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BACKWARD UNIQUENESS OF KOLMOGOROV OPERATORS*

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Abstract. The backward uniqueness of the Kolmogorov operator $L = \sum_{i,k=1}^{n} \partial_{x_i}(a_{i,k}(x,t)\partial_{x_k}) + \sum_{l=1}^{m} x_l \partial_{y_l} - \partial_t$, is proved in this paper. We obtain a weak Carleman inequality via Littlewood-Paley decomposition for the global backward uniqueness. Moreover, a monotonicity inequality is also proved for the Kolmogorov equation.

Key words. Carleman inequality, Kolmogorov operator, backward uniqueness, Littlewood-Paley decomposition.

AMS subject classifications. 35K70, 35A02.

1. Introduction. The Kolmogorov equation is a basic equation in the diffusion process and has many applications in various models (see [PP], [WZ]), for example, the two dimensional Prandtl's boundary layer equations in the Crocco variables and Boltzmann-Landau equations. One of the simplest form of the Kolmogorov operator is given in the following equation

$$\partial_{xx} u + x\partial_y u - \partial_t u = 0.$$

In this paper, we consider the uniqueness problem for the following more general backward Kolmogorov operator:

$$Lu = (\sum_{i,k=1}^{n} \partial_{x_i}(a_{ik}(x,t)\partial_{x_k}) + \sum_{l=1}^{m} x_l \partial_{y_l} + \partial_t)u,$$

where $m \le n$, $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, y_m)$, and $(x, y, t) \in \mathbb{R}^n \times \mathbb{R}^m \times (0, T)$. Assume that the coefficients are symmetric and uniformly elliptic:

$$a_{ik}(x,t) = a_{ki}(x,t), \quad 1 \le i,k \le n; \quad \lambda^{-1}|\xi|^2 \le \sum_{i,k=1}^{n} \xi_i a_{ik}(x,t)\xi_k \le \lambda|\xi|^2, \quad (1.1)$$

for any $(x,t) \in \mathbb{R}^n \times (0,T)$ and $\xi \in \mathbb{R}^n$. Here $\lambda > 1$ is a constant.

Our interest in the backward uniqueness of Kolmogorov operator arises from the study of regularity of Kolmogorov operators and recent progress in the backward uniqueness of the parabolic equations where some important applications have been found. In fact, the backward uniqueness of parabolic operator in half-space is crucial for the proof of smoothness of solutions of Navier-Stokes equations in $L^{3,\infty}$ (see [ESS2]), and some new techniques also are developed in their proof. Their main results state as follows:

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Suppose that u and the generalized derivatives $\partial_t u$ and $\nabla^2 u$ are square integrable over any bounded domain of $\mathbb{R}^N_+ \times (0,T)$,

$$|\Delta u + \partial_t u| \leq M(|u| + |\nabla u|), \quad |u| \leq e^{M|x|^2}, \quad \text{in} \quad \mathbb{R}^N_+ \times (0,T)$$

and $u(x,0) = 0$ in $\mathbb{R}^N_+$. Then $u(x,t) \equiv 0$, in $\mathbb{R}^N_+ \times (0,T)$. (see [ESS])

There is a long history of this type of backward uniqueness of parabolic equations. In the papers like [LPR] and [LA], some type of Carleman inequalities are obtained under the C^2 smoothness assumptions of the coefficients. On the other hand, the well known example of Miller [MI] (where an operator having coefficients which are Hölder-continuous of order 1/6 with respect to t and C^∞ with respect to x does not have the uniqueness property) shows that a certain amount of regularity assumptions on the a_{ik}'s are necessary for the uniqueness.

The main idea in the proof of the backward uniqueness is to obtain a certain type of Carleman inequality, which is also useful for unique continuation (see [LIN], [CHE], [ESC], [EV], [EF], [FER], [ESS], [SP] and so on). On the other hand, the monotonicity inequality of frequency functions (also is called Dirichlet quotient method) could also be used to prove unique continuation, for example, see [Og], [PO], [Ku] [EKPV] and the references therein.

For the backward uniqueness of Kolmogorov operator, an uniform Carleman inequality is necessary. However, it is difficult to obtain such inequalities because of degeneracy of the Kolmogorov operator. By combining the Littlewood-Paley decomposition and the approach of Carleman-type inequality in Escauriaza, Seregin, and Šverák [ESS], we obtain a weak type of Carleman inequality under the assumption that the coefficients are independent of y which implies the backward uniqueness property.

Our main idea is first to establish the Carleman-type inequality for the low frequency part in the degenerate direction. Under the assumption of coefficients which are independent of y, we make use of the Littlewood-Paley decomposition of the solution u in y direction to prove that the L^2 norm of $\triangle_j \partial_y u$ can be controlled by the L^2 norm of $\triangle_j u$. Hence the vanishing property of $\triangle_j u$ for any j implies that u must be vanished.

We assume that u satisfies

$$\begin{cases} Lu = (\displaystyle\sum_{i,k=1}^{n} \partial_{x_i}(a_{ik}(x,t)\partial_{x_k}) + \sum_{l=1}^{m} x_l \partial_{y_l} + \partial_t)u \\ \quad = c(x,t)u + \displaystyle\sum_{i=1}^{n} d(x,t)_i \partial_{x_i} u, \quad (x,y,t) \in \mathbb{R}^{n+m} \times (0,T), \\ u(x,y,0) = 0 \quad (x,y) \in \mathbb{R}^{n+m}, \\ u, \nabla_x u, (\displaystyle\sum_{l=1}^{m} x_l \partial_{y_l} + \partial_t)u \in L^2(\mathbb{R}^{n+m} \times (0,T)), \end{cases} \tag{1.2}$$

where the coefficients are some suitable regular functions.

Let $g = \chi(t)u$, where $\chi(t)$ is a C^∞ smooth function for $0 < t_1 < t_2 < T$ and

$$\chi(t) = \begin{cases} 1, & t \leq t_1, \\ 0, & t > t_2. \end{cases}$$

For fixed $\alpha_0 > 1$, we choose $j_0 = \max\{j \in \mathbb{Z};\, 4^{j+1} \leq \frac{\alpha_0 - 1}{4t}, t \in (0,t_2]\}$. When $\alpha > \alpha_0$, we then choose a small t_2 and obtain the following Carleman-type inequality

$$\begin{aligned}&\int_{\mathbb{R}^{n+m}\times(0,T)} t^{-2\alpha+1}|L\Delta_j g|^2 dxdy\\ &\geq \int_{\mathbb{R}^{n+m}\times(0,T)} \big(\frac{1}{4\lambda}t^{-2\alpha}|\partial_x\Delta_j g|^2+\frac{\alpha-1}{4}t^{-2\alpha-1}|\Delta_j g|^2\big)dxdy,\end{aligned}\tag{1.3}$$

where Δ_j is Littlewood-Paley decomposition operator, $j\in\mathbb{Z}$, $j\leq j_0$. (more details see Section 2.)

The backward uniqueness is proved by applying the above Carleman-type inequality.

THEOREM 1.1. *Suppose that u satisfies the condition (1.2) whose coefficients satisfy (1.1). We assume that $a_{i,k}(x,t)\in C^{0,1}$ and $c(x,t)$, $d(x,t)_l$ are bounded functions, where $1\leq i,k,l\leq n$. Then $u\equiv 0$ in $\mathbb{R}^{n+m}\times(0,T)$.*

REMARK 1.1. *Here $a_{i,k}(x,t)\in C^{0,1}$ means that $\nabla_x a_{i,k}(x,t)$ and $\partial_t a_{i,k}(x,t)$ are bounded. Under the condition of (1.2), the operator L satisfies the well-known Hörmander finite rank condition. The Kolmogorov operator, although degenerated in some sense, still retains most of the properties of the parabolic operator. For example, the interior regularity for weak solutions of the Kolmogorov equation is similar to that of the parabolic operator (see [PP], [ZH] and [WZ]), as well as the backward uniqueness at least under some additional assumptions.*

REMARK 1.2. *For the general Kolmogorov operator, the backward uniqueness problem is still open. For example, if $L_1=\partial_{xx}+x\partial_y+y\partial_z+\partial_t$, we don't know if it has the backward uniqueness property, which is interesting and needs some new idea.*

We also give an alternative proof for the above results by frequency functions method as the parabolic case.

2. Proof of the Main Theorem. We first introduce some of the notations which are used throughout this paper.

Set $g=\chi(t)u$, and $\chi(t)$ is a C^∞ smooth function satisfying

$$\chi(t)=\begin{cases}1, & t\leq t_1,\\ 0, & t>t_2.\end{cases}$$

where $0<t_1<t_2<T$, to be chosen.

Let $\phi(t)=(t+b)^{-\alpha}$ and $f=\phi g$, where $\alpha>0$ and b is a constant satisfying $0<b\leq t_2$.

We introduce the Littlewood-Paley decomposition on $\mathbb{R}^m$. Let $\varphi(\xi)$ be a smooth cut-off function such that

$$\varphi(\xi)=\begin{cases}1, & |\xi|\leq\frac{1}{2},\\ 0, & |\xi|>1.\end{cases}$$

Let $\psi(\xi)=\varphi(\xi/2)-\varphi(\xi)$. For any integer j, as usual, we denote $\triangle_j$ and S_j,

$$\triangle_j h(x)=\mathcal{F}^{-1}(\psi(\frac{\xi}{2^j})\mathcal{F}(h)(\xi)),$$

$$S_j h(x)=\mathcal{F}^{-1}(\varphi(\frac{\xi}{2^j})\mathcal{F}(h)(\xi)),$$

where $h(x) \in \mathcal{S}'(\mathbb{R}^m)$ and $\mathcal{F}^{-1}$ is the inverse of Fourier transformation. For $h \in L^2(\mathbb{R}^m)$, denote $\mathcal{F}(h) = \hat{h}$, and we have

$$\int_{\mathbb{R}^m} |h|^2 dy = \int_{\mathbb{R}^m} |\hat{h}|^2 dy = \int_{\mathbb{R}^m} |\sum_{j \in Z} \Delta_j h|^2 dy.$$

And it is easy to see that there exists $K > 0$ only dependent on the dimension, such that

$$K^{-1} \int_{\mathbb{R}^m} |\sum_{j \in Z} \Delta_j h|^2 dy \leq \int_{\mathbb{R}^m} \sum_{j \in Z} |\Delta_j h|^2 dy \leq K \int_{\mathbb{R}^m} |\sum_{j \in Z} \Delta_j h|^2 dy. \tag{2.1}$$

Let $\Omega_T = \mathbb{R}^{m+n} \times (0, T)$. By our assumption, we may assume that for some positive constant $\lambda > 1$, for all $(x, t) \in \mathbb{R}^n \times (0, T)$ and $\xi \in \mathbb{R}^n$

$$\lambda^{-1} |\xi|^2 \leq \xi_i a_{ij} \xi_j \leq \lambda |\xi|^2$$

and

$$|\nabla_x a_{ij}(x, t)|, |\partial_t a_{ij}(x, t)|, |c(x, t)|, |d(x, t)| \leq \lambda.$$

We make the Littlewood-Paley decomposition in the y direction. For convenience we set

$$f_j = \phi \Delta_j g.$$

We are going to prove a Carleman-type inequality for the function $\Delta_j g$ which enables us to overcome the difficulty of the degeneracy in the y direction.

LEMMA 2.1. *Under the assumptions of Theorem 1.1, for fixed $\alpha_0 > 1$ and $j_0 \in \mathbb{Z}$,*

$$j_0 = max\{j \in \mathbb{Z}; 4^{j+1} \leq \frac{\alpha_0 - 1}{4(t+b)}, t \in (0, t_2]\},$$

and $t_2 \leq (16\lambda^2)^{-1}$. Then the following Carleman inequality about the function $\Delta_j g$

$$\int_{\Omega_T} (t+b)^{-2\alpha+1} |L \Delta_j g|^2 \geq \int_{\Omega_T} \frac{1}{4\lambda} (t+b)^{-2\alpha} |\partial_x \Delta_j g|^2 + \frac{\alpha - 1}{4} (t+b)^{-2\alpha-1} |\Delta_j g|^2,$$

holds for all $\alpha > \alpha_0$ and $j \leq j_0$.

REMARK 2.1. *The function $\Delta_j g$ is the Littlewood-Paley decomposition in y direction of the solution u of problem (1.2). One can easily check that*

$$\Delta_j g = \chi(t) \mathcal{F}^{-1}(\psi(\frac{\xi}{2^j}) \mathcal{F}(u)(\xi)).$$

Proof. We need to estimate the integral of the function $\Delta_j g$ and its derivative in the x direction in terms of

$$I \equiv \int_{\Omega_T} (t+b) |\phi L \Delta_j g|^2 dx dy dt.$$

Recall the notation $\phi(t) = (t+b)^{-\alpha}$, $g = \chi(t)u$ and let $f_j = \phi\Delta_j g$. By the equations of (1.2), we have

$$\begin{aligned}
&\int_{\Omega_T} (t+b)|\phi L\Delta_j g|^2 dxdydt \qquad (2.2)\\
&= \int_{\Omega_T} (t+b)|(\sum_{i,k=1}^n \partial_{x_i}(a_{ik}(x,t)\partial_{x_k}) + \sum_{l=1}^m x_l\partial_{y_l} + \partial_t - \frac{\phi'}{\phi})(\phi\Delta_j g)|^2 dxdydt\\
&\geq \int_{\Omega_T} 2(t+b)(\sum_{i,k=1}^n \partial_{x_i}(a_{ik}(x,t)\partial_{x_k}) - \frac{\phi'}{\phi})f_j(\sum_{l=1}^m x_l\partial_{y_l} + \partial_t)f_j dxdydt\\
&= -\int_{\Omega_T} 2(t+b)\sum_{i,k=1}^n a_{ik}\partial_{x_i}f_j\partial_{y_k}f_j dxdydt\\
&\quad + \int_{\Omega_T}\sum_{i,k=1}^n \big(a_{ik} + (t+b)\partial_t a_{ik}\big)\partial_{x_i}f_j\partial_{x_k}f_j dxdydt
\end{aligned}$$

where we have used the symmetry property of a_{ik}, and $y_k = 0$ for $k > m$.

On the other hand, the usual decomposition gives another lower bound of I. Let $\tilde{\phi} = (t+b)^{\frac12}\phi = (t+b)^{\frac12-\alpha}$, and $\tilde{f}_j = \tilde{\phi}\Delta_j g = (t+b)^{\frac12}f_j$. Then we have

$$\begin{aligned}
I &= \int_{\Omega_T} |(\sum_{i,k=1}^n \partial_{x_i}(a_{ik}(x,t)\partial_{x_k}) + \sum_{l=1}^m x_l\partial_{y_l} + \partial_t - \frac{\tilde{\phi}'}{\tilde{\phi}})(\tilde{\phi}\Delta_j g)|^2 dxdydt\\
&\geq \int_{\Omega_T} 2(\sum_{i,k=1}^n \partial_{x_i}(a_{ik}(x,t)\partial_{x_k}) - \frac{\tilde{\phi}'}{\tilde{\phi}})\tilde{f}_j(\sum_{l=1}^m x_l\partial_{y_l} + \partial_t)\tilde{f}_j dxdydt\\
&= \int_{\Omega_T} \big(-2\sum_{i,k=1}^n a_{ik}\partial_{x_i}\tilde{f}_j\partial_{y_k}\tilde{f}_j + \sum_{i,k=1}^n \partial_t a_{ik}\partial_{x_i}\tilde{f}_j\partial_{x_k}\tilde{f}_j + (\frac{\tilde{\phi}'}{\tilde{\phi}})'|\tilde{f}_j|^2\big)dxdydt\\
&= -\int_{\Omega_T} 2(t+b)\sum_{i,k=1}^n a_{ik}\partial_{x_i}f_j\partial_{y_k}f_j dxdydt + \int_{\Omega_T}\sum_{i,k=1}^n (t+b)\partial_t a_{ik}\partial_{x_i}f_j\partial_{x_k}f_j dxdydt\\
&\quad + \int_{\Omega_T} \frac{\alpha-\frac12}{(t+b)}|f_j|^2 dxdydt \qquad (2.3)
\end{aligned}$$

where in the last equality we used the fact $\tilde{f}_j = (t+b)^{\frac12}f_j$.

Now we choose $t_2 \leq (16\lambda^2)^{-1}$ and then fixed from now on in the definition of function $\chi(t)$. Combining the two inequalities (2.2) with (2.3), a simple calculation yields that

$$\begin{aligned}
I &\geq -\int_{\Omega_T} 2(t+b)\sum_{i,k=1}^n a_{ik}\partial_{x_i}f_j\partial_{y_k}f_j dxdydt\\
&\quad + \int_{\Omega_T}\sum_{i,k=1}^n \big(\frac12 a_{ik} + (t+b)\partial_t a_{ik}\big)\partial_{x_i}f_j\partial_{x_k}f_j dxdydt + \int_{\Omega_T}\frac{\alpha-\frac12}{2(t+b)}|f_j|^2 dxdydt\\
&\geq \int_{\Omega_T} \frac{1}{4\lambda}|\nabla_x f_j|^2 - \lambda(t+b)|\nabla_y f_j|^2 + \frac{\alpha-1}{2(t+b)}|f_j|^2. \qquad (2.4)
\end{aligned}$$

Note that

$$\int_{\mathbb{R}^m} |\partial_y f_j|^2 = \int_{\mathbb{R}^m} \eta^2|\psi(\eta/2^j)\hat{f}(\eta)|^2 \leq 4^{j+1}\int_{\mathbb{R}^m} |\psi(\eta/2^j)\hat{f}(\eta)|^2. \qquad (2.5)$$

Let $\alpha_0 > 1$, and we may assume that the parameter α in the function $\phi(t) = (t+b)^{-\alpha}$ satisfies

$$\alpha > \alpha_0 > 1.$$

Moreover, we choose

$$j_0 = max\{j \in \mathbb{Z}; 4^{j+1} \leq \frac{\alpha_0 - 1}{4(t+b)}, t \in (0, t_2]\}.$$

Hence when $\alpha > \alpha_0$ and $t_2 \leq (16\lambda^2)^{-1}$, for $j \leq j_0$, from (2.4) and (2.5), we obtain the following Carleman-type inequality

$$\begin{aligned}\int_{\Omega_T} (t+b)|\phi L \Delta_j g|^2 &\geq \int_{\Omega_T} \frac{1}{4\lambda}|\nabla_x f_j|^2 + \frac{\alpha - 1}{4(t+b)}|f_j|^2 \\ &= \int_{\Omega_T} \frac{1}{4\lambda}(t+b)^{-2\alpha}|\nabla_x \Delta_j g|^2 + \frac{\alpha-1}{4}(t+b)^{-2\alpha-1}|\Delta_j g|^2.\end{aligned} \tag{2.6}$$

Then we finished the proof of Lemma 2.1. □

Proof of Theorem 1.1. Since $g = \chi(t)u$, then

$$L\Delta_j g = \Delta_j(\chi' u + \chi c(x,t)u + \chi d(x,t) \cdot \nabla_x u).$$

By the above Carleman inequality (2.6), we deduce

$$\begin{aligned}&\int_{\Omega_T} (t+b)^{-2\alpha+1}|(\chi'\Delta_j u + \chi c(x,t)\Delta_j u + \chi d(x,t) \cdot \nabla_x \Delta_j u)|^2 \\ \geq &\int_{\Omega_T} \frac{1}{4\lambda}(t+b)^{-2\alpha}|\chi \nabla_x \Delta_j u|^2 + \frac{\alpha-1}{4}(t+b)^{-2\alpha-1}|\chi \Delta_j u|^2,\end{aligned} \tag{2.7}$$

where we used the assumption $|c(x,t)|, |d(x,t)| \leq \lambda$ and the choice of t_2 which satisfies $t_2 \leq (16\lambda^2)^{-1}$. Consequently

$$\int_{\Omega_T} (t+b)^{-2\alpha+1}|(\chi'\Delta_j u)|^2 \geq \int_{\Omega_T} \frac{\alpha-1}{8}(t+b)^{-2\alpha-1}|\chi \Delta_j u|^2.$$

Summing for all $j \leq j_0$, by the inequality (2.1), we obtain

$$\int_{\Omega_T} (t+b)^{-2\alpha+1}|(\chi' u)|^2 \geq C(K) \int_{\Omega_T} \frac{\alpha-1}{8}(t+b)^{-2\alpha-1}|\chi S_{j_0-1}u|^2. \tag{2.8}$$

Then

$$\int_0^{t_1}\int_{R^{n+m}} |S_{j_0-1}u|^2 \leq \frac{32}{C(K)(\alpha-1)(t_2-t_1)^2}\int_{t_1}^{t_2}\int_{R^{n+m}} \frac{(t_1+b)^{2\alpha+1}}{(t+b)^{2\alpha-1}}|u|^2. \tag{2.9}$$

Now we let $\alpha \to \infty$ in (2.9), then we obtain

$$S_{j_0-1}u \equiv 0.$$

And then $u \equiv 0$ in $R^2 \times (0, t_1)$ by the choice of j_0 and $\alpha_0 \to \infty$. Again we obtain that $u \equiv 0$ in $R^2 \times (0, t_2)$ since t_1 can approach t_2. Finally, we have $u \equiv 0$ in $R^2 \times (0, T)$

after the iteration, since t_2 only depends on L. Hence we have completed the proof of Theorem 1.1. □

Remark 2.2. *The assumption that the coefficients a_{ik}, c and d are independent of y seems to be only a technical assumption. However, we do not know how to remove it in general. The main difficulty for the Kolmogorov operator in our case is the loss of derivative estimates in y direction. On the other hand, the recent regularity result (see [WZ]) shows that one can recover the regularity even in y direction.*

Here we give another proof by using the frequency function method , which is also called Dirichlet quotient method, for example, see [Og], [PO], [Ku] [EKPV] and the references therein. Now we consider the differential inequality

$$|(\sum_{i,k=1}^{n}\partial_{x_i}(a_{ik}(x,t)\partial_{x_k})u+\sum_{l=1}^{m}x_l\partial_{y_l}u+\partial_t u|\le\lambda(|u|+|\nabla_x u|). \tag{2.10}$$

Let

$$e(t)=\int_{\mathbb{R}^{n+m}}u^2dxdy,$$

$$d(t)=\int_{\mathbb{R}^{n+m}}(\sum_{i,k=1}^{n}a_{ik}(x,t)\partial_{x_i}u\partial_{x_k}u)dxdy,$$

and

$$h(t)=\frac{d(t)}{e(t)}.$$

Then we have the following monotonicity inequality lemma.

Lemma 2.2. *Suppose that u satisfies (2.10). In addition to the condition of Theorem 1.1, we assume that for some constant M*

$$\int_{\mathbb{R}^{n+m}}|\nabla_y u|^2\le M\int_{\mathbb{R}^{n+m}}|u|^2. \tag{2.11}$$

Then there exits a constant $C=C(\lambda,M)$ such that

$$\dot h(t)\ge -C(\lambda,M)[h(t)+1]. \tag{2.12}$$

Remark 2.3. *This is the corresponding monotonicity inequality for the Kolmogorov operator. There is an additional assumption (2.11) which seems necessary in our approach. And in application, we again need to make use of the Littlewood-Paley decomposition in y direction.*

Proof. By our assumption, one can calculate directly

$$\dot e(t)=2\int_{\mathbb{R}^{n+m}}uu_t=\int_{\mathbb{R}^{n+m}}uLu+2\int_{\mathbb{R}^{n+m}}u(\partial_t u+\sum_{l=1}^{m}x_l\partial_{y_l}u-\frac{1}{2}Lu),$$

and

$$d(t)=\int_{\mathbb{R}^{n+m}}u(\partial_t u+\sum_{l=1}^m x_l\partial_{y_l}u-\frac{1}{2}Lu)-\frac{1}{2}\int_{\mathbb{R}^{n+m}}uLu.$$

Hence

$$\dot{e}(t)d(t)=2[\int_{\mathbb{R}^{n+m}}u(\partial_t u+\sum_{l=1}^m x_l\partial_{y_l}u-\frac{1}{2}Lu)]^2-\frac{1}{2}[\int_{\mathbb{R}^{n+m}}uLu]^2. \tag{2.13}$$

By our assumption on the coefficient $a_{ik}(x,t)$, we have

$$\begin{aligned}
&\dot{d}(t)\\
&=\int_{\mathbb{R}^{n+m}}\sum_{i,k=1}^n\partial_t a_{ik}(x,t)\partial_{x_i}u\partial_{x_k}u+2\int_{\mathbb{R}^{n+m}}\sum_{i,k=1}^n a_{ik}(x,t)\partial_{x_i}u\partial_{x_k}\partial_t u\\
&=\int_{\mathbb{R}^{n+m}}\sum_{i,k=1}^n\partial_t a_{ik}(x,t)\partial_{x_i}u\partial_{x_k}u-2\int_{\mathbb{R}^{n+m}}\sum_{i,k=1}^n \partial_{x_k}(a_{ik}(x,t)\partial_{x_i}u)\partial_t u\\
&=\int_{\mathbb{R}^{n+m}}\sum_{i,k=1}^n\partial_t a_{ik}(x,t)\partial_{x_i}u\partial_{x_k}u+2\int_{\mathbb{R}^{n+m}}(\partial_t u+\sum_{l=1}^m x_l\partial_{y_l}u)(\partial_t u+\sum_{l=1}^m x_l\partial_{y_l}u-Lu)\\
&\quad-2\int_{\mathbb{R}^{n+m}}\sum_{i,k=1}^n a_{ik}(x,t)\partial_{x_i}u\partial_{y_k}u
\end{aligned}$$

and the right terms are also written as follows:

$$\begin{aligned}
&\int_{\mathbb{R}^{n+m}}\sum_{i,k=1}^n\partial_t a_{ik}(x,t)\partial_{x_i}u\partial_{x_k}u+2\int_{\mathbb{R}^{n+m}}(\partial_t u+\sum_{l=1}^m x_l\partial_{y_l}u-\frac{1}{2}Lu)^2\\
&-\frac{1}{2}\int_{\mathbb{R}^{n+m}}|Lu|^2-2\int_{\mathbb{R}^{n+m}}\sum_{i,k=1}^n a_{ik}(x,t)\partial_{x_i}u\partial_{y_k}u\\
&\geq 2\int_{\mathbb{R}^{n+m}}(\partial_t u+\sum_{l=1}^m x_l\partial_{y_l}u-\frac{1}{2}Lu)^2-\frac{1}{2}\int_{\mathbb{R}^{n+m}}|Lu|^2-C(\lambda,M)[d(t)+e(t)].
\end{aligned}$$

Hence combining with (2.13) we obtain

$$e(t)\dot{d}(t)-d(t)\dot{e}(t)\geq -C(\lambda,M)[d(t)+e(t)]e(t)-\frac{1}{2}e(t)\int_{\mathbb{R}^{n+m}}|Lu|^2. \tag{2.14}$$

Consequently, we have the following monotonicity inequality

$$\dot{h}(t)=\frac{\dot{d}(t)e(t)-\dot{e}(t)d(t)}{e(t)^2}\geq -C(\lambda,M)[h(t)+1]-\frac{\int_{\mathbb{R}^{n+m}}|Lu|^2}{2e(t)}.$$

Together with (2.10), then (2.12) follows easily. Then we proved Lemma 2.2. □

Proof of Theorem 1.1 by the monotonicity inequality. We assume that $e(t)\equiv 0$ for $0\leq t\leq t_1$, and $e(t)>0$ as $t_1<t\leq T$. For $t_1<t<t_2\leq T$, we integrate the above inequality from t to t_2

$$\log\frac{h(t_2)+1}{h(t)+1}\geq -C(\lambda,M)T,$$

which yields

$$h(t) \leq C(\lambda, M, T, h(t_2)). \tag{2.15}$$

Since

$$\frac{\dot{e}(t)}{e(t)} \leq 3h(t) + C(\lambda) \leq C(\lambda, M, T, h(t_2)),$$

integrating from t to t_2, we have

$$e(t_2) \leq e(t)C(\lambda, M, T, h(t_2)). \tag{2.16}$$

Let $t \to t_1$, we get $e(t_2) = 0$ which is a contradiction. Using the same Littlewood-Paley decomposition as Theorem 1.1, we could replace u with $\triangle_j u$, since $\triangle_j u$ satisfies the inequality (2.11). Then the remaining arguments are similar to that of the previous proof of Theorem 1.1. We complete the proof of our theorem by the frequency function method. □

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