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OPERATORS INDUCED BY PRIME NUMBERS*

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Abstract. In this paper, we study operator-theoretic and algebraic structures induced by prime numbers. The Adele ring $\mathbb{A}_{\mathbb{Q}}$ is an algebraic, topological, and measure-theoretic object induced by the p-adic number fields $\{\mathbb{Q}_p\}_{p:prime}$. By determining a Hilbert space $\mathcal{H}$ induced by $\mathbb{A}_{\mathbb{Q}}$, we study how prime numbers act on $\mathcal{H}$. In particular, we are interested in the case where they act like shift operators on $\mathcal{H}$. The main purpose of this paper is to understand the properties of such shift operators a_p induced by prime numbers p. We call them *prime multipliers*. We characterize the von Neumann algebra $vN(\Gamma)$ generated by $\{a_p\}_{p:prime}$. As spectral-theoretic application, we compute the free distributional data of self-adjoint elements $a_p + a_p^*$, for all primes p.

Key words. Adele ring, Hilbert spaces, von Neumann algebras, prime operators, free probability, free moments, free cumulants.

AMS subject classifications. 05E15, 11G15, 11R04, 11R09, 11R47, 11R56, 46L10, 46L40, 46L53, 46L54, 47L15, 47L30, 47L55.

1. Introduction. For the Adele ring $\mathbb{A}_{\mathbb{Q}}$ (e.g., [14]), we construct the corresponding von Neumann algebra $\mathfrak{A} = \overline{\mathbb{C}[\alpha(\mathbb{A}_{\mathbb{Q}})]}^{w}$ acting on the Hilbert space $\mathcal{H} = L^2(\mathbb{A}_{\mathbb{Q}})$, where α is kind of a shifting action. We are interested in the free distributional data of the operators of $\mathfrak{A}$ induced by *prime numbers* (in short, *primes*) on $\mathcal{H}$. Such data provide the operator-theoretic, in particular, spectral-theoretic, data of primes on operator-algebraic Adelic structures. To do that, we will use free probability theory.

1.1. Overview. *Noncommutative free probability* has been studied since mid 1980's. *Voiculescu* extend the classical probability theory to that on noncommutative operator algebraic structures (e.g., [11]). *Speicher* established the combinatorial free probability (e.g., see [12]).

If a given operator a in a (topological) $*$-algebra A, and if $\varphi : A \to \mathbb{C}$ is a fixed (bounded) linear functional, then the free-probabilistic information of a is determined by its *free moments*

$$\left\{ \varphi(a_1 a_2 ... a_n) \in \mathbb{C} \;\middle|\; \begin{array}{c} (a_1, ..., a_n) \in \{a, a^*\}^n, \\ \forall n \in \mathbb{N} \end{array} \right\},$$

where X^n simply means the *Cartesian product* of n-copies of X, for all $n \in \mathbb{N}$.

By the *Moebius inversion*, the *free cumulants*

$$\left\{ k_n(a_1, ..., a_n) \in \mathbb{C} \;\middle|\; \begin{array}{c} (a_1, ..., a_n) \in \{a,\ a^*\}^n \\ \forall n \in \mathbb{N} \end{array} \right\}$$

also represent the same (or equivalent) free-probabilistic information of a (See [12]).

If a is self-adjoint in A, in the sense that $a^* = a$, then the above free-probabilistic data of a represents the spectral information of a. i.e., the free probability measure is identified with the spectral measure of a (under self-adjointness) (e.g., [3], [4], [5], [11] and [12]). Thus, computing the free moments or free cumulants of a provides the spectral data of a.

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By Speicher, it is shown that the free cumulant computation characterizes the free structure in A. i.e., two elements a_1 and a_2 are free A, if and only if all mixed cumulants of $\{a_1, a_1^*\}$ and $\{a_2, a_2^*\}$ vanish in A (See Section 2 below).

In [1], [5], [6] and [7], we use free probability theory to investigate the structure theorems of certain *topological groupoid algebras*. *Groupoids* are algebraic structure with one binary operation having multi-units. For instance, all groups are groupoids with one unit. By characterizing the free blocks of such algebras, we restrict the study of groupoid algebras to the study of free blocks and certain combinatorial connections of those blocks. As application, the free stochastic calculus has been studied (See [2]).

By using the same techniques, we characterize how primes (as operators) act on the Hilbert space generated by the Adele ring. The main purpose of this paper is (i) to study the spectral information of the primes in $\mathfrak{A}$ via free probability, and (ii) to show the inner free-block structures of $\mathfrak{A}$, determined by primes. These will provide a bridge between operator theory and number theory.

1.2. Acknowledgment. After finishing writing this paper, the author realized that there have been attempts to investigate operator algebra theory with / by number-theoretic objects, results and techniques (e.g., [15], [16] and [17]). It is very reasonable but hard approaches by the very historical backgrounds between analysis and number theory, for instance, analytic number theory and L-function theory, etc (e.g., [8] and [9]). In particular, in [15] and [16], number-theoretic objects and results are used for studying type III-factors in von Neumann algebra theory. And, in the book [17], various mathematicians provided results and discussions about interconnection among number theory, theoretical physics, and geometry.

In [15] and [16], Bost and Connes considered Hecke algebras and type III-factors with help of number-theoretic objects and results. The main purposes of them are to study certain von Neumann algebras. Compared with them, the aim of this paper is to construct von Neumann algebraic models directly from number-theoretic objects, and then study number-theoretic problems in terms of operator-algebraic techniques, especially, those from free probability and dynamical systems under representation theory. Here, we concentrate on establishing the base stone, or the beginning steps of the study. In such senses, one can / may distinguish [15], [16] and this paper.

The author also would like to refer [17] to readers for studying connections between number theory and noncommutative-and-commutative operator-algebraic geometry.

2. Motivation and backgrounds. In this section, we introduce basic definitions and backgrounds of our theory. We first provide the motivation for our study in Section 2.1. In Section 2.2, we introduce the Adele ring $\mathbb{A}_{\mathbb{Q}}$. In Sections 2.3, we briefly review free probability theory, which provides main tools for our study.

2.1. Motivation. The readers can learn why we need to study Adelic structures and p-adic structures in mathematics and in other scientific fields, in particular, in physics, from [14]. To emphasize the importance of Adele-and-p-adic analysis, we put several paragraphs from [14]. In (2.1.1) and (2.1.2) below, the sentences in "." are directly from [14]. The main motivation of this paper is introduced in (2.1.3).

2.1.1. To study small worlds, we need non-Archimedean structures and corresponding tools.

"If Δx is an uncertainty in a length measurement, then the inequality

$$\Delta x \geq \sqrt{\frac{hG}{c^3}}$$

takes place. This inequality is stronger than the Heisenberg uncertainty principle. Here, h is the Planck constant, c is the velocity of light, and G is the gravitational constant. ... by virtue of the above inequality, a measurement of distances smaller than the Planck length is impossible. ... According to the Archimedean axiom, any given large segment on a straight line can be surpassed by successive addition of small segments along the same line. Really, this is a physical axiom which concern the process of measurement. ... But as we just discussed, the Planck length is the smallest possible distance that can in principle be measured. So, a suggestion emerges to abandon the Archimedean axiom at very small distance. This leads to a non-Euclidean and non-Riemannian geometry of space at small distances."

Thus, one may need new measuring system, other than the reals $\mathbb{R}$, satisfying non-Archimedean measures. The best possible examples are the p-adic numbers $\mathbb{Q}_p$, for primes p, and the Adele ring $\mathbb{A}_{\mathbb{Q}}$ (See below).

2.1.2. The study of Adelic structures is of great interest from pure mathematical point of view as well as from the physical (and hence scientific) one.

"As possible physical applications we note a consideration of models with non-Archimedean geometry of space-time at very small distances, and also in spectral theory of processes in complicated media. Furthermore, it seems to us that an extension of the formalism of quantum theory to the field of p-adic numbers is of great interest even independent of possible new physical applications because it can lead to better understanding of the formalism of usual quantum theory. ... the investigation of p-adic quantum theory and field theory will be also useful in number theory, representation theory, and p-adic analysis. ... No doubt investigation of p-adic nonlinear interacting systems will provide new deep mathematical results."

2.1.3. We do know how primes act or play their roles in $\mathbb{R}$ (which is an Archimedean structure) from classical number theory. It is natural to ask how primes act on an Adelic structure (which is a non-Archimedean structure). In particular, we are interested in how primes act on certain functions on Adelic structures. By considering a topological space induced by Adelic structure, we act primes on the space in terms of operator theory. And then consider the properties of such operators generated by primes via free probability.

2.2. The Adele Ring $\mathbb{A}_{\mathbb{Q}}$. *Fundamental theorem of arithmetic* says that every positive integer in $\mathbb{N}$ except 1 can be expressed as a usual multiplication of *primes* (or prime numbers), equivalently, all positive integers which are not 1 are *prime-factorized* under multiplication. i.e., the primes are the building blocks of all positive integers except for 1. And hence, all negative integers n, except -1, can be understood as products of -1 and prime-factorizations of $|n|$. Thus, primes are playing key roles in both classical and advanced *number theory.*

The Adele field $\mathbb{A}_{\mathbb{Q}}$ is one of the main topics in advanced number theory connected with other mathematical fields like *algebraic geometry* and *L-function theory*, etc.Throughout this paper, we denote the set of all natural numbers (which are positive integers) by $\mathbb{N}$, the set of all integers by $\mathbb{Z}$, and the set of all rational numbers by $\mathbb{Q}$.

Let's fix a prime p. Define *the p-norm* $|.|_p$ on the rational numbers $\mathbb{Q}$ by

$$|q|_p = \left|p^r \frac{a}{b}\right|_p \overset{def}{=} \frac{1}{p^r},$$

whenever $q = p^r \frac{a}{b} \in \mathbb{Q}^{\times} = \mathbb{Q} \setminus \{0\}$, for some $r \in \mathbb{Z}$, with an additional identity:

$$|0|_p \overset{def}{=} 0 \text{ (for all primes } p).$$

For example,

$$\left|-\frac{24}{5}\right|_2 = \left|2^3 \cdot (-\frac{3}{5})\right|_2 = \frac{1}{2^3} = \frac{1}{8},$$

$$\left|-\frac{24}{5}\right|_3 = \left|3^1 \cdot (-\frac{8}{5})\right| = \frac{1}{3^1} = \frac{1}{3},$$

and

$$\left|\frac{1}{24}\right|_2 = \left|2^{-3} \cdot 3^{-1}\right| = \frac{1}{2^{-3}} = 8,$$

$$\left|\frac{1}{24}\right|_3 = \left|3^{-1} \cdot \frac{1}{8}\right| = \frac{1}{3^{-1}} = 3.$$

It is easy to check that
(i) $|q|_p \geq 0$, for all $q \in \mathbb{Q}$,
(ii) $|q_1 q_2|_p = |q_1|_p \cdot |q_2|_p$, for all $q_1, q_2 \in \mathbb{Q}$
(iii) $|q_1 + q_2|_p \leq \max\{|q_1|_p, |q_2|_p\}$,
for all $q_1, q_2 \in \mathbb{Q}$. In particular, by (iii), we verify that
(iii)$'$ $|q_1 + q_2|_p \leq |q_1|_p + |q_2|_p$,
for all $q_1, q_2 \in \mathbb{Q}$. Thus, by (i), (ii) and (iii)$'$, the p-norm $|.|_p$ is indeed a norm. However, by (iii), this norm is "*non-Archimedean.*" So, the topological pair $(\mathbb{Q}_p, |.|_p)$ is a (non-Archimedean) normed space.

Definition 2.1. *We define a set $\mathbb{Q}_p$ by the norm-closure of the normed space $\mathbb{Q}_p$, for all primes p. We call $\mathbb{Q}_p$, the p-prime field (or the p-adic number field). We can check indeed $\mathbb{Q}_p$ is a field algebraically (See below).*

For a fixed prime p, all elements of the p-prime field $\mathbb{Q}_p$ are formed by

$$p^r \left(\sum_{k=0}^{\infty} a_k p^k\right), \text{ for } 0 \leq a_k < p, \tag{2.2.1}$$

for all $k \in \mathbb{N}$, and for all $r \in \mathbb{Z}$, where $a_k \in \mathbb{N}$. For example,

$$-1 = (p-1)p^0 + (p-1)p + (p-1)p^2 + \cdots.$$

The subset of $\mathbb{Q}_p$, consisting of all elements formed by

$$\sum_{k=0}^{\infty} a_k p^k, \text{ for } 0 \leq a_k < p \text{ in } \mathbb{N},$$

is denoted by $\mathbb{Z}_p$. i.e., for any $x \in \mathbb{Q}_p$, there exists $r \in \mathbb{Z}$, and $x_0 \in \mathbb{Z}_p$, such that

$$x = p^r x_0.$$

Notice that if $x \in \mathbb{Z}_p$, then $|x|_p \leq 1$, and vice versa. i.e.,

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1 = p^0\}. \tag{2.2.2}$$

So, sometimes, the subset $\mathbb{Z}_p$ of (2.2.2) is said to be the *unit disk of* $\mathbb{Q}_p$. Remark that

$$\mathbb{Z}_p \supset p\mathbb{Z}_p \supset p^2\mathbb{Z}_p \supset p^3\mathbb{Z}_p \supset \cdots,$$

moreover, if $x \in p^k\, \mathbb{Z}_p$, then $|x|_p \leq \frac{1}{p^k}$, for all $k \in \mathbb{N}$. It is not difficult to verify that

$$\mathbb{Z}_p \subset p^{-1}\mathbb{Z}_p \subset p^{-2}\mathbb{Z}_p \subset p^{-3}\mathbb{Z}_p \subset \cdots,$$

and hence

$$\mathbb{Q}_p = \bigcup_{k=-\infty}^{\infty} p^k\mathbb{Z}_p, \text{ set-theoretically.} \tag{2.2.3}$$

Consider the boundary U_p of $\mathbb{Z}_p$. By construction, the boundary U_p of $\mathbb{Z}_p$ is identical to

$$U_p = \mathbb{Z}_p \setminus p\mathbb{Z}_p = \{x \in \mathbb{Z}_p : |x|_p = 1 = p^0\}. \tag{2.2.4}$$

Similarly, the subsets p^kU_p are the boundary of $p^k\mathbb{Z}_p$ satisfying

$$p^kU_p = p^k\mathbb{Z}_p \setminus p^{k-1}\mathbb{Z}_p, \text{ for all } k \in \mathbb{Z}.$$

We call the subset U_p of $\mathbb{Q}_p$ in (2.2.4) the *unit circle of* $\mathbb{Q}_p$. And all elements of U_p are said to be *units of* $\mathbb{Q}_p$.

Therefore, by (2.2.3) and (2.2.4), we obtain that

$$\mathbb{Q}_p = \bigsqcup_{k=-\infty}^{\infty} p^kU_p, \text{ set-theoretically,} \tag{2.2.5}$$

where $\sqcup$ means the disjoint union. By [14], whenever $q \in \mathbb{Q}_p$ is given, there always exists

$$q \in a + p^k\mathbb{Z}_p, \text{ for } a,\ k \in \mathbb{Z}.$$

FACT (See [14]). *The p-prime field* $(\mathbb{Q}_p, |.|_p)$ *is a Banach space. And it is locally compact. In particular, the unit disk* $\mathbb{Z}_p$ *is compact in* $\mathbb{Q}_p$. □

Define now the addition on $\mathbb{Q}_p$ by

$$\left(\sum_{n=-N_1}^{\infty} a_np^n\right) + \left(\sum_{n=-N_2}^{\infty} b_np^n\right) = \sum_{n=-\max\{N_1,N_2\}}^{\infty} c_np^n, \tag{2.2.6}$$

for $N_1,\, N_2 \in \mathbb{N}$, where the summands c_np^n satisfies that

$$c_np^n \overset{def}{=} \begin{cases} (a_n + b_n)p^n & \text{if } a_n + b_n < p \\ p^{n+1} & \text{if } a_n + b_n = p \\ s_np^{n+1} + r_np^n & \text{if } a_n + b_n = s_np + r_n, \end{cases}$$

for all $n \in \{-\max\{N_1,\, N_2\},\, ...,\, 0,\, 1,\, 2,\, ...\}$. Clearly, if $N_1 > N_2$ (resp., $N_1 < N_2$), then, for all $j = -N_1, ..., -(N_1 - N_2 + 1)$, (resp., $j = -N_2, ..., -(N_2 - N_1 + 1)$),

$$c_j = a_j \text{ (resp., } c_j = b_j).$$

And define the multiplication on $\mathbb{Z}_p$ by

$$\left(\sum_{k_1=0}^{\infty} a_{k_1} p^{k_1}\right)\left(\sum_{k_2=0}^{\infty} b_{k_2} p^{k_2}\right) = \sum_{n=-N}^{\infty} c_n p^n, \tag{2.2.7}$$

where

$$c_n = \sum_{k_1+k_2=n} \left(r_{k_1,k_2} i_{k_1,k_2} + s_{k_1-1,k_2}\ i^c_{k_1-1,k_2} + s_{k_1,k_2-1} i^c_{k_1,k_2-1} + s_{k_1-1,k_2-1} i^c_{k_1-1,\ k_2-1}\right),$$

where

$$a_{k_1} b_{k_2} = s_{k_1,k_2} p + r_{k_1,k_2},$$

by the division algorithm, and

$$i_{k_1,k_2} = \begin{cases} 1 & \text{if } a_{k_1} b_{k_2} < p \\ 0 & \text{otherwise,} \end{cases}$$

and

$$i^c_{k_1,k_2} = 1 - i_{k_1,k_2},$$

for all $k_1,\, k_2 \in \mathbb{N}$, and hence, on $\mathbb{Q}_p$, the multiplication is extended to

$$\left(\sum_{k_1=-N_1}^{\infty} a_{k_1} p^{k_1}\right)\left(\sum_{k_2=-N_2}^{\infty} b_{k_2} p^{k_2}\right) \tag{2.2.7$'$}$$

$$= \left(p^{-N_1}\right)\left(p^{-N_2}\right)\left(\sum_{k_1=0}^{\infty} a_{k_1-N_1} p^{k_1}\right)\left(\sum_{k_2=0}^{\infty} b_{k_1-N_2} p^{k_2}\right).$$

Then, under the addition (2.2.6) and the multiplication (2.2.7)$'$, the algebraic triple $(\mathbb{Q}_p,\, +,\, \cdot)$ becomes a field, for all prime p. Thus the p-prime fields $\mathbb{Q}_p$ are algebraically fields.

FACT. *Every p-prime field $\mathbb{Q}_p$, with the binary operations* (2.2.6) *and* (2.2.7)$'$ *is a field.* □

Moreover, the Banach filed $\mathbb{Q}_p$ is also a (unbounded) Haar-measure space $(\mathbb{Q}_p,\, \sigma(\mathbb{Q}_p),\, \rho_p)$, for all primes p, where $\sigma(\mathbb{Q}_p)$ means the σ-algebra of $\mathbb{Q}_p$, consisting of all measurable subsets of $\mathbb{Q}_p$. Moreover, this measure ρ_p satisfies that

$$\begin{aligned} \rho_p\left(a + p^k \mathbb{Z}_p\right) &= \rho_p\left(p^k \mathbb{Z}_p\right) \\ &= \frac{1}{p^k} \\ &= \rho\left(p^k \mathbb{Z}_p^\times\right) = \rho\left(a + p^k \mathbb{Z}_p^\times\right), \end{aligned} \tag{2.2.8}$$

for all $a \in \mathbb{N}$, and $k \in \mathbb{Z}$. Also, one has

$$\begin{aligned} \rho_p(a+U_p) &= \rho_p\left(U_p\right) = \rho_p\left(\mathbb{Z}_p \ \setminus \ p\mathbb{Z}_p\right) \\ &= \rho_p\left(\mathbb{Z}_p\right) - \rho_p\left(p\mathbb{Z}_p\right) \\ &= 1 - \tfrac{1}{p}, \end{aligned}$$

for all $a \in \mathbb{N}$. Similarly, we obtain that

$$\rho_p\left(a + p^k U_p\right) = \rho\left(p^k U_p\right) = \frac{1}{p^k} - \frac{1}{p^{k+1}}, \tag{2.2.9}$$

for all $a \in \mathbb{N}$, and $k \in \mathbb{Z}$ (See Chapter IV of [14]).

FACT. *The Banach field $\mathbb{Q}_p$ is an unbounded Haar-measure space, where ρ_p satisfies* (2.2.8) *and* (2.2.9), *for all primes p.* □

The computations (2.2.8) and (2.2.9) are typically important for us, by (2.2.3) and (2.2.5) (See Section 3 below).

The above three facts show that $\mathbb{Q}_p$ is a unbounded Haar-measured, locally compact Banach field, for all primes p.

DEFINITION 2.2. *Let $\mathcal{P} = \{p : p \text{ is a prime}\} \cup \{\infty\}$. The Adele ring $\mathbb{A}_\mathbb{Q} = (\mathbb{A}_\mathbb{Q}, +, \cdot)$ is defined by the set*

$$\{(x_p)_{p\in\mathcal{P}} : x_p \in \mathbb{Q}_p, \ \textit{almost all } x_p \in U_p, x_\infty \in \mathbb{R}\}, \tag{2.2.10}$$

equipped with

$$(x_p)_p + (y_p)_p = (x_p + y_p)_p,$$

and

$$(x_p)_p(y_p)_p = (x_p y_p)_p,$$

for all $(x_p)_p$, $(y_p)_p \in \mathbb{A}_\mathbb{Q}$. The Adele ring $\mathbb{A}_\mathbb{Q}$ has its product topology of the usual topology on $\mathbb{R}$ and p-adic topologies of $\mathbb{Q}_p$, for all primes p. For convenience, we put $\mathbb{R} = \mathbb{Q}_\infty$.

Indeed, the algebraic structure $\mathbb{A}_\mathbb{Q}$ is a ring. Also, by construction, and under the product topology, the Adele ring $\mathbb{A}_\mathbb{Q}$ is also a locally compact Banach space equipped with the product measure. Set-theoretically,

$$\mathbb{A}_\mathbb{Q} = \prod_{p\in\mathcal{P}} \mathbb{Q}_p = \mathbb{R} \times \left(\prod_{p:prime} \mathbb{Q}_p\right),$$

by identifying $\mathbb{Q}_\infty = \mathbb{R}$. We identify $|.|_\infty = |.|$, where $|.|$ means the norm, the absolute value on $\mathbb{R} = \mathbb{Q}_\infty$.

The product measure ρ of the Adele ring $\mathbb{A}_\mathbb{Q}$ is

$$\rho = \underset{p\in\mathcal{P}}{\times} \rho_p,$$

by identifying $\rho_\infty = \rho_\mathbb{R}$, the usual distance-measure (induced by $|.|_\infty$) on $\mathbb{R}$.

FACT. *The Adele ring $\mathbb{A}_\mathbb{Q}$ is a unbounded-measured locally compact Banach ring.* □

2.3. Free probability. We refer to the readers [10] and [11], for more about free probability theory. In this section, we briefly introduce Speicher's combinatorial free probability.

The free probability is originally introduced by Voiculescu in analytic way (with combinatorial observation) (e.g., [29]). Since we will study certain group von Neumann algebras, we concentrate on the case where all given operator algebras are von Neumann algebras. Recall that *von Neumann algebras* are the weak $*$-topology closure of a $*$-algebra. Without loss of generality, a von Neumann algebra A generated by a subset S of a fixed operator algebra $B(H)$ is $\overline{\mathbb{C}[S]}^{w}$, containing the identity operator 1_H on H, where $\overline{Y}^{w}$ means the weak $*$-topology closure of $Y \subseteq B(H)$, where $B(H)$ is the operator algebra, consisting of all bounded (or equivalently, continuous) linear operators on a Hilbert space H.

Let $B \subset A$ be von Neumann algebras with $1_B = 1_A$ and assume that there exists a *conditional expectation* $E_B : A \to B$ satisfying that

(i) $E_B(b) = b$, for all $b \in B$,
(ii) $E_B(b\ a\ b') = b\ E_B(a)\ b'$, for all $b,\ b' \in B$ and $a \in A$,
(iii) E_B is bounded (or continuous), and
(iv) $E_B(a^*) = E_B(a)^*$, for all $a \in A$.

Then the pair $(A,\ E_B)$ is called a *B-valued (amalgamated) W^*-probability space (with amalgamation over B)*.

For any fixed B-valued random variables $a_1,\ ...,\ a_s$ in $(A,\ E_B)$, we can have the *B-valued free distributional data* of them;

∘ *$(i_1,\ ...,\ i_n)$-th B-valued joint $*$-moments*:

$$E_B\left(b_1 a_{i_1}^{r_1}\ b_2 a_{i_2}^{r_2}\ ...\ b_n\ a_{i_n}^{r_n}\right)$$

∘ *$(j_1,\ ...,\ j_m)$-th B-valued joint $*$-cumulants*:

$$k_m^B\left(b_1' a_{j_1}^{t_1},\ b_2' a_{j_2}^{t_2},\ ...,\ b_m' a_{j_m}^{t_m}\right),$$

which provide the equivalent B-valued free distributional data of $a_1,\ ...,\ a_s$, for all $(i_1,\ ...,\ i_n) \in \{1,\ ...,\ s\}^n$, $(j_1,\ ...,\ j_m) \in \{1,\ ...,\ s\}^m$, for all $n,\ m \in \mathbb{N}$, where $b_1,\ ...,\ b_n,\ b_1',\ ...,\ b_m' \in B$ are arbitrary and $r_1,\ ...,\ r_n,\ t_1,\ ...,\ t_m \in \{1,\ *\}$. By the *Moebius inversion*, indeed, they provide the same B-valued free distributional data of $a_1,\ ...,\ a_s$. i.e., they satisfy

$$E_B\left(b_1 a_{i_1}^{r_1}\ ...\ b_n a_{i_n}^{r_n}\right) = \sum_{\pi \in NC(n)} k_\pi^B\left(b_1 a_{i_1}^{r_1},\ ...,\ b_n a_{i_n}^{r_n}\right)$$

and

$$k_m^B\left(b_1' a_{j_1}^{r_1},\ ...,\ b_m' a_{j_m}^{t_m}\right) = \sum_{\theta \in NC(m)} E_{B:\theta}\left(b_1' a_{j_1}^{t_1}, ..., b_m' a_{j_m}^{t_m}\right) \mu(\theta, 1_m),$$

where $NC(k)$ is the *lattice of all noncrossing partitions over* $\{1,\ ...,\ k\}$, for $k \in \mathbb{N}$, and $k_\pi^B(...)$ and $E_{B:\theta}(...)$ are the *partition-depending cumulant* and the *partition-depending moment* and where μ is the *Moebius functional* in the *incidence algebra* I_2.

Recall that the *partial ordering on* $NC(k)$ is defined by

$$\pi \leq \theta \overset{def}{\Longleftrightarrow} \forall \text{ blocks } V \text{ in } \pi,\ \exists \text{ blocks } B \text{ in } \theta \text{ s.t. } V \subseteq B.$$

Under this partial ordering, $NC(k)$ is a lattice with its *maximal element* $1_k = \{(1, ..., k)\}$ and its *minimal element* $0_k = \{(1), (2), ..., (k)\}$. The notation $(...)$ inside partitions $\{...\}$ means the *blocks of the partitions*. For example, 1_k is the one-block partition and 0_k is the k-block partition, for $k \in \mathbb{N}$. Also, recall that the *incidence algebra* I_2 is the collection of all functionals

$$\xi : \cup_{k=1}^{\infty} (NC(k) \times NC(k)) \to \mathbb{C},$$

satisfying $\xi(\pi, \theta) = 0$, whenever $\pi > \theta$, with its usual function addition $(+)$ and its *convolution* $(*)$ defined by

$$\xi_1 * \xi_2(\pi, \theta) \overset{def}{=} \sum_{\pi \leq \sigma \leq \theta} \xi_1(\pi, \sigma)\xi_2(\sigma, \theta),$$

for all $\xi_1, \xi_2 \in I_2$. Then this algebra I_2 has the *zeta functional* ζ, defined by

$$\zeta(\pi, \theta) \overset{def}{=} \begin{cases} 1 & \text{if } \pi \leq \theta \\ 0 & \text{otherwise.} \end{cases}$$

The *Moebius functional* μ is the convolution-inverse of ζ in I_2. So, it satisfies

$$\mu(0_1, 1_1) = 1, \text{ and } \sum_{\pi \in NC(k)} \mu(\pi, 1_k) = 0,$$

for all $k \in \mathbb{N} \setminus \{1\}$, and

$$\mu(0_k, 1_k) = (-1)^{k-1} c_{k-1},$$

where $c_m \overset{def}{=} \frac{1}{m+1} \begin{pmatrix} 2m \\ m \end{pmatrix}$ is the *m-th Catalan number*, for all $m \in \mathbb{N}$.

The amalgamated freeness is characterized by the amalgamated $*$-cumulants. Let (A, E_B) be given as above. Two W^*-subalgebras A_1 and A_2 of A, having their common W^*-subalgebra B, are *free over B in (A, E_B)*, if and only if their mixed $*$-cumulants vanish. Two subsets X_1 and X_2 of A are *free over B in (A, E_B)*, if $vN(X_1, B)$ and $vN(X_2, B)$ are free over B in (A, E_B), where $vN(S_1, S_2)$ means the von Neumann algebra generated by S_1 and S_2. In particular, two B-valued random variable x_1 and x_2 are *free over B in (A, E_B)*, if $\{x_1\}$ and $\{x_2\}$ are free over B in (A, E_B).

Suppose two W^*-subalgebras A_1 and A_2 of A, containing their common W^*-subalgebra B, are free over B in (A, E_B). Then we can construct a W^*-subalgebra $vN(A_1, A_2)$ of A generated by A_1 and A_2. Such W^*-subalgebra of A is denoted by $A_1 *_B A_2$. If there exists a family $\{A_i : i \in I\}$ of W^*-subalgebras of A, containing their common W^*-subalgebra B, satisfying $A = \underset{i \in I}{*_B} A_i$, then we call A, the *B-valued free product algebra of* $\{A_i : i \in I\}$.

Assume now that the W^*-subalgebra B is $*$-isomorphic to $\mathbb{C} = \mathbb{C} \cdot 1_A$. Then the conditional expectation E_B becomes a linear functional on A. By φ, denote E_B. Then, for $a_1, ..., a_n \in (A, \varphi)$,

$$k_n(a_1, ..., a_n) = \sum_{\pi \in NC(n)} \varphi_\pi(a_1, ..., a_n)\mu(\pi, 1_n)$$

by the Moebius inversion

$$= \sum_{\pi \in NC(n)} \left(\underset{V \in \pi}{\Pi} \varphi_V(a_1, ..., a_n) \right) \mu(\pi, 1_n)$$

since the image of φ are in $\mathbb{C}$ (For example, if $\pi = \{(1, 3), (2), (4, 5)\}$ in $NC(5)$, then

$$\begin{aligned} \varphi_\pi(a_1, ..., a_5) &= \varphi(a_1 \varphi(a_2) a_3) \varphi(a_4 a_5) \\ &= \varphi(a_1 a_3) \varphi(a_2) \varphi(a_4 a_5). \end{aligned}$$

Remark here that, if φ is an arbitrary conditional expectation E_B, and if $B \neq \mathbb{C} \cdot 1_A$, then the above second equality does not hold in general.)

$$= \sum_{\pi \in NC(n)} \left(\underset{V \in \pi}{\Pi} \varphi_V(a_1, \ ..., \ a_n) \mu(0_{|V|}, \ 1_{|V|}) \right)$$

by the multiplicativity of μ. i.e., If (A, φ) is a $\mathbb{C}$-valued W^*-probability space, then

$$k_n(a_1, ..., a_n) = \sum_{\pi \in NC(n)} \left(\underset{V \in \pi}{\Pi} \varphi_V(a_1, ..., a_n) \mu\left(0_{|V|}, \ 1_{|V|}\right) \right),$$

for all $n \in \mathbb{N}$, where $a_1, ..., a_n \in (A, \varphi)$.

3. Adele von Neumann algebra $\mathfrak{A}$. In this section, we consider the von Neumann algebra $\mathfrak{A}$ generated by the Adele ring $\mathbb{A}_\mathbb{Q}$. Since the Adele ring $\mathbb{A}_\mathbb{Q}$ is a measure space, naturally, one can construct the L^2-space $\mathcal{H}$ induced by this measure space, consisting of all square integrable function, as a Hilbert space. i.e., let

$$\mathcal{H} = L^2(\mathbb{A}_\mathbb{Q}, \rho). \tag{3.1}$$

Notice that all elements g of $\mathcal{H}$ has its form

$$g = \sum_{S \in Supp(g)} t_S \chi_S, \text{ with } t_S \in \mathbb{C}, \tag{3.2}$$

where χ_S is the characteristic functions, where *Supp*(g) means the support of g,

$$Supp(g) \overset{def}{=} \{S \in \sigma(\mathbb{A}_\mathbb{Q}) : t_S \neq 0\}. \tag{3.3}$$

Now, define an action α of $\mathbb{A}_\mathbb{Q}$ acting on $\mathcal{H}$ by

$$\alpha : x \in \mathbb{A}_\mathbb{Q} \longmapsto \alpha(x) \overset{denote}{=} \alpha_x \in B(\mathcal{H}), \tag{3.4}$$

satisfying

$$\alpha_x(\chi_S) = \chi_{xS}, \text{ for all } S \in \sigma(\mathbb{A}_\mathbb{Q}),$$

where $xS = \{xs : s \in S\}$, for all $S \subseteq \mathbb{A}_\mathbb{Q}$, and where $B(\mathcal{H})$ means the operator algebra consisting of all (bounded linear) operators on the Hilbert space $\mathcal{H}$.

Definition 3.1. *The L^2-Hilbert space $\mathcal{H} = L^2(\mathbb{A}_\mathbb{Q}, \rho)$ of* (3.1) *is called the Adele Hilbert space. Let α be a action* (3.4) *of the Adele ring $\mathbb{A}_\mathbb{Q}$, acting on $\mathcal{H}$. Then the* *corresponding von Neumann algebra* $\mathfrak{A} = \overline{\mathbb{C}[\alpha(\mathbb{A}_{\mathbb{Q}})]}^{w}$ *induced by the Adele ring* $\mathbb{A}_{\mathbb{Q}}$ *is called the Adele von Neumann algebra.*

By the very definition, if a is an element of the Adele von Neumann algebra $\mathfrak{A}$, then it has its expression,

$$a = \sum_{x \in \mathbb{A}_{\mathbb{Q}}} t_x \alpha_x, \text{ with } t_x \in \mathbb{C},$$

where $x = (x_p)_p \in \mathbb{A}_{\mathbb{Q}}$.

Extend the action α of the Adele ring $\mathbb{A}_{\mathbb{Q}}$ acting on the Adele Hilbert space $\mathcal{H}$ to the representation of the von Neumann algebra $\mathfrak{A}$ acting on $\mathcal{H}$. Let's denote this extended action of $\mathfrak{A}$ on $\mathcal{H}$ by α, too. i.e.,

$$\alpha_a \left(\sum_{S \in \sigma(\mathbb{A}_{\mathbb{Q}})} r_S \; \chi_S \right) = \sum_{x \in \mathbb{A}_{\mathbb{Q}}} \sum_{S \in \sigma(\mathbb{A}_{\mathbb{Q}})} t_x r_S \chi_{xS}, \tag{3.5}$$

for all $a = \sum_{x \in \mathbb{A}_{\mathbb{Q}}} t_x \; \alpha_x \in \mathfrak{A}$, where

$$\alpha_x(\chi_S) = (x_p)_p S = (x_p)_p \left(\underset{p}{\Pi} S_p \right) = \underset{p}{\Pi} x_p S_p. \tag{3.6}$$

Definition 3.2. *The pair* $(\mathcal{H}, \alpha)$ *of the Adele Hilbert space* $\mathcal{H}$ *and the von Neumann algebra representation* α *of* (3.5) *of the Adele von Neumann algebra* $\mathfrak{A}$ *is called the canonical representation of* $\mathfrak{A}$ *acting on* $\mathcal{H}$.

Define now elements a_p of the Adele von Neumann algebra $\mathfrak{A}$ by

$$a_p = \alpha \left((1, \; 1, \; ..., \; 1, \underset{p\text{-th entry}}{p}, \; 1, \; 1, ...) \right) \in \mathfrak{A}, \tag{3.7}$$

for all primes p, where

$$(1, 1, ..., 1, p, 1, ...) \in \mathbb{A}_{\mathbb{Q}}.$$

Then this element a_p of $\mathfrak{A}$ satisfies that

$$\begin{aligned} a_p \left(\sum_{S \in \sigma(\mathbb{A}_{\mathbb{Q}})} t_S \chi_S \right) &= \sum_{S \in \sigma(\mathbb{A}_{\mathbb{Q}})} t_S \chi_{pS} \\ &= \sum_{S \in \sigma(\mathbb{A}_{\mathbb{Q}})} t_S \chi_{\left(\underset{q \in \mathcal{P}}{\Pi} S'_q \right)}, \end{aligned}$$

for all $\sum_{S \in \sigma(\mathbb{A}_{\mathbb{Q}})} t_S \; \chi_S \in \mathcal{H}$, where

$$S'_q = \begin{cases} S_q & \text{if } q \neq p \\ pS_p & \text{if } q = p, \end{cases}$$

for primes q. Notice here that if S is a measurable subset of $\mathbb{A}_{\mathbb{Q}}$, then S is of the form

$$S = \underset{q \in \mathcal{P}}{\Pi} S_q, \text{ for } S_q \in \sigma(\mathbb{Q}_q),$$

for all $q \in \mathcal{P}$, but for almost of all primes q, $S_q \subseteq \mathbb{Z}_q$ (by the very definition of the Adele ring $\mathbb{A}_{\mathbb{Q}}$).

DEFINITION 3.3. *The elements a_p of the Adele von Neumann algebra $\mathfrak{A}$ defined in* (3.7) *are called the p-prime multiplier, for all primes p. The family $\{a_p\}_{p:prime}$ in $\mathfrak{A}$ is called the prime multipliers.*

Remark that

$$e \overset{def}{=} (1, 1, 1, ...) \in \mathbb{A}_{\mathbb{Q}} \subset \mathfrak{A},$$

is the unity of $\mathbb{A}_{\mathbb{Q}}$ (and hence, the $\alpha(e) = \alpha_e$ is the identity operator in $\mathfrak{A}$).

Let a_p be the p-prime multiplier for a fixed prime p. Then the adjoint a_p^* of a_p is identified with $a_{p^{-1}}$, i.e.,

$$a_p^* = \alpha \left((1,\ 1,\ ...,\ 1,\ \underset{p\text{-th}}{p^{-1}},\ 1,\ 1,\ ...) \right), \tag{3.8}$$

for all primes p. Indeed, for any fixed $S_k = \underset{q \in \mathcal{P}}{\Pi} S_{q:k} \in \sigma(\mathbb{A}_{\mathbb{Q}})$, with $S_{q:k} \in \sigma(\mathbb{Q}_q)$ (for almost of all $S_{q:k} \subseteq \mathbb{Z}_q$), for $k = 1, 2$,

$$\begin{aligned} < a_p \chi_{S_1}, \chi_{S_2} > &= < p\chi_{S_1}, \chi_{S_2} > \\ &= < \chi_{pS_1}, \chi_{S_2} >= \int_{\mathbb{A}_{\mathbb{Q}}} \chi_{pS_1} \chi_{S_2} d\rho \\ &= \int_{\mathbb{A}_{\mathbb{Q}}} \chi_{pS_1 \cap S_2} d\rho = \rho \left(pS_1 \cap S_2 \right) \\ &= \rho \left(S_1 \cap \frac{1}{p} S_2 \right) = \int_{\mathbb{A}_{\mathbb{Q}}} \chi_{S_1 \cap \frac{1}{p} S_2} d\rho \\ &= \int_{\mathbb{A}_{\mathbb{Q}}} \chi_{S_1} \chi_{\frac{1}{p} S_2} d\rho =< \chi_{S_1}, \frac{1}{p} \chi_{S_2} >, \end{aligned}$$

and hence the adjoint a_p^* is identical to the right-hand side of (3.8). We denote the element which is represented by the right-hand side of (3.8) by $a_{p^{-1}}$ or $a_{\frac{1}{p}}$, for primes p. Then the relation (3.8) can be re-written by

$$a_p^* = a_{p^{-1}} \text{ in } \mathfrak{A}, \text{ for all primes } p.$$

Now, let p and q be arbitrarily fixed primes, and let $x \in \{p, p^{-1}\}$, and $y \in \{q, q^{-1}\}$. If $p \leq q$ in primes, then

$$\begin{aligned} a_x a_y &= \alpha \left((1,\ 1,\ ...,\ 1,\ \underset{p\text{-th}}{x},\ 1,\ 1,\ ...) \right) \alpha \left((1,\ 1,\ ...,\ 1,\ \underset{q\text{-th}}{y},\ 1,\ 1,\ ...) \right) \\ &= \begin{cases} \alpha \left(\left(1,\ ...,\ 1,\ \underset{p\text{-th}}{x},\ 1,\ ...,\ 1,\ \underset{q\text{-th}}{y},\ 1,\ ... \right) \right) & \text{if } p < q \\ \alpha \left((1,\ 1,\ ...,\ 1,\ \underset{p\text{-th}}{xy},\ 1,\ 1,\ ...) \right) & \text{if } p = q. \end{cases} \end{aligned} \tag{3.9}$$

If a_p is the p-prime multiplier in the Adele von Neumann algebra $\mathfrak{A}$, for a prime p, then

$$\begin{aligned} a_p^n &= (\alpha(1,\ 1,\ ...,\ 1,\ p,\ 1,\ 1,\ ...))^n \\ &= \alpha\left((1,\ 1,\ ...,\ 1,\ p^n,\ 1,\ 1,\ ...)\right), \end{aligned} \tag{3.10}$$

by (3.9), for all $n \in \mathbb{N}$. By a little abuse of notation, we denote the resulted operator

$$\alpha\left((1,\ 1,\ ...,\ 1,\ p^n,\ 1,\ 1,\ ...)\right)$$

of (3.10) by a_{p^n}, alternatively.

Define now new elements T_p in the Adele von Neumann algebra $\mathfrak{A}$ by

$$T_p = a_p + a_p^* = a_p + a_{p^{-1}},$$

for all prime p. Then T_p is a well-defined self-adjoint element of $\mathfrak{A}$.

DEFINITION 3.4. *The element $T_p = a_p + a_{p^{-1}}$ of the Adele von Neumann algebra $\mathfrak{A}$ induced by the p-prime multiplier a_p and its adjoint a_p^* is called the p-prime operator of $\mathfrak{A}$, for all prime p.*

Notice that

$$a_p a_{p^{-1}} = \alpha_e = 1_{\mathfrak{A}} = a_{p^{-1}} a_p, \tag{3.11}$$

the identity element of $\mathfrak{A}$. By (3.11) we have the following lemma.

LEMMA 3.1. *Every n-power of the p-prime multiplier a_p^n is unitary on the Adele Hilbert space $\mathcal{H}$, for all $n \in \mathbb{N}$, for all primes p.*

Proof. If $n = 1$, the p-prime multiplier a_p is unitary in $\mathfrak{A}$, by (3.11).

Now, assume that $n > 1$. Then, by (3.10), $a_p^n = a_{p^n}$, for all $n \in \mathbb{N}$. So, it is not difficult to check

$$(a_{p^n})^* = a_{p^{-n}},\ \text{for } n \in \mathbb{N}.$$

Therefore,

$$(a_p^n)^*(a_p^n) = (a_{p^n})^*(a_{p^n}) = a_{p^{-n}} a_{p^n} = a_{p^{-n}p^n}$$

by (3.7)

$$\begin{aligned} &= \alpha_e = 1_{\mathfrak{A}} \\ &= a_{p^n p^{-n}} = a_{p^n} a_{p^{-n}} = (a_{p^n})(a_{p^n})^* = (a_p^n)(a_p^n)^*. \end{aligned}$$

Therefore, a_p^n is unitary in $\mathfrak{A}$, for all $n > 1$. So, the elements a_p^n are unitaries in $\mathfrak{A}$, for all $n \in \mathbb{N}$, for primes p. ∎

By the above lemma, we obtain the following theorem.

THEOREM 3.2. *Let $\{a_p\}_{p:prime}$ be the prime multipliers in the Adele von Neumann algebra $\mathfrak{A}$. Then the subfamily $\{a_p,\ a_p^*\}$ generates a group Γ_p under operator multiplication on $\mathfrak{A}$, for each prime p. Moreover, the group Γ_p is group-isomorphic to the infinite cyclic abelian group $\mathbb{Z}$. i.e.,*

$$\Gamma_p \overset{Group}{=} \mathbb{Z}, \tag{3.12}$$

where "$\overset{Group}{=}$" *means "being group-isomorphic," for all primes* p.

Proof. By the previous lemma, all (powers) of p-prime multipliers a_p are unitaries, for all primes p, on the Adele Hilbert space $\mathcal{H}$. Therefore, their adjoints $a_p^* = a_{p^{-1}}$ are unitaries on $\mathcal{H}$, too, as inverses of a_p, for all primes p. So, the family $\{a_p, a_{p^{-1}}\}$ is a unitary family in the Adele von Neumann algebra $\mathfrak{A}$. Moreover, the set Γ_p

$$\Gamma_p \overset{def}{=} \{a_{p^n}, a_{p^{-n}} : n \in \mathbb{Z}\}$$

becomes a group, with identification

$$a_{p^0} = \alpha_e = 1_{\mathfrak{A}} = a_{p^{-0}},$$

under the operator multiplication on $\mathfrak{A}$. Indeed, the operator multiplication on $\mathfrak{A}$ is closed in Γ_p. i.e., whenever $a_{p^{n_1}}, a_{p^{n_2}} \in \Gamma_p$, for $n_1, n_2 \in \mathbb{Z}$,

$$a_{p^{n_1}} a_{p^{n_2}} = a_{p^{n_1}p^{n_2}} = a_{p^{n_1+n_2}} \in \Gamma_p,$$

too. Also, the operation is associative, and commutative. The operation-identity $\alpha_e = a_{p^0}$ exists in Γ_p;

$$a_{p^n} \alpha_e = a_{p^{n+0}} = a_{p^n}, \text{ for all } n \in \mathbb{Z}.$$

For $n \in \mathbb{Z} \setminus \{0\}$, there always exists $-n \in \mathbb{Z}$, such that

$$a_{p^n} a_{p^{-n}} = a_{p^{n-n}} = \alpha_e.$$

So, for any $a_{p^n} \in \Gamma_p \setminus \{\alpha_e\}$, there always exists $a_{p^{-n}} = a_p^*$, the operation-inverse.

Therefore, the set Γ_p, equipped with the operator multiplication, forms a group. For convenience, we denote this group $(\Gamma_p, \cdot)$ simply by Γ_p.

Define a map

$$\Phi : \Gamma_p \to \mathbb{Z}$$

by

$$\Phi(a_{p^n}) = n, \text{ for all } n \in \mathbb{Z},$$

where $\mathbb{Z} = (\mathbb{Z}, +)$ is the infinite cyclic abelian group, consisting of all integers. Then it is bijective, by the very definition of Γ_p. Moreover,

$$\Phi(a_{p^{n_1}} a_{p^{n_2}}) = \Phi\left(a_{p^{n_1+n_2}}\right) = n_1 + n_2 = \Phi(a_{p^{n_1}}) + \Phi(a_{p^{n_2}}),$$

for all $n_1, n_2 \in \mathbb{Z}$. Thus, the map Φ is a group-homomorphism. By the bijectivity of Φ, it is in fact a group-isomorphism. i.e.,

$$\Gamma_p \overset{\text{Group}}{=} \mathbb{Z}.$$

□

The above theorem shows that in the Adele von Neumann algebra $\mathfrak{A}$, each p-prime multiplier a_p, which is a unitary, generates a group Γ_p, under the operator multiplication on $\mathfrak{A}$, and the corresponding subgroup Γ_p of $\mathfrak{A}$ is group-isomorphic to $\mathbb{Z}$. It shows that there are infinite copies of embedded cyclic abelian group $\mathbb{Z}$ in $\mathfrak{A}$.

DEFINITION 3.5. *Let Γ_p be a group generated by the p-prime multiplier a_p, for all primes p. Then we call Γ_p, the p-prime subgroup of $\mathfrak{A}$, for all primes p. Sometimes, to emphasize the generator of the p-prime subgroup Γ_p, we write $\Gamma_p = < a_p >$, for all primes p.*

Now, notice that all prime subgroups $\Gamma_p = < a_p >$, for all primes p, share the group-identity α_e in the Adele von Neumann algebra $\mathfrak{A}$. So, one obtains the following corollary.

COROLLARY 3.3. *Let $\{\Gamma_p\}_{p:prime}$ be the family of all prime subgroups of $\mathfrak{A}$. Then the group Γ generated by $\{\Gamma_p\}_{p:prime}$, under the operator multiplication on the Adele von Neumann algebra $\mathfrak{A}$, is group-isomorphic to the free group F_∞ with countably infinite generators.*

Proof. Let Γ_p be p-prime subgroups of $\mathfrak{A}$, and let Γ be the group generated by $\{\Gamma_p\}_{p:prime}$, under the operator multiplication on $\mathfrak{A}$. It is true that, then Γ is the group generated by all p-prime multipliers $\{a_p\}_{p:prime}$, i.e.,

$$\Gamma =< \{a_p\}_{p:prime} >=< \{a_p, a_{p^{-1}}\}_{p:prime} > .$$

Let F_∞ be the free group with ∞-generators $\{g_n\}_{n=1}^{\infty}$, with their inverses $\{g_n^{-1}\}_{n=1}^{\infty}$. It is well-known that there are countably infinitely many primes. i.e., there exists a bijection from $\{p\}_{p:prime}$ onto $\mathbb{N}$. Say h.

Define now a map

$$\Psi : \Gamma \to F_\infty$$

satisfying

$$\Psi\left(a_p^k\right) \overset{def}{=} g_{h(p)}^k \text{ and } \Psi\left(a_p^{*k}\right) = g_{h(p)}^{-k},$$

for all $k \in \mathbb{N}$, and for all primes p, where $g_{h(p)}^{-k}$ means the group-inverse of $g_{h(p)}^k$.

Notice that if p, q are primes and k_1, $k_2 \in \mathbb{Z}$, then the operator $a_{p^{k_1}}\ a_{q^{k_2}}$ satisfies that

$$\begin{aligned}
&\left(a_{p^{k_1}} a_{q^{k_2}}\right)^* \left(a_{p^{k_1}} a_{q^{k_2}}\right) \\
&= a_{q^{k_2}}^* a_{p^{k_1}}^* a_{p^{k_1}} a_{q^{k_2}} = a_{q^{-k_2}} a_{p^{-k_1}} a_{p^{k_1}} a_{q^{k_2}} \\
&= a_{q^{-k_2}} \left(a_{p^{-k_1+k_1}}\right) a_{q^{k_2}} = a_{q^{-k_2+k_2}} = \alpha_e,
\end{aligned}$$

and hence $a_{p^{k_1}} a_{q^{k_2}}$ is a unitary in $\mathfrak{A}$, too. Thus, the products $\prod_{n=1}^{N} a_{p_n^{k_n}}$ are unitaries, too. i.e., all elements of Γ are unitaries on the Adele Hilbert space $\mathcal{H}$.

So, the map Ψ is a well-defined group-homomorphism from Γ to F_∞. Moreover, by the bijectivity of h, it is bijective, too. So, Ψ is a group-isomorphism. Therefore,

$$\Gamma \overset{\text{Group}}{=} F_\infty. \tag{3.13}$$

□

Let F_n be the free group with n-generators, for $n \in \mathbb{N} \cup \{\infty\}$. The group von Neumann algebra $L(F_n)$ acting on the group Hilbert space $L^2(F_n)$ is called the free group factor. By the above corollary, one can recognize that the Adele von Neumann algebra $\mathfrak{A}$ contains free group factors as its W^*-subalgebras.

Corollary 3.4. *Let $\mathfrak{A}$ be the Adele von Neumann algebra, and let $L(F_n)$ be a free group factor, for $n \in \mathbb{N} \setminus \{\infty\}$. Then the free group factor $L(F_n)$ is ($*$-isomorphic to) a W^*-subalgebra of $\mathfrak{A}$.*

Proof. Fix $n \in \mathbb{N} \cup \{\infty\}$, and take $X = \{p_1, ..., p_n\}$, a set of distinct n-primes. Then we have

$$\{a_{p_1}, ..., a_{p_n}\} \subset \mathfrak{A}.$$

Consider the subgroup Γ_X, generated by $\{a_{p_1}, ..., a_{p_n}\}$. By the above corollary,

$$\Gamma_X \overset{\text{Group}}{=} \underset{k=1}{\overset{n}{*}} \Gamma_{p_k} \overset{\text{Group}}{=} \underbrace{\mathbb{Z} * \;....\; * \;\mathbb{Z}}_{n\text{-times}} \overset{\text{Group}}{=} F_n.$$

Therefore, the W^*-subalgebra $vN(\Gamma_X)$ satisfies

$$vN(\Gamma_X) \overset{*\text{-iso}}{=} L(F_n) \text{ in } \mathfrak{A}. \tag{3.14}$$

□

The above corollary provides the motivation of this paper. i.e., our Adele von Neumann algebra $\mathfrak{A}$ has certain inner free structure, determined by primes. In Sections 4, 5, and 6, we consider such free structure of $\mathfrak{A}$ in detail.

It is well-known that, if Γ_1 and Γ_2 are groups and $\Gamma = \Gamma_1 * \Gamma_2$ is the (group-)free product group of Γ_1 and Γ_2, then the group von Neumann algebra $vN(\Gamma)$ is $*$-isomorphic to the (operator-algebraic-)free product von Neumann algebra $vN(\Gamma_1) *_{\mathbb{C}} vN(\Gamma_2)$ of the group von Neumann algebras $vN(\Gamma_1)$ and $vN(\Gamma_2)$. So, the free group factors $L(F_n)$ is $*$-isomorphic to

$$\underbrace{L(\mathbb{Z}) *_{\mathbb{C}} ... *_{\mathbb{C}} L(\mathbb{Z})}_{n\text{-times}},$$

for all $n \in \mathbb{N} \cup \{\infty\}$ (See [12], [1], [2], and [4]). As we have seen above, the Adele von Neumann algebra $\mathfrak{A}$ contains $L(F_n)$, for $n \in \mathbb{N} \cup \{\infty\}$. It shows that there does exist the free structures in $\mathfrak{A}$, induced by

$$L(\mathbb{Z}) \overset{*\text{-iso}}{=} vN\left(\Gamma_p\right), \text{ for all primes } p.$$

Now, we provide the following proposition which gives us a useful tool for studying free structure of $\mathfrak{A}$, in Sections 4, 5, and 6.

Proposition 3.5. *Let p be a prime, and let $T_p = a_p + a_p^*$ be the p-prime operator of the Adele von Neumann algebra $\mathfrak{A}$. Then*

$$T_p^n = (a_p + a_p^*)^n = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} a_{p^{2k-n}}, \tag{3.15}$$

for all $n \in \mathbb{N}$, where $\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{k!(n-k)!}$, for all $k \leq n \in \mathbb{N}$.

Proof. The proof of (3.13) is straightforward, by the binomial expansion. In fact, it is enough to show that

$$a_{p^n} a_{p^m} = a_{p^m} a_{p^n}, \tag{3.16}$$

for all n, $m \in \mathbb{Z}$, with identification $a_{p^0} = \alpha_e = 1_{\mathfrak{A}}$. Since

$$p^n p^m = p^{n+m} = p^{m+n} = p^m p^n \text{ in } \mathbb{Q}_p,$$

for all primes p, the relation (3.14) holds true. So,

$$T_p^n = (a_p + a_{p^{-1}})^n = \sum_{k=0}^{n} \binom{n}{k} a_{p^k} a_{p^{-(n-k)}}.$$

□

Definition 3.6. *Let T_p be the p-prime operator in the Adele von Neumann algebra $\mathfrak{A}$. The element*

$$T_o \overset{def}{=} \sum_{p:prime} T_p$$

of $\mathfrak{A}$ is called the prime radial operator of the Adele von Neumann algebra $\mathfrak{A}$.Let X be a subset of the set of all primes. Then the X-radial operator T_X is defined by

$$T_X \overset{def}{=} \sum_{p \in X} T_p.$$

4. Free Moments of Prime Operators. Let $\mathbb{A}_{\mathbb{Q}}$ be the Adele ring and let $\mathfrak{A}$ be the Adele von Neumann algebra $\overline{\mathbb{C}[\alpha(\mathbb{A}_{\mathbb{Q}})]}^{w}$. And let

$$\{a_p = \alpha\left((1,\ 1,\ ...,\ 1,\ p,\ 1,\ 1,\ ...)\right)\}_{p:prime}$$

be the prime multipliers in the sense of (3.5), which are the unitaries on the Adele Hilbert space $\mathcal{H} = L^2(\mathbb{A}_{\mathbb{Q}},\ \rho)$, and

$$\{T_p = a_p + a_{p^{-1}}\}_{p:prime},$$

the prime operators, which are self-adjoint operators on $\mathcal{H}$. Also, let

$$T_o = \sum_{p:prime} T_p$$

be the prime radial operator of $\mathfrak{A}$.

Define now the bounded (or continuous) linear functional φ on $\mathfrak{A}$ by

$$\varphi\left(\sum_{x \in \mathbb{A}_{\mathbb{Q}}} t_x\ \alpha_x\right) \overset{def}{=} t_e, \tag{4.1}$$

where $e = (1, 1, ...)$ is the unity of the Adele ring $\mathbb{A}_{\mathbb{Q}}$. Then it is a well-defined linear functional on $\mathfrak{A}$. Indeed,

$$\begin{aligned}\varphi\left(\sum_{x \in \mathbb{A}_{\mathbb{Q}}} t_x\ \alpha_x\ +\ \sum_{y \in \mathbb{A}_{\mathbb{Q}}} s_y\ \alpha_y\right) &= \varphi\left(\sum_{x \in \mathbb{A}_{\mathbb{Q}}} (t_x\ + s_x)\ \alpha_x\right)\\ &= t_e + s_e\\ &= \varphi\left(\sum_{x \in \mathbb{A}_{\mathbb{Q}}} t_x\ \alpha_x\right) + \varphi\left(\sum_{y \in \mathbb{A}_{\mathbb{Q}}} s_y\ \alpha_y\right),\end{aligned}$$

and

$$\varphi\left(t\left(\sum_{x\in\mathbb{A}_{\mathbb{Q}}} t_x\ \alpha_x\right)\right) = \varphi\left(\sum_{x\in\mathbb{A}_{\mathbb{Q}}} tt_x\ \alpha\right) = tt_e = t\varphi\left(\sum_{X\in\mathbb{A}_{\mathbb{Q}}} t_x\ \alpha_x\right),$$

for all $\sum_{x\in\mathbb{A}_{\mathbb{Q}}} t_x\ \alpha_x,\ \sum_{y\in\mathbb{A}_{\mathbb{Q}}} s_y\ \alpha_y \in \mathfrak{A}$. And

$$\varphi\left(\left(\sum_{X\in\mathbb{A}_{\mathbb{Q}}} t_x\ \alpha_x\right)^*\right) = \varphi\left(\sum_{x\in\mathbb{A}_{\mathbb{Q}}} \overline{t_x}\ \alpha_{x^{-1}}\right),$$

where

$$x^{-1} = ((x_p)_p)^{-1} = (x_p^{-1})_p \text{ in } \mathbb{A}_{\mathbb{Q}},$$

and

$$\alpha_x^* = \alpha_{x^{-1}},$$

for all $x = (x_p)_p \in \mathbb{A}_{\mathbb{Q}} = \overline{t_e} = \left(\varphi\left(\sum_{x\in\mathbb{A}_{\mathbb{Q}}} t_x\ \alpha_x\right)\right)^*$, for all $\sum_{x\in\mathbb{A}_{\mathbb{Q}}} t_x\ \alpha_x \in \mathfrak{A}$. Moreover, it is bounded by the very definition (4.1). So, the pair $(\mathfrak{A},\ \varphi)$ is a well-defined ($\mathbb{C}$-valued) W^*-probability space in the sense of [12].

DEFINITION 4.1. *Let $\mathfrak{A}$ be the Adele von Neumann algebra and let φ be a bounded linear functional on $\mathfrak{A}$, introduced in* (4.1)*. Then the W^*-probability space $(\mathfrak{A},\ \varphi)$ is called the Adele W^*-probability space.*

So, all elements of the Adele von Neumann algebra $\mathfrak{A}$ are understood as free random variables in the Adele W^*-probability space $(\mathfrak{A},\ \varphi)$. In this section, we are interested in the free distributional data of the prime operators $\{T_p\}_{p:prime}$ and the prime radial operator T_o.

By (3.13), if $T_p = a_p + a_{p^{-1}}$ is the p-prime operator in $\mathfrak{A}$, then

$$T_p^n = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} a_{p^{2k-n}}, \tag{4.2}$$

for all $n \in \mathbb{N}$, and for all prime p.

THEOREM 4.1. *For a fixed prime p, let $T_p = a_p + a_p^*$ be the p-prime operator in the Adele von Neumann algebra $\mathfrak{A}$, where a_p is the p-prime multiplier and $a_p^* = a_{p^{-1}}$ is the adjoint of a_p. Then*

$$\varphi(T_p^n) = \begin{cases} \begin{pmatrix} n \\ \frac{n}{2} \end{pmatrix} = (n!)\left(\frac{n}{2}!\right)^{-2} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd,} \end{cases} \tag{4.3}$$

for all $n \in \mathbb{N}$. Therefore, the p-prime operator T_p has its free distributional data $\left\{\begin{pmatrix} 2n \\ n \end{pmatrix}\right\}_{n=1}^{\infty}$.

Proof. By (4.2), we have that

$$T_p^n = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} a_{p^{2k-n}}, \text{ for all } n \in \mathbb{N}.$$

Thus,

$$\varphi(T_p^n) = \begin{cases} \begin{pmatrix} n \\ k \end{pmatrix} = \begin{pmatrix} 2k \\ k \end{pmatrix} = \begin{pmatrix} n \\ \frac{n}{2} \end{pmatrix} & \text{if } n = 2k \\ 0 & \text{otherwise,} \end{cases}$$

for all $n \in \mathbb{N}$. □

By (4.3), we have

$$\varphi(T_p^{2n}) = \begin{pmatrix} 2n \\ n \end{pmatrix} = \frac{(2n)!}{(n!)^2}, \text{ for all } n \in \mathbb{N},$$

and

$$\varphi(T_p^{2n-1}) = 0, \text{ for all } n \in \mathbb{N}.$$

Definition 4.2. *Let (A, ψ) be an arbitrary W^*-probability space, and let a_1, $a_2 \in (A, \psi)$ be free random variables. Assume more that two free random variables a_1 and a_2 are self-adjoint in the von Neumann algebra A. These two free random variables a_1 and a_2 are identically (free-)distributed, if*

$$\psi(a_1^n) = \psi(a_2^n), \text{ for all } n \in \mathbb{N}.$$

i.e., the free random variables a_1 and a_2 have equivalent free-moment data $\{\psi(a_1^n)\}_{n=1}^{\infty}$ and $\{\psi(a_2^n)\}_{n=1}^{\infty}$.

By (4.3), we obtain the following corollary.

Corollary 4.2. *Let $(\mathfrak{A}, \varphi)$ be the Adele W^*-probability space and let $\{T_p\}_{p:prime}$ be the prime operators in the Adele von Neumann algebra $\mathfrak{A}$. Then they are identically distributed.*

Proof. By (4.3), for all prime p,

$$\varphi(T_p^{2n}) = \begin{pmatrix} 2n \\ n \end{pmatrix}, \text{ and } \varphi(T_p^{2n-1}) = 0,$$

for all $n \in \mathbb{N}$. Therefore, the p-prime operators T_p's are all identically distributed. □

Now, let $p_1, ..., p_k$ be primes, for some $k \in \mathbb{N}$. For a mixed n-tuple $(p_{i_1}, ..., p_{i_n})$ of $\{p_1, ..., p_k\}$, we can compute the mixed free moments

$$\varphi\left(T_{p_{i_1}} T_{p_{i_2}} \dots T_{p_{i_n}}\right).$$

Before computing the mixed free moments of $T_{p_1}, ..., T_{p_k}$ in the Adele W^*-probability space $(\mathfrak{A}, \varphi)$, we consider the following lemma, first.

LEMMA 4.3. *Let p, q be distinct prime, and let T_p and T_q be corresponding prime operators in $(\mathfrak{A}, \varphi)$. Then*

$$T_p T_q = T_q T_p. \tag{4.4}$$

Proof. Let T_p and T_q be prime operators in the Adele W^*-probability space $(\mathfrak{A}, \varphi)$. Then

$$\begin{aligned} T_p T_q &= \left(a_p + a_{p^{-1}}\right)\left(a_q + a_{q^{-1}}\right) \\ &= a_p a_q + a_p a_{q^{-1}} + a_{p^{-1}} a_q + a_{p^{-1}} a_{q^{-1}} \\ &= \alpha\left(X_p\right)\alpha(X_q) + \alpha(X_p)\alpha(X_{q^{-1}}) + \alpha(X_{p^{-1}})\alpha(X_q) + \alpha(X_{p^{-1}})\alpha(X_{q^{-1}}) \end{aligned}$$

by the definition of prime multipliers, where

$$X_r = (1, ..., 1, \underset{r\text{-th}}{r}, 1, 1, ...), \text{ for all primes } r$$

in $\mathbb{A}_{\mathbb{Q}}$

$$= \alpha(X_p X_q) + \alpha(X_p X_{q^{-1}}) + \alpha(X_{p^{-1}} X_q) + \alpha(X_{p^{-1}} X_{q^{-1}}) \tag{4.5}$$

since α is an action.

Assume that $p < q$ in primes. Then the formula (4.5) becomes that:

$$\begin{aligned} &\alpha\left([1,\ ...,\ 1,\ p,\ 1,\ ...,\ 1,\ q,\ 1,\ 1,\ ...]\right) \\ &\qquad + \alpha\left([1,\ ...,\ 1,\ p,\ 1,\ ...,\ 1,\ q^{-1},\ 1,\ 1,\ ...]\right) \\ &\qquad + \alpha\left([1,\ ...,\ 1,\ p^{-1},\ 1,\ ...,\ 1,\ q,\ 1,\ 1,\ ...]\right) \\ &\qquad + \alpha\left([1,\ ...,\ 1,\ p^{-1},\ 1,\ ...,\ 1,\ q^{-1},\ 1,\ 1,\ ...]\right) \\ &= \alpha\left(X_q X_p\right) + \alpha\left(X_{q^{-1}} X_p\right) + \alpha\left(X_q X_{p^{-1}}\right) + \alpha\left(X_{q^{-1}} X_{p^{-1}}\right) \\ &= a_q a_p + a_{q^{-1}} a_p + a_q a_{p^{-1}} + a_{q^{-1}} a_{p^{-1}} \\ &= \left(a_q + a_{q^{-1}}\right)\left(a_p + a_{p^{-1}}\right) \\ &= T_q T_p. \end{aligned}$$

Therefore, we obtain that

$$T_p T_q = T_q T_p.$$

□

By the commutativity of $\{T_p\}_{p:prime}$, as in (4.5), one can verify that

$$T_p^n T_q^m = T_q^m T_p^n, \tag{4.6}$$

for all n, $m \in \mathbb{N}$, and for all primes p, q. Therefore, we obtain the following theorem.

THEOREM 4.4. *Let p_1, ..., p_k be arbitrarily fixed primes, for some $k \in \mathbb{N} \setminus \{1\}$. Let $(p_{i_1}, ..., p_{i_n})$ be the mixed n-tuple of $\{p_1, ..., p_k\}$, for $n \in \mathbb{N} \setminus \{1\}$, and assume that $\#(p_j)$ means the number of p_j's in the given n-tuple $(p_{i_1}, ..., p_{i_n})$ in $\mathbb{N} \cup \{0\}$, for $j = 1, ..., k$. Then we obtain that: if $\#(p_j)$ are even or 0, for all $n \in \mathbb{N}$, and if there exists at least one j such that $\#(p_j) \neq 0$, then*

$$\begin{aligned} \varphi\left(T_{p_{i_1}} \ ...\ T_{p_{i_n}}\right) &= \varphi\left(T_{p_1}^{\#(p_1)}\ T_{p_2}^{\#(p_2)}\ ...\ T_{p_k}^{\#(p_k)}\right) \\ &= \prod_{j=1}^{k} \begin{bmatrix} \#(p_j) \\ \\ \frac{\#(p_j)}{2} \end{bmatrix}, \end{aligned} \tag{4.7}$$

where

$$\left[\begin{array}{c} \#(p_j) \\ \\ \frac{\#(p_j)}{2} \end{array}\right] \overset{def}{=} \begin{cases} \left(\begin{array}{c} \#(p_j) \\ \\ \frac{\#(p_j)}{2} \end{array}\right) & \text{if } \#(p_j) \in 2\mathbb{N} \\ \\ 0 & \text{otherwise,} \end{cases} \tag{4.8}$$

for $j = 1, ..., k$*. By (4.7), if* n *is odd in* $\mathbb{N}$*, then*

$$\varphi\left(T_{p_{i_1}} \ ... \ T_{p_{i_n}}\right) = 0.$$

Proof. Let $(p_{i_1}, ..., p_{i_n})$ be the mixed n-tuple of $\{p_1, ..., p_k\}$, for some $k \in \mathbb{N} \setminus \{1\}$, for all $n \in \mathbb{N} \setminus \{1\}$. Assume first that there exists at least one nonzero $\#(p_j)$, and suppose all $\#(p_1)$, ..., $\#(p_k)$ in $(p_{i_1}, ..., p_{i_n})$ are either even or 0. Then

$$\varphi\left(T_{p_{i_1}} \ ... \ T_{p_{i_n}}\right) = \varphi\left(T_{p_1}^{\#(p_1)} \ ... \ T_{p_k}^{\#(p_k)}\right)$$

by (4.6)

$$= \varphi\left(\prod_{j=1}^{k}\left(\left[\begin{array}{c} \#(p_j) \\ \\ \frac{\#(p_j)}{2} \end{array}\right]\alpha_e + \sum_{k=1}^{\#(p_{i_j})}\left(\begin{array}{c} \#(p_{i_j}) \\ k \end{array}\right) a_{p_{i_j}^{2k-\#(p_{i_j})}}\right)\right)$$

$$= \varphi\left(\prod_{j=1}^{k}\left(\left[\begin{array}{c} \#(p_j) \\ \\ \frac{\#(p_j)}{2} \end{array}\right]\alpha_e\right) + \sum\right)$$

where

$$\sum \overset{denote}{=} \prod_{j=1}^{k}\left(\sum_{k=1}^{\#(p_j)}\left(\begin{array}{c} \#(p_j) \\ k \end{array}\right) a_{p_j^{2k-\#(p_j)}}\right)$$

then

$$= \varphi\left(\prod_{j=1}^{k}\left(\left[\begin{array}{c} \#(p_j) \\ \\ \frac{\#(p_j)}{2} \end{array}\right]\alpha_e\right)\right) + 0$$

since $a_{p^{n_1}} \ a_{q^{n_2}}$ does not becomes α_e, whenever $p \neq q$ in primes

$$= \prod_{j=1}^{k}\left[\begin{array}{c} \#(p_j) \\ \\ \frac{\#(p_j)}{2} \end{array}\right].$$

where

$$\left[\begin{array}{c} \#(p_j) \\ \\ \frac{\#(p_j)}{2} \end{array}\right] \overset{def}{=} \left(\begin{array}{c} \#(p_j) \\ \\ \frac{\#(p_j)}{2} \end{array}\right), \text{ whenever } \#(p_j) \in 2\mathbb{N},$$

and it becomes 0, otherwise, for all $j = 1, ..., n$.

Suppose now n is odd in $\mathbb{N}$. Then there exists at least one j in $\{1, ..., k\}$, such that $\#(p_j)$ is odd. Therefore,

$$T_{p_{i_1}} ... T_{p_{i_n}} = T_{p_1}^{\#(p_1)} ... T_{p_k}^{\#(p_k)}$$

does not contain the α_e-terms, and hence

$$\varphi\left(T_{p_{i_1}} \,...\, T_{p_{i_n}}\right) = 0.$$

$\square$

The above theorem characterizes the mixed moment computations of prime operators.

EXAMPLE 4.1. *(1) Let* $(p,\, q,\, q,\, q,\, q,\, p)$ *be a mixed* 6*-tuple of fixed distinct primes* $\{p,\, q\}$*. Then*

$$\begin{aligned}\varphi\left(T_p T_q T_q T_q T_q T_p\right) &= \varphi(T_p^2 T_q^4) \\ &= \varphi\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix}\alpha_e\right) \\ &= \begin{pmatrix} 2 \\ 1 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix} = 12.\end{aligned}$$

(2) Let $(p,\, q,\, q,\, p,\, q)$ *be a mixed* 5*-tuple of fixed distinct primes* $\{p,\, q\}$*. By the above theorem, one can verify that*

$$\varphi\left(T_p T_q T_q T_p T_q\right) = 0,$$

since 5 *is odd in* $\mathbb{N}$.

(3) Let $(p,\, p,\, q,\, q,\, q,\, p,\, q,\, q)$ *be a mixed* 8*-tuple of fixed distinct primes* $\{p,\, q\}$*. Then*

$$\varphi\left(T_p T_p T_q T_q T_q T_p T_q T_q\right) = \varphi\left(T_p^3 T_q^5\right) = 0,$$

by (4.8).

(4) Let $(p,\, q,\, p,\, p,\, r,\, q,\, p,\, r)$ *be a mixed* 8*-tuple of fixed distinct primes* $\{p,\, q,\, r\}$*. Then*

$$\begin{aligned}\varphi\left(T_p T_q T_p T_p T_r T_q T_p T_r\right) &= \varphi\left(T_p^4 T_q^2 T_r^2\right) \\ &= \begin{pmatrix} 4 \\ 2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ &= 24.\end{aligned}$$

5. Freeness of prime operators. In this section, we also let $\mathbb{A}_{\mathbb{Q}}$ be the Adele ring, a locally compact measured topological ring, and $(\mathfrak{A},\, \varphi)$, the Adele W^*-probability space. And let a_p be the p-prime multipliers and $T_p = a_p + a_p^*$, the p-prime operators, for all prime p. In Section 4, we computed the free moments and mixed free moments of p-prime operators T_p. In particular, we obtain that the computation (4.3):

$$\varphi\left(T_p^{2n}\right) = \begin{pmatrix} 2n \\ n \end{pmatrix}, \text{ and } \varphi\left(T_p^{2n-1}\right) = 0,$$

for all $n \in \mathbb{N}$, for all primes p.

Moreover, for any mixed n-tuple $(p_{i_1},\, ...,\, p_{i_n})$ of a fixed family $\{p_1,\, ...,\, p_k\}$ of distinct primes, the computation (4.7) shows that

$$\varphi\left(T_{p_{i_1}} \,...\, T_{p_{i_n}}\right) = \prod_{j=1}^{k} \left[\begin{array}{c} \#(p_j) \\ \\ \frac{\#(p_j)}{2} \end{array}\right],$$

whenever n is even in $\mathbb{N}$, where

$$\left[\begin{array}{c} k \\ \\ \frac{k}{2} \end{array}\right] = \begin{cases} \begin{pmatrix} k \\ \\ \frac{k}{2} \end{pmatrix} & \text{if } k \text{ is even} \\ \\ 0 & \text{if } k \text{ is odd,} \end{cases}$$

and it vanishes whenever n is odd in $\mathbb{N}$.

In this section, we will compute the equivalent free distributional data determined by free cumulants. By the Moebius inversion, for any $a \in (\mathfrak{A},\ \varphi)$, we have that

$$k_n(a, ..., a) = \sum_{\pi \in NC(n)} \left(\underset{V \in \pi}{\Pi}\ \varphi(a^{|V|})\ \mu\left(0_{|V|},\ 1_{|V|}\right)\right), \tag{5.1}$$

for all $n \in \mathbb{N}$ (See Section 2.2 or [12]).

THEOREM 5.1. *Let $T_p = a_p + a_{p^{-1}}$ be the p-prime operator in $\mathfrak{A}$, for a fixed prime p. Then*

$$k_n\left(T_p,\ ...,\ T_p\right) = \sum_{\pi \in NCE(n)} \left(\underset{V \in \pi}{\Pi} \left(\frac{|V|!}{\left(\frac{|V|}{2}!\right)^2} \right) \mu\left(0_{|V|},\ 1_{|V|}\right) \right), \tag{5.2}$$

for all $n \in \mathbb{N}$, where

$$NCE(n) \overset{def}{=} \{\pi \in NC(n) : |V| \in 2\mathbb{N}, \forall V \in \pi\}. \tag{5.3}$$

We call the subset $NCE(n)$, the even noncrossing partition set. In particular, if $NCE(n)$ is empty in the noncrossing partition set $NC(n)$, then

$$k_n(T_p, ..., T_p) = 0, \ \textit{whenever } n \textit{ is odd.}$$

Especially, if n is odd, then the free cumulant vanishes.

Proof. For $n \in \mathbb{N}$, we have that

$$k_n(T_p, ..., T_p) = \sum_{\pi \in NC(n)} \left(\underset{V \in \pi}{\Pi} \varphi\left(T_p^{|V|}\right)\ \mu\left(0_{|V|},\ 1_{|V|}\right)\right)$$

by (5.1)

$$= \sum_{\pi \in NC(n)} \left(\underset{V \in \pi,\, |V| \in 2\mathbb{N}}{\Pi}\ \varphi\left(T_p^{|V|}\right) \mu\left(0_{|V|},\ 1_{|V|}\right)\right)$$

by (4.3)

$$= \sum_{\pi \in NCE(n)} \left(\underset{V \in \pi}{\Pi}\ \varphi\left(T_p^{|V|}\right)\ \mu\left(0_{|V|},\ 1_{|V|}\right)\right), \tag{5.4}$$

by the definition of the even noncrossing partition set $NCE(n)$. Therefore, we obtain the desired computation (5.2).

It is clear that, if the subset $NCE(n)$ of $NC(n)$ is empty, then the formula (5.4) vanishes, and hence the free cumulant vanishes. As a special case, assume now that n is odd. Then $NCE(n)$ is empty. Indeed, for any $\pi \in NC(n)$, there always exists at least one block $V_o \in \pi$, such that $|V_o|$ is odd in $\mathbb{N}$. Therefore, for all $\pi \in NC(n)$, the corresponding block-depending summand

$$\underset{V\in\pi}{\Pi}\varphi\left(T_p^{|V|}\right)\mu\left(0_{|V|},\ 1_{|V|}\right)$$

of $k_n(T_p, ..., T_p)$ satisfies that:

$$\begin{aligned}&\underset{V\in\pi}{\Pi}\varphi\left(T_p^{|V|}\right)\mu\left(0_{|V|},\ 1_{|V|}\right)\\&=\left(\varphi\left(T_p^{|V_o|}\right)\mu\left(0_{|V_o|},\ 1_{|V_o|}\right)\right)\cdot\left(\underset{V\in\pi,\ V\neq V_o}{\Pi}\varphi\left(T_p^{|V|}\right)\mu\left(0_{|V|},\ 1_{|V|}\right)\right)\\&=0,\end{aligned}$$

and hence, if n is odd in $\mathbb{N}$, then

$$k_n(T_p, ..., T_p) = 0.$$

Therefore, only if n is even in $\mathbb{N}$,

$$k_n(T_p, ..., T_p) = \sum_{\pi\in NCE(n)}\left(\underset{V\in\pi}{\Pi}\varphi\left(T_p^{|V|}\right)\ \mu\left(0_{|V|},\ 1_{|V|}\right)\right),$$

by (5.4)

$$= \sum_{\pi\in NCE(n)}\left(\underset{V\in\pi}{\Pi}\left(\left(\begin{array}{c}|V|\\ \frac{|V|}{2}\end{array}\right)\right)\mu\left(0_{|V|},\ 1_{|V|}\right)\right),$$

by (4.3). □

In Section 4, we defined the identically (free-)distributedness of self-adjoint elements in W^*-probability spaces. And it is shown that all prime operators T_p are identically distributed by (4.3). The above theorem (or (5.2)) again demonstrates that T_p are identically distributed via the Moebius inversion. i.e., the free moments and the free cumulants contain the equivalent free-distributional data. So, the computation (5.2) (for all prime p) guarantees the identically distributedness of prime operators, too.

In the rest of this section, let's consider the "mixed" cumulants of $T_{p_{i_1}}, ..., T_{p_{i_n}}$, where

$$(p_{i_1}, ..., p_{i_n}) \in \{p_1, ..., p_k\}^n,$$

for some fixed $k \in \mathbb{N} \setminus \{1\}$, for all $n \in \mathbb{N} \setminus \{1\}$.

Let $p \neq q$ be fixed distinct primes, and let T_p and T_q be corresponding prime operators in the Adele W^*-probability space $(\mathfrak{A}, \varphi)$. Then

$$k_2(T_p, T_q) = 0. \tag{5.5}$$

Observe that

$$\begin{aligned}k_2(T_p, T_q) &= \sum_{\pi\in NC(2)}\varphi_\pi(T_p, T_q)\mu(\pi, 1_2)\\&= \varphi(T_pT_q) - \varphi(T_p)\varphi(T_q) = 0.\end{aligned}$$

Also, we get that

$$k_4\left(T_p,\ T_q,\ T_p,\ T_q\right)$$
$$= k_4(T_p, T_p, T_q, T_q)$$

since $T_pT_q = T_qT_p$, by (4.4)

$$= k_2\left(T_p^2,\ T_q^2\right)$$

by Speicher (See [11] and [12]), and by (4.7)

$$\begin{aligned} &= \sum_{\pi \in NC(2)} \varphi_\pi\left(T_p^2,\ T_q^2\right)\mu(0_\pi,\, 1_2) \\ &= \varphi\left(T_p^2T_q^2\right) - \varphi\left(T_p^2\right)\varphi\left(T_q^2\right) \\ &= \begin{pmatrix} 2 \\ 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ &= 0. \end{aligned}$$

Thus, by the commutativity of $\{T_p\}_{p:prime}$,

$$k_4(T_{r_1}, T_{r_2}, T_{r_3}, T_{r_4}) = 0, \qquad (5,5)'$$

whenever $(r_1,\, r_2,\, r_3,\, r_4)$ are the mixed quadruples of $\{p,\, q\}$ satisfying

$$\#(p) = 2 = \#(q).$$

REMARK 5.1. *By* $(5.5)'$, *we have*

$$\varphi\left(T_p^2T_q^2\right) = \varphi\left(T_p^2\right)\varphi\left(T_q^2\right).$$

So, it is possible that one may be tempted to believe φ *is a character on* $\{T_p\}_{p:prime}$, *in the sense that*

$$\varphi(T_p^{n_1}T_q^{n_2}) = \varphi(T_p^{n_1})\varphi(T_q^{n_2}),$$

for all primes p, q, *and* $n_1,\, n_2 \in \mathbb{N}$. *However, it does not hold in general. For instance, if the above multiplicativity of* φ *were true, then we must have*

$$\varphi\left(T_p^6\right) = \begin{pmatrix} 6 \\ 3 \end{pmatrix} = 0 = \varphi(T_p)\varphi(T_p^5),$$

which is definitely not true. Also, if φ *were a character, then*

$$\begin{aligned} \varphi\left(T_p^6\right) &= \begin{pmatrix} 6 \\ 3 \end{pmatrix} = 20 \\ &= 12 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix} \\ &= \varphi\left(T_p^2\right)\varphi\left(T_p^4\right), \end{aligned}$$

which is not true, either. So, of course, the linear functional φ *cannot be a character on* $\{T_p\}_{p:prime}$.

However, we remark that

$$\varphi\left(T_p^{2n_1}T_q^{2n_2}\right) = \begin{pmatrix} 2n_1 \\ n_1 \end{pmatrix}\begin{pmatrix} 2n_2 \\ n_2 \end{pmatrix} = \varphi\left(T_p^{2n_1}\right)\varphi\left(T_q^{2n_2}\right),$$

for all n_1, $n_2 \in \mathbb{N}$, *and for all "distinct" primes* $p \neq q$.

Motivated by (5.5) and $(5.5)'$, we obtain the following computations. In fact, the proof of the following theorem is not directly related to free cumulant computations as usual. It is proven theoretically based on the results from the structure theorems in Section 3 (See (3.13) and (3.14)).

THEOREM 5.2. *Let* p_1, ..., p_k *be fixed distinct primes, for* $k \in \mathbb{N} \setminus \{1\}$, *and let* $(p_{i_1}, ..., p_{i_n})$ *be a mixed n-tuple of* $\{p_1, ..., p_k\}$, *for all* $n \in \mathbb{N} \setminus \{1\}$. *Then*

$$k_n \left(T_{p_{i_1}}, \ ..., \ T_{p_{i_n}}\right) = 0, \tag{5.6}$$

for all $n \in \mathbb{N}$.

Proof. By (3.13), the group Γ generated by the p-prime subgroups $\Gamma_p = < a_p >$ of the Adele von Neumann algebra $\mathfrak{A}$ is group-isomorphic to the free group F_∞. And hence, by (3.14), the free group factor $L(F_\infty)$ satisfies

$$L(F_\infty) \overset{*\text{-iso}}{=} \underset{n=1}{\overset{\infty}{*_{\mathbb{C}}}} L(\mathbb{Z}) \overset{*\text{-iso}}{=} \underset{p:prime}{*_{\mathbb{C}}} vN(\Gamma_p) \overset{*\text{-iso}}{=} vN(\Gamma) \tag{5.7}$$

is a W^*-subalgebra of $\mathfrak{A}$, where $vN(\Gamma_p)$ and $vN(\Gamma)$ mean the group von Neumann algebras generated by Γ_p and Γ, respectively.

The first $*$-isomorphic relation of (5.7) is came from [11] and [12], under the canonical group-von Neumann-algebra linear functional τ

$$\tau\left(\sum_{g \in F_\infty} t_g g\right) \overset{def}{=} t_e, \text{ for all } \sum_{g \in F_\infty} t_g g \in L(F_\infty).$$

The second $*$-isomorphic relation of (5.7) holds, because our linear functional φ and the canonical group-von Neumann-algebra linear functional τ are equivalent on $vN(\Gamma)$, respectively on $L(F_\infty)$ (See [12], [3], and [4]). Thus, the third $*$-isomorphic relation of (5.7) holds because of (3.13). i.e., $vN(\Gamma_p)$ and $vN(\Gamma_q)$ are free in $vN(\Gamma)$, whenever $p \neq q$.

Since p_1, ..., p_k are distinct from each other, there exist W^*-algebras $vN(\Gamma_1)$, ..., $vN(\Gamma_k)$, such that they are free in $(\mathfrak{A}, \varphi)$. By definition, the corresponding p-prime operators T_{p_j} are contained in $vN(\Gamma_{p_j})$, for $j = 1$, ..., k. Therefore, they are free from each other in $(\mathfrak{A}, \varphi)$, too. Equivalently, all mixed free cumulants of them vanish.

As an inner structure, one can verify that there is an inner free structure induced by p-prime operators $\{T_p\}_{p:prime}$. □

COROLLARY 5.3. *Let* T_p *be prime operators in the Adele von Neumann algebra* $\mathfrak{A}$, *and let* A_p *be the* W^**-subalgebras* $\overline{\mathbb{C}[\{T_p\}]}^w$ *of* $\mathfrak{A}$, *generated by* T_p, *for all primes p. Then* A_p*'s are free from each other, for all prime p, in the Adele* W^**-probability space* $(\mathfrak{A}, \varphi)$. *Therefore, the von Neumann subalgebra* A_{prime} *of* $\mathfrak{A}$, *generated by* $\{T_p :$ *all primes* $p\}$, *is* $*$*-isomorphic to*

$$A_{prime} \overset{*\text{-}iso}{=} \underset{p:prime}{*_{\mathbb{C}}} A_p. \tag{5.8}$$

Proof. It suffices to check A_p are W^*-subalgebra of $vN(\Gamma_p)$, for all primes p. But, by the very definition, it is trivial. Therefore, by (5.7), we obtain the relation (5.8). □

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