

**Proof.** Let \(z_{0} = (x_{0},t_{0}) \in Q_{T}\). Let \(\xi(x)\) be a function in \(C^{\infty}_{0}(B_{2}(x_{0}))\) such that \(0 \leq \xi \leq 1\), \(\xi = 1\) in \(B_{1}(x_{0})\) and \(|\nabla \xi| \leq 2\). We also denote by \(\xi_{R} \) the function \(\xi_{2R}(x) = \xi(\frac{x}{2R})\). As in \([6]\), for a function \(H^{i}(x,t)\) in \(B_{2R}(x_{0})\) as

\[ \tilde{H}^{i}_{x_{0},2R}(t) = \frac{\int_{B_{2R}(x_{0})} H^{i}(x,t) \xi_{2R}^{2} dx}{\int_{B_{2R}(x_{0})} \xi_{2R}^{2} dx}, \]

Then we define

\[ \tilde{H}_{x_{0},2R}(t) = \sum_{i} \tilde{H}^{i}_{x_{0},2R}(t) dx_{i}. \]
Let \( \tau \in C^\infty(\mathbb{R}, \mathbb{R}) \) be a function only in \( t \) and satisfy \( 0 \leq \tau \leq 1, \tau \equiv 1 \) on \([t_0 - R^2, t_0]\), \( \tau \equiv 0 \) on \( t < t_0 - (2R)^2 \). By the above choice, we note

\[
\int_{t_0 - 4R^2}^{t_0} \left[ \int_{B_{2R}(x_0)} (H^1 - H_{2R}^1(t)) \xi^2 \, dx \right] \partial_t H_{2R}^1(t) \tau^2 \, dt = 0. \tag{4.3}
\]

Let \( I_{(-\infty, t_0)} \) be the characteristic function of the interval \((-\infty, t_0)\). Testing \( \phi = (H - H_{2R}(t)) \xi_{2R}^2 \tau^2 I_{(-\infty, t_0)} \) and noting (4.3), we have

\[
\int_{B(x_0, 2R) \times \{t_0\}} |H - \dot{H}_{x_0, 2R}(t)|^2 \xi^2 \tau^2 \, dx + \int_{Q_{2R}(z_0)} \sigma(u) |dH|^2 \xi^2 \tau^2 \, dz 
\leq 2 \int_{Q_{2R}(z_0)} |H - H_{x_0, 2R}(t)|^2 \xi^2 \tau \, dz 
\]

\[= 2 \int_{Q_{2R}(z_0)} \sigma(u) \langle dH, \xi d\xi \wedge (H - \dot{H}_{x_0, 2R}(t)) \rangle \tau^2 \, dz. \]

It follows from (4.4) that

\[
\int_{Q_{2R}} |dH|^2 \xi^2 \tau^2 \, dz \leq \frac{C}{R^2} \int_{Q_{2R}(z_0)} |H - \dot{H}_{x_0, 2R}(t)|^2 \, dz.
\]

A similar argument to Lemma 1 yields

\[
\int_{t_0 - R^2}^{t_0} \int_{B_{2R}(x_0)} |dH|^2 \xi^2 \, dx \tau^2 \, dt 
= \int_{t_0 - R^2}^{t_0} \left( \int_{B_{2R}(x_0)} |\nabla H|^2 \xi^2 \, dx - \sum_{i,j=1}^{n} \int_{B_{2R}(x_0)} \frac{\partial H^i}{\partial x_j} \frac{\partial H^j}{\partial x_i} \xi^2 \, dx \right) \tau^2 \, dt 
\]

\[= \int_{t_0 - R^2}^{t_0} \int_{B_{2R}(x_0)} \left( |\nabla H|^2 \xi^2 + 2 \sum_{i,j=1}^{n} \frac{\partial H^i}{\partial x_j} \frac{\partial \xi}{\partial x_i} [H^j - \dot{H}^j_{x_0, 2R}(t)] \right) \, dx \tau^2 \, dt 
+ \int_{t_0 - R^2}^{t_0} \int_{B_{2R}(x_0)} \sum_{i,j=1}^{n} \xi^2 [H^j - \dot{H}^j_{x_0, 2R}(t)] \frac{\partial^2 H^i}{\partial x_j \partial x_i} \, dx \tau^2 \, dt. \]

By using \( d^*H = 0 \), the last term in above identity is zero. This proves our claim. \( \square \)

We have the following \( L^p \)-estimate:

**Lemma 8.** Let \((u, H)\) be a weak solution to the system (4.1)-(4.2) with the assumptions of Theorem B. Then there exists an exponent \( p > 2 \) such that \( \nabla H \in L^p_{\text{loc}}(Q_T) \); moreover for all \( Q_R(z_0) \subset Q_{4R}(z_0) \subset Q_T \) we have

\[
\int_{Q_{4R}(z_0)} |\nabla H|^p \, dz \leq C \left( \int_{Q_{4R}(z_0)} |\nabla H|^2 \, dz \right)^{\frac{p}{2}}
\]

and \( u \in W^{2,1}_{p/2, \text{loc}}(Q_T) \).

For the proof of Lemma 8, the same proof as in [5] gives the desired \( L^p \)-estimate for \( H \) by using the reverse Hölder inequality as in Proposition 3. The fact \( u \in
Let a 1-form $H$.

Then for all $\rho < R$ and $\rho < R$

where $W$.

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Let $v$.

By a slight modification of arguments in [12] (for the details, see [14]), we have

Proof of Theorem B. By a similar proof as in Section 3, we have

Now we complete the proof of Theorem B.

Proof of Theorem B. For any $z_0 \in Q_T$, choose $R_0$ with $Q_{R_0}(z_0) \subset Q_T$. Let $S_{R_0}(z_0)$ be the parabolic boundary of $Q_{R}(z_0)$ defined by

Let a 1-form $H_1 \in V_{2,0}^1(Q_R(z_0))$ be the weak solution of the following parabolic problem:

$$
\partial_t H_1 = \sigma(u_{z_0,R}) \Delta H_1, \quad \text{in } Q_R(z_0),
$$

$$
H_1|_{S_R(z_0)} = H|_{S_R(z_0)}, \quad \text{on } S_R(z_0).
$$

For all $0 < R \leq R_0$, we have

$$
\int_{Q_R(z_0)} |\nabla H|^2 \, dz \leq C \left(\frac{\rho}{R}\right)^{n+2} \int_{Q_R(z_0)} |\nabla H|^2 \, dz + C \int_{Q_R(z_0)} |\nabla W|^2 \, dz
$$

with $W = H - H_1$. By a similar proof as in Section 3, we have

$$
\int_{Q_R(z_0)} |\nabla W|^2 \, dz \leq C \omega^{\frac{n+1}{2}} \left(\frac{C_R^{-n}}{Q_R(z_0)} \int_{Q_{4R}(z_0)} |u - u_{z_0,R}|^{p/2} \, dz \right) \int_{Q_{2R}(z_0)} |\nabla H|^2 \, dz.
$$

Let $v \in W_{p/2}^{2,1}(Q_R(z_0))$ be a weak solution of

$$
\partial_t v = \Delta v, \quad \text{in } Q_R(z_0),
$$

$$
v|_{S_R(z_0)} = u|_{S_R(z_0)}, \quad \text{on } S_R(z_0).
$$

Then for all $0 < R \leq R_0$, we have

$$
\int_{Q_R(z_0)} |\partial_t u|^{p/2} \, dz \leq C \left(\frac{\rho}{R}\right)^{n+2} \int_{Q_R(z_0)} |\partial_t u|^{p/2} \, dz + C \int_{Q_R(z_0)} |\partial_t w|^{p/2} \, dz
$$

and

$$
\int_{Q_R(z_0)} |\nabla u|^{p/2} \, dz \leq C \left(\frac{\rho}{R}\right)^{n+2} \int_{Q_R(z_0)} |\nabla u|^{p/2} \, dz + C \int_{Q_R(z_0)} |\nabla w|^{p/2} \, dz,
$$

where $w = u - v$ satisfies

$$
\partial_t w = \Delta w + \sigma(u)|dH|^2, \quad \text{in } Q_R(z_0),
$$

$$
w = 0 \quad \text{on } S_R(z_0).
$$
Since $|\nabla H|^2$ is locally in $L^{p/2,p/2}(Q_T)$, we have from Theorem 9.1 of [10; Chapter IV] and Lemma 8 that

$$(4.7) \int_{Q_R(z_0)} \left( |\nabla^2 w|^{p/2} + |\partial_t u|^{p/2} \right) dz \leq C \int_{Q_R(z_0)} |\nabla H|^p dz$$

$$\leq CR^{n+2} \left( \int_{Q_4R(z_0)} |\nabla H|^2 dx \right)^{p/2}.$$ 

By the Sobolev inequality and using $L^p$-estimates in (4.7), we know

$$\int_{Q_R(z_0)} |\nabla u|^{p/2} dz \leq CR^{p/2} \int_{Q_R(z_0)} |\nabla^2 w|^{p/2} dz$$

$$\leq CR^{n+2-\frac{p}{2}(n+1)} \left( \int_{Q_4R(z_0)} |\nabla H|^2 dx \right)^{p/2}.$$ 

By a version of the Sobolev-Poincare inequality, we have

$$(4.8) \int_{Q_R(z_0)} |u - u_{z_0,R}|^{p/2} dz \leq C \left[ R^{p/2} \int_{Q_R(z_0)} |\nabla u|^{p/2} dz + R^p \int_{Q_R(z_0)} |\partial_t u|^{p/2} dz \right].$$

For any $z_0 \in Q_T$ and $r$ with $Q_{z_0,r} \subset Q_T$, we denote

$$\Phi(z_0, r) = r^{-n} \int_{Q_r(z_0)} |\nabla H|^2 dz, \quad \xi(z_0, r) = r^{-n-2+p/2} \int_{Q_r(z_0)} |\nabla u|^{p/2} dz,$$

$$\eta(z_0, r) = r^{-n-2+p} \int_{Q_r(z_0)} |\partial_t u|^{p/2} dz.$$ 

Then for all $\tau < 1$, we have

$$\Phi(z_0, \tau R) \leq C_1 [1 + \omega^{\frac{n-2}{p}}(C_2[\xi(z_0, R) + \eta(z_0, R)])] \tau^{-(n+2)} \tau^2 \Phi(x_0, R),$$

$$\xi(z_0, \tau R) \leq C_1 \tau^2 \xi(x_0, R) + \tau^{\frac{p}{2}-(n+2)} \Phi^\xi(x_0, R)$$

and

$$\eta(z_0, \tau R) \leq C_1 \tau^p \eta(z_0, R) + \tau^{p-(n+2)} \Phi^\eta(z_0, R).$$

If there exists a constant $\varepsilon_0$ such that $\Phi(z_0, r) + \xi(z_0, r) + \eta(z_0, r) < \varepsilon_0$ for some $r \leq R_0$, then a similar iteration step as in Section 3 yields

$$\phi(z_0, \rho) + \xi(z_0, \rho) + \eta(z_0, \rho) \leq C \rho^{2\alpha}$$

for all $\alpha < 1$ and $\rho \leq r \leq R_0$. Using the Sobolev inequality (4.8) and Lemma 9, we obtain through the Campanato space that $u(x, t)$ and $H(x, t)$ are Hölder continuous in $\alpha$ locally in $Q$ where $Q$ is an open subset of $Q_T$. A similar argument as in Section 3 yields that $u(x, t)$ and $H(x, t)$ are also in $C^{1,\gamma}_{loc}(Q_T)$ for some $\gamma < 1$.

Since $u$ is in $W^{2,1}_{p,2;loc}(Q_T)$, we have $\nabla u \in L_{q_3}\loc(Q_T)$, $q_3 = \frac{(n+2)p}{n+2-2p}$ by the parabolic type Sobolev inequality (see [10; Lemma 3.3, page 80]). Moreover, Hölder’s
inequality gives
\[ \xi(z_0, R) \leq \left( R^{q_3-n-2} \int_{Q_{z_0,R}} |\nabla u|^{q_3} \, dz \right)^{\frac{p}{2q_3}}. \]

We have \( Q_T \setminus \tilde{Q} \subset \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \) where
\[ \Sigma_1 = \{ z_0 \in Q_T : \liminf_{R \to 0^+} R^{-n} \int_{Q_{z_0,R}} |\nabla H|^2 \, dz > 0 \}, \]
\[ \Sigma_2 = \{ z_0 \in Q_T : \liminf_{R \to 0^+} R^{q_3-n-2} \int_{Q_{z_0,R}} |\nabla u|^{q_3} \, dz > 0 \}, \]
and
\[ \Sigma_3 = \{ z_0 \in Q_T : \liminf_{R \to 0^+} R^{p-n-2} \int_{Q_{z_0,R}} |\partial_t u|^{p/2} \, dz > 0 \}. \]

Since \( \nabla H \in L_{2,loc}(Q_T, \mathbb{R}^n) \) and \( \partial_t u \in L_{p/2}(Q_T, \mathbb{R}^n) \), we have
\[ \mathcal{H}^{n+2-q_3}(Q_T \setminus \tilde{Q}) = 0, \]
where \( \mathcal{H}^{n+2-q_3} \) denotes \( (n+2-q_3) \)-Hausdorff measure. This proves our claim. \( \square \)

REFERENCES
