UNIQUENESS OF SOLUTIONS FOR AN ELLIPTIC EQUATION MODELING MEMS∗
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Abstract. We show among other things, that for small voltage, the stable solution of the basic nonlinear eigenvalue problem modelling a simple electrostatic MEMS is actually the unique solution, provided the domain is star-shaped and the dimension is larger or equal than 3. In two dimensions, we need the domain to be either strictly convex or symmetric. The case of a power permittivity profile is also considered. Our results, which use an approach developed by Schaaf [13], extend and simplify recent results by Guo and Wei [7], [8].

Key words. MEMS, Stable solutions, Quenching branch.

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1. Introduction. We study the effect of the parameter λ, the dimension N, the profile f and the geometry of the domain Ω ⊂ \mathbb{R}^N, on the question of uniqueness of the solutions to the following elliptic boundary value problem with a singular nonlinearity:

\[
\begin{aligned}
-\Delta u &= \frac{\lambda f(u)}{(1-u)^3} \quad &\text{in } \Omega \\
0 < u < 1 &\quad &\text{in } \Omega \\
u &= 0 &\quad &\text{on } \partial \Omega.
\end{aligned}
\]

This equation has been proposed as a model for a simple electrostatic Micro-Electromechanical System (MEMS) device consisting of a thin dielectric elastic membrane with boundary supported at 0 below a rigid ground plate located at height \(z = 1\). See [10, 11]. A voltage – directly proportional to the parameter \(\lambda\) – is applied, and the membrane deflects towards the ground plate and a snap-through may occur when it exceeds a certain critical value \(\lambda^*\), the pull-in voltage.

In [9] a fine ODE analysis of the radially symmetric case with a constant profile \(f \equiv 1\) on a ball \(B\), yields the following bifurcation diagram that describes the \(L^\infty\)-norm of the solutions \(u\) – which in this case necessarily coincides with \(u(0)\) – in terms of the corresponding voltage \(\lambda\).

The question whether the diagram above describes realistically the set of all solutions in more general domains and for non-constant profiles, and whether rigorous mathematical proofs can be given for such a description, has been the subject of many recent investigations. See [3, 4, 5, 7, 8].

We summarize in the following two theorems some of the established results concerning Figure 1. First, for every solution \(u\) of \((S)_{\lambda,f}\), we consider the linearized operator

\[
L_{u,\lambda} = -\Delta - \frac{2\lambda f}{(1-u)^3}
\]
and its eigenvalues \( \{ \mu_{k, \lambda}(u); k = 1, 2, \ldots \} \) (with the convention that eigenvalues are repeated according to their multiplicities). The Morse index \( m(u, \lambda) \) of a solution \( u \) is the largest \( k \) for which \( \mu_{k, \lambda}(u) \) is negative. A solution \( u \) of \((S)_{\lambda, f}\) is said to be stable (resp., semi-stable) if \( \mu_{1, \lambda}(u) > 0 \) (resp., \( \mu_{1, \lambda}(u) \geq 0 \)).

A description of the first stable branch and of the higher unstable ones is given in the following.

**Theorem A** [3, 4, 5]. Suppose \( f \) is a smooth nonnegative function in \( \Omega \). Then, there exists a finite \( \lambda^* > 0 \) such that

1. If \( 0 \leq \lambda < \lambda^* \), there exists a (unique) minimal solution \( u_{\lambda} \) of \((S)_{\lambda, f}\) such that \( \mu_{1, \lambda}(u_{\lambda}) > 0 \). It is also unique in the class of all semi-stable solutions.
2. If \( \lambda > \lambda^* \), there is no solution for \((S)_{\lambda, f}\).
3. If \( 1 \leq N \leq 7 \), then \( u^* = \lim_{\lambda \uparrow \lambda^*} u_{\lambda} \) is a solution of \((S)_{\lambda^*, f}\) such that \( \mu_{1, \lambda}(u^*) = 0 \), and \( u^* \) — referred to as the extremal solution of problem \((S)_{\lambda, f}\) — is the unique solution.
4. If \( 1 \leq N \leq 7 \), there exists \( \lambda_2^* \) with \( 0 < \lambda_2^* < \lambda^* \) such that for any \( \lambda \in (\lambda_2^*, \lambda^*) \), problem \((S)_{\lambda, f}\) has a second solution \( U_{\lambda} \) with \( \mu_{1, \lambda}(U_{\lambda}) < 0 \) and \( \mu_{2, \lambda}(U_{\lambda}) > 0 \). Moreover, at \( \lambda = \lambda_2^* \) there exists a second solution \( U^* := \lim_{\lambda \downarrow \lambda_2^*} U_{\lambda} \) with
   \[
   \mu_{1, \lambda_2^*}(U^*) < 0 \quad \text{and} \quad \mu_{2, \lambda_2^*}(U^*) = 0.
   \]
5. Given a more specific potential \( f \) in the form
   \[
   f(x) = \left( \prod_{i=1}^{k} |x - p_i|^{\alpha_i} \right) h(x), \quad \inf_{\Omega} h > 0, \quad (1)
   \]
   with points \( p_i \in \Omega \), \( \alpha_i \geq 0 \), and given \( u_n \) a solution of \((S)_{\lambda_n, f}\), we have the equivalence
   \[
   \|u_n\|_\infty \to 1 \iff m(u_n, \lambda_n) \to +\infty
   \]
   as \( n \to +\infty \).
It was also shown in [4] that the permittivity profile \( f \) can dramatically change the bifurcation diagram, and totally alter the critical dimensions for compactness. Indeed, the following theorem summarizes the result related to the effect of power law profiles.

**Theorem B** [4]. Assume \( \Omega \) is the unit ball \( B \) and \( f \) in the form

\[
f(x) = |x|^\alpha h(|x|), \quad \inf_B h > 0.
\]

Then we have

1. If \( N \geq 8 \) and \( \alpha > \alpha_N := \frac{3N-14-4\sqrt{6}}{4+2\sqrt{6}} \), the extremal solution \( u^* \) is again a classical solution of \((S)_{\lambda^*,f}\) such that \( \mu_{1,\lambda^*}(u^*) = 0 \).
2. If \( N \geq 8 \) and \( \alpha > \alpha_N := \frac{3N-14-4\sqrt{6}}{4+2\sqrt{6}} \), the conclusion of Theorem A-(4) still holds true.
3. On the other hand, if either \( 2 \leq N \leq 7 \) or \( N \geq 8, 0 \leq \alpha \leq \alpha_N = \frac{3N-14-4\sqrt{6}}{4+2\sqrt{6}} \), for \( f(x) = |x|^\alpha \) necessarily we have that

\[
u^*(x) = 1 - |x|^{\frac{2+\alpha}{\alpha}}, \quad \lambda^* = \frac{(2+\alpha)(3N + 4 - 4\alpha)}{9}.
\]

The bifurcation diagram suggests the following conjectures:

1. For \( 2 \leq N \leq 7 \) there exists a curve \((\lambda(t), u(t))_{t \geq 0}\) in the solution set

\[
\mathcal{V} = \left\{(\lambda, u) \in (0, +\infty) \times C^1(\Omega) : u \text{ is a solution of } (S)_{\lambda,f}\right\}, \tag{2}
\]

starting from \((0,0)\) at \( t = 0 \) and going to "infinity": \( \|u(t)\|_{\infty} \to 1 \) as \( t \to +\infty \), with infinitely many bifurcation or turning points in \( \mathcal{V} \).
2. In dimension \( N \geq 2 \) and for any profile \( f \), there exists a unique solution for small voltages \( \lambda \).
3. For \( 2 \leq N \leq 7 \) there exist exactly two solutions for \( \lambda \) in a small left neighborhood of \( \lambda^* \).

Conjectures 1 and 2 have been established for power law profiles in the radially symmetric case [7], and for the case where \( f \equiv 1 \) and \( \Omega \) is a suitably symmetric domain in \( \mathbb{R}^2 \) [8]. Indeed, in these cases Guo and Wei first show that

\[
\lambda_* = \inf\left\{\lambda > 0 : (S)_{\lambda,f} \text{ has a non-minimal solution}\right\} > 0,
\]

and then apply the fine bifurcation theory developed by Buffoni, Dancer and Toland [1] to verify the validity of Conjecture 1 in that case. The fact that \( \lambda_* > 0 \) then allows them to carry out some limiting argument and to prove that the Morse index of \( u(t) \) blows up as \( t \to +\infty \), which is crucial for showing that infinitely many bifurcation or turning points occur along the curve. Thanks to Theorem A-(5), we shall be able in Section 2 to show the validity of Conjecture 1 in general domains \( \Omega \), by circumventing the need to prove that \( \lambda_* > 0 \). On the other hand, we shall prove in Section 3 that indeed \( \lambda_* > 0 \) for a large class of domains, and therefore we have uniqueness for small voltage. Our proofs simplify considerably those of Guo and Wei [7, 8], and extend them to general star-shaped domains \( \Omega \) and power law profiles \( f(x) = |x|^\alpha, \alpha \geq 0 \).

Conjecture 3 has been shown in [3] in the class of solutions \( u \) with \( m(u,\lambda) \leq k \), for every given \( k \in \mathbb{N} \), and is still open in general.
2. A quenching branch of solutions. The first global result on the set of solutions in general domains was proved by the first author in [3]. By using a degree argument (repeated below), he showed the following result.

**Theorem 2.1.** Assume $2 \leq N \leq 7$ and $f$ be as in (1). There exist a sequence $\{\lambda_n\} \subset \mathbb{N}$ and associated solution $u_n$ of $(S)_{\lambda_n, f}$ so that

$$m(u_n, \lambda_n) \to +\infty \quad \text{as} \; n \to +\infty.$$

We now introduce some notation from Section 2.1 of [1]. Set

$$X = Y = \{u \in C^1(\bar{\Omega}) : u = 0 \text{ on } \partial \Omega\}, \quad U = (0, +\infty) \times \{u \in X : \|u\|_\infty < 1\},$$

and define the real analytic function $F : \mathbb{R} \times U \to Y$ as $F(\lambda, u) = u - \lambda K(u)$, where $K(u) = -\Delta^{-1} (f(x)(1 - u)^{-2})$ is a compact operator on every closed subset in $\{u \in X : \|u\|_\infty < 1\}$ and $\Delta^{-1}$ is the Laplacian resolvent with homogeneous Dirichlet boundary condition. The solution set $V$ given in (2) rewrites as

$$V = \{(\lambda, u) \in U : F(\lambda, u) = 0\},$$

and the projection of $V$ onto $X$ is defined as

$$\Pi_X V = \{u \in X : \exists \lambda \text{ so that } (\lambda, u) \in V\}.$$

**Proof.** In view of Theorem A-(5), we have the equivalence

$$\sup_{(\lambda, u) \in V} \max_{\Omega} u = 1 \iff \sup_{(\lambda, u) \in V} m(u, \lambda) = +\infty.$$

Arguing by contradiction, we can assume that

$$\sup_{(\lambda, u) \in V} \max_{\Omega} u \leq 1 - 2\delta, \quad \sup_{(\lambda, u) \in V} m(u, \lambda) < +\infty \quad (3)$$

for some $\delta \in (0, \frac{1}{2})$. By Theorem 1.3 in [3] one can find $\lambda_1, \lambda_2 \in (0, \lambda^*)$, $\lambda_1 < \lambda_2$, so that $(S)_{\lambda, f}$ possesses

- for $\lambda_1$, only the (non degenerate) minimal solution $u_{\lambda_1}$ which satisfies $m(u_{\lambda_1}, \lambda_1) = 0$;
- for $\lambda_2$, only the two (non degenerate) solutions $u_{\lambda_2}, U_{\lambda_2}$ satisfying $m(u_{\lambda_2}, \lambda_2) = 0$ and $m(U_{\lambda_2}, \lambda_2) = 1$, respectively.

Consider a $\delta$-neighborhood $\mathcal{V}_\delta$ of $\Pi_X V$:

$$\mathcal{V}_\delta := \{u \in X : \text{dist}_X(u, \Pi_X V) \leq \delta\}.$$

Note that (3) gives that $V$ is contained in a closed subset of $\{u \in X : \|u\|_\infty < 1\}$:

$$\mathcal{V}_\delta \subset \{u \in X : \|u\|_\infty \leq 1 - \delta\}.$$

We can now define the Leray-Schauder degree $d_\lambda$ of $F(\lambda, \cdot)$ on $\mathcal{V}_\delta$ with respect to zero, since by definition of $\Pi_X V$ (the set of all solutions) $\partial V_\delta$ does not contain any solution of $(S)_{\lambda, f}$ for any value of $\lambda$. Since $d_\lambda$ is well defined for any $\lambda \in [0, \lambda^*)$, by homotopy $d_{\lambda_1} = d_{\lambda_2}$. To get a contradiction, let us now compute $d_{\lambda_1}$ and $d_{\lambda_2}$. Since the only
zero of $F(\lambda_1, \cdot)$ in $V_0$ is $u_{\lambda_1}$ with Morse index zero, we have $d_{\lambda_1} = 1$. Since $F(\lambda_2, \cdot)$ has in $V_0$ exactly two zeroes $u_{\lambda_2}$ and $U_{\lambda_2}$ with Morse index zero and one, respectively, we have $d_{\lambda_1} = 1 - 1 = 0$. This contradicts $d_{\lambda_1} = d_{\lambda_2}$, and the proof is complete. □

We can now combine Theorem A-(5) with the fine bifurcation theory in [1] to establish a more precise multiplicity result. See also [2].

Observe that $A_0 := \{(\lambda, u_\lambda) : \lambda \in (0, \lambda^*)\}$ is a maximal arc-connected subset of

$$S := \{(\lambda, u) \in U : F(\lambda, u) = 0 \text{ and } \partial_u F(\lambda, u) : X \rightarrow Y \text{ is invertible}\}$$

with $A_0 \subset S$. Assume that the extremal solution $u^*$ is a classical solution so to have $u^* \in (S \cap U) \setminus S$. Assumption (C1) of Section 2.1 in [1] does hold in our case. As far as condition (C2):

$$\{(\lambda, u) \in U : F(\lambda, u) = 0\} \text{ is open in } \{(\lambda, u) \in \mathbb{R} \times X : F(\lambda, u) = 0\},$$

let us stress that it is a weaker statement than requiring $U$ to be an open subset in $\mathbb{R} \times X$. In our case, the map $F(\lambda, u)$ is defined only in $U$ (and not in the whole $X$), and then condition (C2) does not make sense. However, we can replace it with the new condition (C2):

$$U \text{ is an open set in } \mathbb{R} \times X,$$

which does hold in our context. Since (C2) is used only in Theorem 2.3-(iii) in [1] to show that $S$ is open in $S$, our new condition (C2) does not cause any trouble in the arguments of [1].

Since $\partial_u F(\lambda, u)$ is a Fredholm operator of index 0, by a Lyapunov-Schmidt reduction we have that assumptions (C3)-(C5) do hold in our case (let us stress that these conditions are local and $U$ is an open set in $\mathbb{R} \times X$).

Setting $\lambda = 0$ and defining the map $\nu : U \rightarrow [0, +\infty)$ as $\nu(\lambda, u) = \frac{1}{\|u\|_\infty}$, conditions (C6)-(C8) do hold in view of the property $\lambda \in [0, \lambda^*]$. Theorem 2.4 in [1] then applies and gives the following.

**Theorem 2.2.** Assume $u^*$ a classical solution of $(S)_{\lambda^*, f}$. Then there exists an analytic curve $(\hat{\lambda}(t), \hat{u}(t))_{t \geq 0}$ in $V$ starting from $(0, 0)$ and so that $\|\hat{u}(t)\|_\infty \rightarrow 1$ as $t \rightarrow +\infty$. Moreover, $\hat{u}(t)$ is a non-degenerate solution of $(S)_{\hat{\lambda}(t), f}$ except at isolated points.

By the Implicit Function Theorem, the curve $(\hat{\lambda}(t), \hat{u}(t))$ can only have isolated intersections. If we now use the usual trick of finding a minimal continuum in $(\{\hat{\lambda}(t), \hat{u}(t)) : t \geq 0\}$ joining $(0, 0)$ to “infinity”, we obtain a continuous curve $(\lambda(t), u(t))$ in $V$ with no self-intersections which is only piecewise analytic. Clearly, $\partial_u F(\lambda, u) : X \rightarrow Y$ is still invertible along the curve except at isolated points.

Let now $2 \leq N \leq 7$ and $f$ be as in (1). By the equivalence in Theorem A-(5) we get that $m(\hat{\lambda}(t), \hat{u}(t)) \rightarrow +\infty$ as $t \rightarrow +\infty$, and then $\mu_{k, \hat{\lambda}(t)}(\hat{u}(t)) < 0$ for $t$ large, for every $k \geq 1$. Since $\mu_{k, \lambda(0)}(u(0)) = \mu_{k, 0}(0) > 0$ and $u(t)$ is a non-degenerate solution of $(S)_{\lambda(t), f}$ except at isolated points, we find $t_k > 0$ so that $\mu_{k, \lambda(t)}(u(t))$ changes from positive to negative sign across $t_k$. Since $\mu_{k+1, \lambda(t)}(u(t)) \geq \mu_{k, \lambda(t)}(u(t))$, we can choose $t_k$ to be non-increasing in $k$ and to have $t_k \rightarrow +\infty$ as $k \rightarrow +\infty$.

To study secondary bifurcations, we will use the gradient structure in the problem.
Setting \((\lambda_k, u_k) := (\lambda(t_k), u(t_k))\), we have that \((\lambda_k, u_k) \notin S\). Choose \(\delta > 0\) small so that \(\|u_k\|_{\infty} < 1 - \delta\), and replace the nonlinearity \((1 - u)^{-2}\) with a regularized one:

\[
f_\delta(u) = \begin{cases} (1 - u)^{-2} & \text{if } u \leq 1 - \delta, \\ \delta^{-2} & \text{if } u \geq 1 - \delta, \end{cases}
\]

and the map \(F(\lambda, u)\) with the corresponding one \(F_\delta(\lambda, u)\). We replace \(X\) and \(Y\) with \(H^2(\Omega) \cap H^1_0(\Omega)\) and \(L^2(\Omega)\), respectively. The map \(F_\delta(\lambda, u)\) can be considered as a map from \(\mathbb{R} \times X \to Y\) with a gradient structure:

\[
\partial_u J_\delta(\lambda, u)[\varphi] = \langle F_\delta(\lambda, u), \varphi \rangle_{L^2(\Omega)}
\]

for every \(\lambda \in \mathbb{R}\) and \(u, \varphi \in X\), where \(J_\delta : \mathbb{R} \times X \to \mathbb{R}\) is the functional given by

\[
J_\delta(\lambda, u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx - \lambda \int_{\Omega} f(x)G_\delta(u) \, dx, \quad G_\delta(u) = \int_0^u f_\delta(s) \, ds.
\]

Assumptions (G1)-(G2) in Section 2.2 of [1] do hold. We have that \((\lambda(t), u(t)) \in S\) for \(t\) close to \(t_k\) and \(m(\lambda(t), u(t))\) changes across \(t_k\). If \(\lambda(t)\) is injective, by Proposition 2.7 in [1] we have that \((\lambda(t_k), u(t_k))\) is a bifurcation point. Then we get the validity of Conjecture 1 as claimed below.

**Theorem 2.3.** Assume \(2 \leq N \leq 7\) and \(f\) be as in (1). Then there exists a continuous, piecewise analytic curve \((\lambda(t), u(t))_{t \geq 0}\) in \(\mathcal{V}\), starting from \((0, 0)\) and so that \(\|\hat{u}(t)\|_{\infty} \to 1\) as \(t \to +\infty\), which has either infinitely many turning points, i.e. points where \((\lambda(t), u(t))\) changes direction (the branch locally “bends back”), or infinitely many bifurcation points.

**Remark 2.1.** In [7] the above analysis is performed in the radial setting to obtain a curve \((\lambda(t), u(t))_{t \geq 0}\), as given by Theorem 2.3, composed by radial solutions and so that \(m_\epsilon(\lambda(t), u(t)) \to +\infty\) as \(t \to +\infty\), \(m_\epsilon(\lambda, u)\) being the radial Morse index of a solution \((\lambda, u)\). In this way, it can be shown that bifurcation points can’t occur and then \((\lambda(t), u(t))_{t \geq 0}\) exhibits infinitely many turning points. Moreover, they can also deal with the case where \(N \geq 8\) and \(\alpha > \alpha_N\).

### 3. Uniqueness of solutions for small voltage in star-shaped domains.

We address the issue of uniqueness of solutions of the singular elliptic problem

\[
\begin{cases}
-\Delta u = \frac{\lambda|x|^\alpha}{(1-u)^2} & \text{in } \Omega \\
0 < u < 1 & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\]

for \(\lambda > 0\) small, where \(\alpha \geq 0\) and \(\Omega\) is a bounded domain in \(\mathbb{R}^N, N \geq 2\). We shall make crucial use of the following extension of Pohozaev’s identity due to Pucci and Serrin [12].

**Proposition 3.1.** Let \(v\) be a solution of the boundary value problem

\[
\begin{cases}
-\Delta v = f(x, v) & \text{in } \Omega \\
v = 0 & \text{on } \partial\Omega.
\end{cases}
\]
Then for any \(a \in \mathbb{R}\) and any \(h \in C^2(\Omega; \mathbb{R}^N) \cap C^1(\bar{\Omega}; \mathbb{R}^N)\), the following identity holds

\[
\int_{\Omega} [\text{div}(h)F(x, v) - avf(x, v) + \langle \nabla_x F(x, v), h \rangle] \, dx = \int_{\Omega} \left[ \left(\frac{1}{2}\text{div}(h) - a\right)|\nabla v|^2 - \langle Dh \nabla v, \nabla v \rangle \right] \, dx + \frac{1}{2} \int_{\partial\Omega} |\nabla v|^2 \langle h, v \rangle d\sigma,
\]

(5)

where \(F(x, s) = \int_0^s f(x, t) \, dt\).

An application of the method in [13] leads to the following result.

**Theorem 3.1.** Let \(\Omega \subset \mathbb{R}^N\) be a star-shaped domain with respect to 0. If \(N \geq 3\), then for \(\lambda\) small, the stable solution \(u_\lambda\) is the unique solution of equation (4).

**Proof.** Since \(u_\lambda\) is the minimal solution of (4) for \(\lambda \in (0, \lambda^*)\), setting \(v = u - u_\lambda\) equation (4) rewrites equivalently as

\[
\begin{cases}
-\Delta v = \lambda|x|^\alpha g_\lambda(x, v) & \text{in } \Omega \\
0 \leq v < 1 - u_\lambda & \text{in } \Omega \\
v = 0 & \text{on } \partial\Omega,
\end{cases}
\]

(6)

where

\[
g_\lambda(x, s) = \frac{1}{(1 - u_\lambda(x) - s)^2} - \frac{1}{(1 - u_\lambda(x))^2}.
\]

(7)

It then suffices to prove that the solutions of (6) must be trivial for \(\lambda\) small enough. First compute \(G_\lambda(x, s)\):

\[
G_\lambda(x, s) = \int_0^s g_\lambda(x, t) \, dt = \frac{1}{1 - u_\lambda(x) - s} - \frac{1}{1 - u_\lambda(x)} - \frac{s}{(1 - u_\lambda(x))^2}.
\]

Since the validity of the relation

\[
\nabla_x \left( |x|^\alpha G_\lambda(x, s) \right) = \alpha |x|^{\alpha - 2} G_\lambda(x, s) + |x|^\alpha \nabla_x G_\lambda(x, s),
\]

for \(h(x) = \frac{x}{N}\) and \(f(x, v) = |x|^\alpha g_\lambda(x, v)\) we apply the Pohozaev identity (5) to a solution \(v\) of (6) to get

\[
\lambda \int_{\Omega} |x|^\alpha \left[ (1 + \frac{\alpha}{N}) G_\lambda(x, v(x)) - av(x)g_\lambda(x, v(x)) + \langle \nabla_x G_\lambda(x, v(x)), \frac{x}{N} \rangle \right] \, dx
\]

\[
= \int_{\Omega} \left[ \left(\frac{1}{2} - a\right)|\nabla v|^2 - \langle D(\frac{x}{N}) \nabla v, \nabla v \rangle \right] dx + \frac{1}{2N} \int_{\partial\Omega} |\nabla v|^2 \langle h, v \rangle \, d\sigma
\]

(8)

\[
\geq \left(\frac{1}{2} - a - \frac{1}{N}\right) \int_{\Omega} |\nabla v|^2 dx.
\]

Since easy calculations show that

\[
\frac{G_\lambda(x, s)}{g_\lambda(x, s)} = \frac{1 - u_\lambda(x) - s}{1 - u_\lambda(x)} - \frac{1}{(1 - u_\lambda(x))^2} + \frac{s}{(1 - u_\lambda(x))^2}.
\]
and 

$$\frac{\nabla_x G_\lambda(x, s)}{g_\lambda(x, s)} = \frac{1 - \frac{(1-u_\lambda(x) - s)^2(1-u_\lambda(x) - 2s)}{(1-u_\lambda(x))^2}}{1 - \frac{(1-u_\lambda(x) - s)^2}{(1-u_\lambda(x))^2}} \nabla u_\lambda(x),$$

we obtain

$$\left| \frac{G_\lambda(x, s)}{g_\lambda(x, s)} \right| \leq C_0 |1 - u_\lambda(x) - s| \quad \text{and} \quad \left| \frac{\nabla_x G_\lambda(x, s)}{g_\lambda(x, s)} - \nabla u_\lambda \right| \leq C_0 |1 - u_\lambda(x) - s|^2 |\nabla u_\lambda|$$

for some $C_0 > 0$, provided $\lambda$ is away from $\lambda^*$. Since $u_\lambda \to 0$ in $C^1(\bar{\Omega})$ as $\lambda \to 0^+$, for $a > 0$ from (9) we deduce that for any $(x, s)$ satisfying $|1 - u_\lambda(x) - s| \leq \delta$

$$(1 + \frac{\alpha}{N})G_\lambda(x, s) - asg_\lambda(x, s) + \langle \nabla_x G_\lambda(x, s), \frac{x}{N} \rangle \leq g_\lambda(x, s) \left[ C_0 (1 + \frac{\alpha}{N}) \delta - a(1 - u_\lambda(x) - \delta) + \langle \nabla u_\lambda, \frac{x}{N} \rangle + \frac{C_0}{N} \delta^2 |\nabla u_\lambda||x| \right] \leq 0,$$

provided $\delta$ and $\lambda$ are sufficiently small (depending on $a$). Since $N \geq 3$, we can pick $0 < a < \frac{1}{2} - \frac{1}{N}$, and then by (8), (10) get that

$$\lambda \int_{\{0 \leq v \leq 1 - u_\lambda - \delta\}} |x|^{\alpha} \left[ (1 + \frac{\alpha}{N})G_\lambda(x, v(x)) - av(x)g_\lambda(x, v(x)) + \langle \nabla_x G_\lambda(x, v(x)), \frac{x}{N} \rangle \right] dx \\ \geq \left( \frac{1}{2} - a - \frac{1}{N} \right) \int_{\Omega} |\nabla v|^2 dx \geq C_s \left( \frac{1}{2} - a - \frac{1}{N} \right) \int_{\Omega} v^2 dx$$

for $\delta$ and $\lambda$ sufficiently small, where $C_s$ is the best constant in the Sobolev embedding of $H^1_0(\Omega)$ into $L^2(\Omega)$.

On the other hand, since $G_\lambda(x, s), s g_\lambda(x, s)$ and $\nabla_x G_\lambda(x, s)$ are quadratic with respect to $s$ as $s \to 0$ (uniformly in $\lambda$ away from $\lambda^*$), there exists a constant $C_\delta > 0$ such that

$$(1 + \frac{\alpha}{N})G_\lambda(x, v(x)) - av g_\lambda(x, v(x)) + \langle \nabla_x G_\lambda(x, v(x)), \frac{x}{N} \rangle \leq C_\delta v^2(x)$$

for $x \in \{0 \leq v \leq 1 - u_\lambda - \delta\}$, uniformly for $\lambda$ away from $\lambda^*$. Combining (11) and (12) we get that

$$C_s \left( \frac{1}{2} - a - \frac{1}{N} \right) \int_{\{0 \leq v \leq 1 - u_\lambda - \delta\}} v^2 dx \leq \lambda C_\delta \int_{\{0 \leq v \leq 1 - u_\lambda - \delta\}} |x|^{\alpha} v^2 dx.$$ 

Therefore, for $\lambda$ sufficiently small we conclude that $v \equiv 0$ in $\{0 \leq v \leq 1 - u_\lambda - \delta\}$. This implies that $v \equiv 0$ in $\Omega$ for sufficiently small $\lambda$, and we are done. 

We now refine the above argument so as to cover other situations. To this aim, we consider the – potentially empty – set

$$H(\Omega) = \left\{ h \in C^1(\bar{\Omega}, \mathbb{R}^N) : \text{div}(h) \equiv 1 \text{ and } \langle h, \nu \rangle \geq 0 \text{ on } \partial\Omega \right\},$$

and the corresponding parameter

$$M(\Omega) := \inf \left\{ \sup_{x \in \Omega} \tilde{\mu}(h, x) : h \in H(\Omega) \right\},$$
where

\[
\bar{\mu}(h, x) = \frac{1}{2} \sup_{|\xi| = 1} ((Dh(x) + Dh(x)^T)\xi, \xi).
\]

The following is an extension of Theorem 3.1.

**Theorem 3.2.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \) such that \( M(\Omega) < \frac{1}{2} \). Then, for \( \lambda \) small the minimal solution \( u_\lambda \) is the unique solution of problem (4), provided either \( N \geq 3 \) or \( \alpha > 0 \).

**Proof.** As above, we shall prove that equation (6), with \( g_\lambda \) as in (7), has only trivial solutions for \( \lambda \) small. For a solution \( v \) of (6) the Pohozaev identity (5) with \( h \in H(\Omega) \) yields

\[
\lambda \int_\Omega |x|^{\alpha} [G_\lambda(x, v(x))(1 + \alpha(x|x|^2, h)) - av(x)g_\lambda(x, v(x)) + \langle \nabla_x G_\lambda(x, v(x)), h \rangle] \, dx
\]

\[
= \int_\Omega \left[ \left( \frac{1}{2} - a \right) |\nabla v|^2 - \frac{1}{2} \langle (Dh + Dh^T)\nabla v, \nabla v \rangle \right] \, dx + \frac{1}{2} \int_{\partial\Omega} |\nabla v|^2 \langle h, v \rangle \, d\sigma
\]

\[
\geq \left( \frac{1}{2} - a - \sup_{x \in \Omega} \bar{\mu}(h, x) \right) |\nabla v|^2 \, dx.
\] (13)

Fix \( 0 < a < \frac{1}{2} - M(\Omega) \) and choose \( h \in H(\Omega) \) such that

\[
\frac{1}{2} - a - \sup_{x \in \Omega} \bar{\mu}(h, x) > 0.
\]

It follows from (9) that for any \( (x, s) \) satisfying \( |1 - u_\lambda(x) - s| \leq \delta|x| \) there holds

\[
G_\lambda(x, s)(1 + \alpha(x|x|^2, h)) - av(x)g_\lambda(x, v(x)) + \langle \nabla_x G_\lambda(x, v(x)), h \rangle
\]

\[
\leq g_\lambda(x, s)\left[ C_0 \delta|x| + \alpha C_0 |h| - a(1 - u_\lambda - \delta|x|) + \langle \nabla u_\lambda, h \rangle + C_0 \delta^2 |x|^2 |\nabla u_\lambda||h| \right] \leq 0
\] (14)

provided \( \lambda \) and \( \delta \) are sufficiently small. It then follows from (13) and (15) that

\[
\lambda \int_{\{0 \leq v \leq 1 - u_\lambda - \delta|x| \}} |x|^{\alpha} [G_\lambda(x, v(x))(1 + \alpha(x|x|^2, h))]
\]

\[
- av(x)g_\lambda(x, v(x)) + \langle \nabla_x G_\lambda(x, v(x)), h \rangle \, dx
\]

\[
\geq \left( \frac{1}{2} - a - \sup_{x \in \Omega} \bar{\mu}(h, x) \right) \int_\Omega |\nabla v|^2 \, dx.
\] (15)

On the other hand, there exists a constant \( C_\delta > 0 \) such that

\[
G_\lambda(x, v(x))(1 + \alpha(x|x|^2, h(x))) - av(x)g_\lambda(x, v(x)) + \langle \nabla_x G_\lambda(x, v(x)), h(x) \rangle >
\]

\[
= \frac{v^2(x)}{(1 - u_\lambda(x) - v(x))(1 - u_\lambda(x))} \left( 1 + \alpha(x|x|^2, h(x)) + \frac{av^2(x)v(x) - 2u_\lambda(x)}{(1 - u_\lambda(x) - v(x))^2} \right)
\]

\[
+ \frac{v^2(x)(3 - 3u_\lambda(x) - 2v(x))}{(1 - u_\lambda(x) - v(x))^2} < \nabla u_\lambda(x), h(x) \geq C_\delta \frac{v^2(x)}{|x|^2}
\]
for $x \in \{0 \leq v \leq 1 - u_\lambda - \delta|x|\}$, uniformly for $\lambda$ away from $\lambda^*$.

If now $N \geq 3$, then Hardy’s inequality combined with (15) implies

$$\frac{(N-2)^2}{4} \left(1 - a - \sup_{x \in \Omega} \bar{\mu}(h, x)\right) \int_{\{0 \leq v \leq 1 - u_\lambda - \delta|x|\}} \frac{v^2}{|x|^2} \, dx \leq \lambda C_\delta \int_{\{0 \leq v \leq 1 - u_\lambda - \delta|x|\}} \frac{v^2}{|x|^2} \, dx.$$  

On the other hand, when $N = 2$ the space $H^1_0(\Omega)$ embeds continuously into $L^p(\Omega)$ for every $p > 1$, and then, by Hölder inequality, for $\alpha > 0$ we get that

$$\int_{\Omega} \frac{v^2}{|x|^{2-\alpha}} \, dx \leq \left( \int_{\Omega} |x|^{-(2-\alpha)\frac{p}{p-2}} \, dx \right)^{\frac{p-2}{2}} \left( \int_{\Omega} |v|^p \, dx \right)^{\frac{2}{p}} \leq C_N^{-1} \int_{\Omega} |\nabla v|^2 \, dx$$

provided $(2 - \alpha)\frac{p}{p-2} < 2$, which is true for $p$ large depending on $\alpha$ (see [6] for some very general Hardy inequalities). It combines with (15) to yield

$$C_{N,\alpha} \left(1 - a - \sup_{x \in \Omega} \bar{\mu}(h, x)\right) \int_{\{0 \leq v \leq 1 - u_\lambda - \delta|x|\}} \frac{v^2}{|x|^{2-\alpha}} \, dx \leq \lambda C_\delta \int_{\{0 \leq v \leq 1 - u_\lambda - \delta|x|\}} \frac{v^2}{|x|^{2-\alpha}} \, dx.$$  

In both cases, we can conclude that for $\lambda$ sufficiently small $v \equiv 0$ for $x \in \{0 \leq v \leq 1 - u_\lambda - \delta|x|\}$, for some $\delta > 0$ small. Since we can assume $\delta$ and $\lambda$ sufficiently small to have

$$1 - u_\lambda - \delta|x| \geq \frac{1}{2} \text{ in } \{x \in \Omega : |x| \geq \frac{1}{2} \text{dist}(0, \partial \Omega)\},$$

we then have

$$v \equiv 0 \text{ in } \{x \in \Omega : v(x) \leq \frac{1}{2}\} \cap \{x \in \Omega : |x| \geq \frac{1}{2} \text{dist}(0, \partial \Omega)\}.$$  

Since $v = 0$ on $\partial \Omega$ and the domain $\{x \in \Omega : |x| \geq \frac{1}{2} \text{dist}(0, \partial \Omega)\}$ is connected, the continuity of $v$ gives that

$$v \equiv 0 \text{ in } \{x \in \Omega : |x| \geq \frac{1}{2} \text{dist}(0, \partial \Omega)\}.$$  

Therefore, the maximum principle for elliptic equations implies $v \equiv 0$ in $\Omega$, which completes the proof of Theorem 3.2.  

**Remark 3.1.** In [13] examples of dumbbell shaped domains $\Omega \subset \mathbb{R}^N$ which satisfy condition $M(\Omega) < \frac{1}{2}$ are given for $N \geq 3$. When $N \geq 4$, there even exist topologically nontrivial domains with this property. Let us stress that in both cases $\Omega$ is not starlike, which means that the assumption $M(\Omega) < \frac{1}{2}$ on a domain $\Omega$ is more general than being star-shaped.

The remaining case $N = 2$ and $\alpha = 0$, is a bit more delicate. We have the following result.

**Theorem 3.3.** If $\Omega$ is either a strictly convex or a symmetric domain in $\mathbb{R}^2$, then $(S)_{\lambda,1}$ has the unique solution $u_\lambda$ for small $\lambda$.

**Proof.** The crucial point here is the following inequality: for every solution $v$ of (6) there holds

$$\int_{\partial \Omega} |\nabla v|^2 \, d\sigma \geq l(\partial \Omega)^{-1} \left( \int_\Omega |\Delta v| \, dx \right)^2.$$
Indeed, we have that
\[
\int_{\partial \Omega} |\nabla v|^2 \, d\sigma \geq l(\partial \Omega)^{-1} \left( \int_{\partial \Omega} |\nabla v| \, d\sigma \right)^2 = l(\partial \Omega)^{-1} \left( \int_{\partial \Omega} \partial_v v \, d\sigma \right)^2 = l(\partial \Omega)^{-1} \left( \int_{\Omega} |\nabla v| \, dx \right)^2 ,
\]
where \(l(\partial \Omega)\) is the length of \(\partial \Omega\). Note that \(-\Delta v = \lambda g_\lambda(x,v) \geq 0\) for every solution \(u_\lambda + v\) of \((S)_{\lambda,1}\), in view of the minimality of \(u_\lambda\).

By Lemma 4 in [13] for \(\lambda\) small there exists \(x_\lambda \in \Omega\) so that
\[
\langle \nabla u_\lambda(x), x - x_\lambda \rangle \leq 0 \quad \forall \ x \in \Omega . \tag{16}
\]
In particular, for \(\lambda\) small \(x_\lambda\) lies in a compact subset of \(\Omega\) and, when \(\Omega\) is symmetric, coincides exactly with the center of symmetries. In both situations, then we have that there exists \(c_0 > 0\) so that
\[
\langle x - x_\lambda, \nu(x) \rangle \geq c_0 \quad \forall \ x \in \partial \Omega .
\]

We use now the Pohozaev identity (5) with \(a = 0\) and \(h(x) = \frac{x - x_\lambda}{2}\). For every solution \(v\) of (6) it yields
\[
\lambda \int_{\Omega} \left[ G_\lambda(x,v(x)) + \langle \nabla_x G_\lambda(x,v(x)), \frac{x - x_\lambda}{2} \rangle \right] \, dx = \frac{1}{4} \int_{\partial \Omega} |\nabla v|^2 \langle x - x_\lambda, \nu \rangle \, d\sigma \geq \frac{c_0}{4} \left( \int_{\Omega} |\Delta v| \, dx \right)^2 . \tag{17}
\]
Since
\[
\nabla_x G_\lambda(x,s) = (1 - u_\lambda(x) - s)^{-2} \left[ 1 - \frac{(1 - u_\lambda(x) - s)^2(1 - u_\lambda(x) + 2s)}{(1 - u_\lambda(x))^3} \right] \nabla u_\lambda(x) ,
\]
by (16) we easily see that
\[
\langle \nabla_x G_\lambda(x,s), x - x_\lambda \rangle \leq 0
\]
for \(\lambda\) and \(\delta\) small, provided \((x,s)\) satisfies \(|1 - u_\lambda(x) - s| \leq \delta\). Since \(G_\lambda(x,s), \nabla_x G_\lambda(x,s)\) are quadratic with respect to \(s\) as \(s \to 0\) (uniformly in \(\lambda\) small), there exists a constant \(C_\delta > 0\) such that
\[
G_\lambda(x,v(x)) \leq C_\delta v^2(x) , \quad \langle \nabla_x G_\lambda(x,v(x)), \frac{x - x_\lambda}{2} \rangle \leq C_\delta v^2(x)
\]
for \(x \in \{ 0 \leq v \leq 1 - u_\lambda - \delta \} ,\) uniformly for \(\lambda\) small.

Since on two-dimensional domains
\[
\left( \int_{\Omega} |v|^p \, dx \right)^{\frac{1}{p}} \leq C_p \int_{\Omega} |\Delta v| \, dx
\]
for every \(p \geq 1\) and \(v \in W^{2,1}(\Omega)\) so that \(v = 0\) on \(\partial \Omega\), we get that
\[
\lambda \int_{\Omega} \langle \nabla_x G_\lambda(x,v(x)), \frac{x - x_\lambda}{2} \rangle \, dx \leq \lambda C_\delta \int_{\Omega} v^2 \, dx \leq \lambda C_\delta C_2^2 \left( \int_{\Omega} |\Delta v| \, dx \right)^2 . \tag{18}
\]
As far as the term with \( G_\lambda(x, v(x)) \), fix \( b \in (0, 1) \) and split \( \Omega \) as the disjoint union of \( \Omega_1 = \{ v \leq b \} \) and \( \Omega_2 = \{ v > b \} \). On \( \Omega_1 \) we have that

\[
\lambda \int_{\Omega_1} G_\lambda(x, v(x)) \, dx \leq \lambda C_4 \int_{\Omega} v^2 \, dx \leq \lambda C_4 C_2^2 \left( \int_{\Omega} |\Delta v| \, dx \right)^2
\]

provided \( \lambda \) and \( \delta \) are small to satisfy \( b \leq 1 - u_\lambda - \delta \) in \( \Omega_1 \).

Since for \( \lambda \) small

\[
\frac{G_\lambda(x, s)^2}{g_\lambda(x, s)} \leq C \quad \forall \ b \leq s \leq 1,
\]

we have that

\[
\lambda \int_{\Omega_2} G_\lambda(x, v(x)) \, dx \leq \lambda D_1 \int_{\Omega} |v(x)|^{\frac{\lambda}{2}} g_\lambda^{\frac{1}{2}}(x, v(x)) \, dx
\]

\[
\leq \lambda D_2 \left( \int_{\Omega} |v|^3 \, dx \right)^{\frac{\lambda}{6}} \left( \int_{\Omega} g_\lambda(x, v(x)) \, dx \right)^{\frac{1}{2}}
\]

\[
\leq \lambda^{\frac{\lambda}{2}} D_3 \left( \int_{\Omega} |\Delta v| \, dx \right)^2
\]

for some positive constants \( D_1, D_2 \) and \( D_3 \). So we get that

\[
\lambda \int_{\Omega} G_\lambda(x, v(x)) \, dx \leq \left( \lambda C_4 C_2^2 + \lambda^{\frac{\lambda}{2}} D_3 \right) \left( \int_{\Omega} |\Delta v| \, dx \right)^2.
\]

(19)

Inserting (18)-(19) into (17) finally we get that

\[
\left( 2\lambda C_4 C_2^2 + \lambda^{\frac{\lambda}{2}} D_3 - \frac{c_0}{4} \right) \left( \int_{\Omega} |\Delta v| \, dx \right)^2 \geq 0,
\]

and then \( v \equiv 0 \) for \( \lambda \) small. \( \square \)

REFERENCES
