Abstract. If $\Omega_j \in \mathbb{R}^d (d \geq 2)$ are bounded open subsets and $\Phi \in C^1(\Omega_1 ; \Omega_2)$ respects Lebesgue measure and satisfies $F \circ \Phi \in BV(\Omega_1)$ for all $F \in BV(\Omega_2)$ then $\Phi$ is uniformly Lipschitzian.

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The problem addressed in this note is motivated by the study of the propagation of regularity in the transport by vector fields with bounded divergence,

$$\frac{\partial u}{\partial t} + \sum_{j=1}^{d} a_j(t, x) \frac{\partial u}{\partial x_j} = 0, \quad x \in \mathbb{R}^d, \quad d \geq 2, \quad t > 0, \quad (1)$$

where $x = (x_1, x_2, \cdots, x_d)$ and,

$$a := (a_1, \cdots, a_d) \in L^\infty([0, T] \times \mathbb{R}^d), \quad \text{div}_x a = \sum_{j=1}^{d} \partial_{x_j} a_j(t, x) \in L^\infty([0, T] \times \mathbb{R}^d). \quad (2)$$

The recent result of [Am] shows that this suffices to guarantee the uniqueness of $L^\infty$ solutions of Cauchy problem if the vector field $a$ is of BV regularity.

Then, for arbitrary initial data $u_0(x) \in L^\infty(\mathbb{R}^d)$ there is a unique solution $u(t, x) \in L^\infty([0, T] \times \mathbb{R}^d)$ with $u|_{t=0} = u_0$. With the same hypotheses, there is a well defined flow $\Phi_t$ and the solution is given by $u(t) = u_0 \circ \Phi_{-t}$. The flow respects Lebesgue measure in the sense of (3) below.

We have given examples [CLR2] which show that such transport equations do not in general propagate either Hölder or $BV$ regularity. The counterexamples had flows which were mostly smooth with small singular sets. Thus there were large open sets on which the flows were $C^1$ maps. On those sets, the following result shows that $BV$ preservation implies that the flow must of necessity be uniformly Lipschitzian. In the examples in [CLR2], it is easily seen that the the flows are not uniformly Lipschitzian. The example (shown to us by L. Ambrosio) of a measure preserving $\Phi : [0, 2] \to [1, 11]$

$$\Phi(x) = x \quad \text{for} \quad 0 < x < 1, \quad \Phi(x) = x - 2 \quad \text{for} \quad 1 < x < 2,$$

shows that measure preserving maps which are smooth except for jumps, can preserve $BV$ without being Lipschitzian. The following result shows that this cannot happen for $C^1$ maps. The result applies as well to maps which respect but do not preserve measure.
Theorem 1. Suppose that \( \Omega_j \) are bounded open subsets of \( \mathbb{R}^d \) (\( d \geq 2 \)) and \( \Phi \in C^1(\Omega_1; \Omega_2) \) has the following two properties;

1. \( \exists \gamma > 0, \forall \text{ Borel subsets } A \subset \Omega_2, \quad \frac{1}{\gamma} |\Phi^{-1}(A)| \leq |A| \leq \gamma |\Phi^{-1}(A)|. \)

where \( |\cdot| \) denotes Lebesgue measure, and,

2. \( \forall F \in BV(\Omega_2), \quad F \circ \Phi \in BV(\Omega_1). \)

Then \( \Phi \in W^{1,\infty}(\Omega_1) \).

The proof of Theorem 1 consists of two lemmas.

Lemma 2. If \( \Phi \in C^1 \) but not in \( W^{1,\infty} \), then for any positive number \( M > 0 \), there exists an \( F \in C_0^\infty(\Omega_2) \) such that

\[
\left\| (F \circ \Phi)' \right\|_{L^1(\Omega_1)} \geq M \left\| F' \right\|_{L^1(\Omega_2)}. 
\]

Proof. The chain rule implies that for any \( F \in C_0^1 \) and \( 1 \leq i \leq d \),

\[
\int_{\Omega_1} \left| \frac{\partial (F \circ \Phi)(x)}{\partial x_i} \right| \, dx = \int_{\Omega_1} \left| \sum_{j=1}^{d} \frac{\partial F}{\partial y_j} (\Phi(x)) \frac{\partial \Phi_j(x)}{\partial x_i} \right| \, dx.
\]

Since \( \Phi' \) is not bounded, there is for any \( M > 0 \), an \( \bar{x} \in \Omega_1 \) such that

\[
\max_{1 \leq i, j \leq d} \left| \frac{\partial \Phi_i}{\partial x_j}(\bar{x}) \right| \geq \frac{8M}{\gamma}.
\]

Without loss of generality, we may assume that

\[
\left| \frac{\partial \Phi_1}{\partial x_1}(\bar{x}) \right| = \max_{1 \leq i, j \leq d} \left| \frac{\partial \Phi_i}{\partial x_j}(\bar{x}) \right| \geq \frac{8M}{\gamma}.
\]

Let \( \bar{y} = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_d) := \Phi(\bar{x}) \). For \( 0 < \epsilon \) small,

\[
N_\epsilon := \{ y \in \mathbb{R}^d : |y_1 - \bar{y}_1| < \epsilon, \quad |y_j - \bar{y}_j| < \sqrt{\epsilon} \text{ for } 2 \leq j \} \subset \Omega_2.
\]

Define

\[
M_\epsilon := \Phi^{-1}(N_\epsilon).
\]

For \( \epsilon \) small and \( x \in M_\epsilon \),

\[
\left| \frac{\partial \Phi_1}{\partial x_1}(x) \right| \geq \frac{1}{2} \left| \frac{\partial \Phi_1}{\partial x_1}(\bar{x}) \right|, \quad \text{and for } j \geq 2, \quad \left| \frac{\partial \Phi_1}{\partial x_j}(x) \right| \leq 2 \left| \frac{\partial \Phi_1}{\partial x_1}(\bar{x}) \right|.
\]

Choose \( \phi \in C_0^\infty([-1, 1]) \) satisfying

\[
\int_{-\infty}^{\infty} |\phi(z)| \, dz = 1.
\]
Define
\[ F := \phi \left( \frac{y_1 - \tilde{y}_1}{\epsilon} \right) \prod_{j=2}^d \phi \left( \frac{y_j - \tilde{y}_j}{\sqrt{\epsilon}} \right). \]

Then,
\[ ||F'||_{L^1(\Omega_2)} := \int_{\Omega_2} \left| \sum_{j=1}^d \frac{\partial F(y)}{\partial y_j} \right| dy = \int_{\Omega_2} \left| \sum_{j=1}^d \frac{\partial y_j}{\partial y_j} \right| dy \]
\[ = \epsilon^{(d-1)/2} (1 + (d-1)\sqrt{\epsilon}) \int_{-\infty}^{\infty} |\phi'(z)| dz. \]

For \( \epsilon \) small,
\[ ||F'||_{L^1(\Omega_2)} \leq 2 \epsilon^{(d-1)/2} \int_{-\infty}^{\infty} |\phi'(z)| dz. \]

In view of (6), (9) and (10), we have
\[ \int_{\Omega_1} \left| \frac{\partial (F \circ \Phi)(x)}{\partial x_1} \right| \; dx = \int_{\Omega_1} \left| \sum_{j=1}^d \frac{\partial F}{\partial y_j}(\Phi(x)) \frac{\partial \Phi_j(x)}{\partial x_1} \right| \; dx \]
\[ \geq \int_{\Omega_1} \left| \frac{\partial F(\Phi(x))}{\partial y_1} \frac{\partial \Phi_1(x)}{\partial x_1} \right| \; dx - \int_{\Omega_1} \left| \sum_{j=2}^d \frac{\partial F(\Phi(x))}{\partial y_j} \frac{\partial \Phi_j(x)}{\partial x_1} \right| \; dx \]
\[ \geq \left| \frac{\partial \Phi_1(x)}{\partial x_1} \right| \left[ \gamma \int_{\Omega_1} \left| \frac{\partial F(y)}{\partial y_1} \right| \; dy - \frac{2}{\gamma} \int_{\Omega_1} \left| \sum_{j=2}^d \frac{\partial F(y)}{\partial y_j} \right| dy \right] \]
\[ = \left| \frac{\partial \Phi_1(x)}{\partial x_1} \right| \left( \frac{\gamma}{2} - \frac{2}{\gamma} \epsilon(d-1) \right) \int_{-\infty}^{\infty} |\phi'(z)|dz. \]

Thus, for \( \epsilon \) small
\[ \int_{\Omega_1} \left| \frac{\partial (F \circ \Phi)(x)}{\partial x_1} \right| \; dx \geq \frac{\gamma}{4} \left| \frac{\partial \Phi_1(x)}{\partial x_1} \right| \epsilon^{(d-1)/2} \int_{-\infty}^{\infty} |\phi'(z)|dz. \]

Estimates (13) and (14) imply
\[ \int_{\Omega_1} \left| \frac{\partial (F \circ \Phi)(x)}{\partial x_1} \right| \; dx \geq \frac{\gamma}{8} \left| \frac{\partial \Phi_1(x)}{\partial x_1} \right| ||F'||_{L^1(\Omega_2)}. \]

(5) follows from (8) and (15). \( \square \)
The next lemma completes the proof.

**Lemma 3.** If $\Phi \in C^1(\Omega_1 : \Omega_2)$ satisfies hypotheses (3) and (4) of Theorem 1, then there is a constant $C > 0$ so that for all $F \in BV(\Omega_2)$

$$\|(F \circ \Phi)′\|_{\text{Var}} \leq C \|F′\|_{\text{Var}}.$$  

**Proof.** The space of functions $H$ belonging to $BV(\Omega_j)$ (modulo the constants) is a Banach space normed by $\|H′\|_{\text{Var}}$. Using the Closed Graph Theorem, it suffices to verify that the map from $BV(\Omega_2)$ to $BV(\Omega_1)$ which sends $F$ to $F \circ \Phi$ has closed graph.

To that end, suppose that $F_n \to F$ in $BV(\Omega_2)$, and

$$F_n \circ \Phi \to G \text{ in } BV(\Omega_1).$$  

(16) 

It suffices to show that $G′ = (F \circ \Phi)′$ in the sense of distributions.

Choose representatives $\tilde{F}_n$ of $F_n$ and $\tilde{F}$ of $F$ so that,

$$\int_{\Omega_2} \tilde{F}_n \, dy = 0, \quad \int_{\Omega_2} \tilde{F} \, dy = 0.$$

This together with $BV$ convergence implies that

$$\tilde{F}_n \to \tilde{F} \text{ in } L^1(\Omega_2).$$  

(17) 

Since

$$|A| = |\Phi(\Phi^{-1}(A))| \geq \gamma|\Phi^{-1}(A)|,$$

one sees, starting with $g = \chi_A$, that the map sending $g$ to $g \circ \Phi$ is continuous from $L^1(\Omega_2)$ to $L^1(\Omega_1)$. Therefore,

$$\tilde{F}_n \circ \Phi \to \tilde{F} \circ \Phi \text{ in } L^1(\Omega_1).$$

Therefore

$$(\tilde{F}_n \circ \Phi)' \to (\tilde{F} \circ \Phi)' \text{ in the sense of distributions, } D'(\Omega_1).$$

On the other hand (16) implies that,

$$(\tilde{F}_n \circ \Phi)' \to G' \text{ in } D'(\Omega_1).$$

Therefore $(F \circ \Phi)' = G'$ which completes the proof. $\blacksquare$
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