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VISCOUS COMPRESSIBLE MULTI-FLUIDS: MODELING AND MULTI-D EXISTENCE*

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1. Introduction. Description of multi-fluid systems is an interesting developing area of physics as well as mathematics. Numerous models arising here form several families. Whereas the mail goal of the article is mathematical study of one of these families, this Section is devoted to a general discussion of models.

1.1. Three families of multi-fluid models. A first family concern mixtures made of different chemical species that are mixed at the molecular level and share the same velocity and temperature. The corresponding models are usually termed (multicomponent) mixtures [13], [21]. Here individual velocities of constituents are determined from the average hydrodynamic velocity by means of diffusion laws which come from the kinetic theory [9], [30].

A second family concern multiphase flows or heterogeneous flows where the different phases are immiscible and only occupy a fraction of the total volume. By using averaging techniques over the phases these fluids lead to classical multifluid models [3], [22], and this case can be termed heterogeneous (multiphase) multifluid. An additional difficulty which arises here is that the phase volume fractions are extra unknowns.

Finally, a third family corresponds to fluids made of different chemical species that are mixed at the molecular level but do not share the same velocity and/or temperature [29], [30]. This family can be termed homogeneous (interpenetrating) multifluid. These models are somewhat close mathematically to multiphase fluid models with a constant volume fractions.

1.2. Homogeneous bifluids. We deal only with the third family of models and (without loss of generality) suppose that the number of constituents equals 2. That means, in particular, that both fluids fill the whole volume, i. e. in each point both constituents are present, and they are at the same phase. Our goal is to study two definite models of homogeneous bifluids. We do not discuss the process of derivation of these models since it is a subject of monographs [22] and [28]. However, in [28] models without temperature are mainly considered, and we are interested in models which consider temperature. Following [22] and [28], we suppose that each constituent of the bifluid possesses its own velocity, and consider two cases: with two (separated) temperatures or one (common) temperature.

1.3. Mathematical well-posedness as an expansion of mono-fluid NS/NSF theory. Mathematical results, anyway, are based on the methods developed for mono-fluid systems, described by Navier—Stokes (NS) or Navier—Stokes—

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Fourier (NSF) equations. We believe that there is no need to observe global theory developed for NS and NSF, and just refer to monographs [4], [6], [17], [24] and [27]. Anyway, one should distinguish between barotropic and heat-conductive models, and, on the other hand, between stationary and non-stationary problems. The difference is in the threshold value of adiabatic coefficient for which global results are obtained, and also in generality of constitutive equation for the pressure. For instance, the equation of ideal gas $p = \rho\theta$ is still a serious mathematical problem, which can be avoided considering the pressure as a sum of cold and warm terms, e. g. like that: $p = \rho^\gamma + \rho\theta$ [19], [20], [23], or $p = \rho^{5/3} + \rho\theta + \theta^4$ [5], [6]. It is reasonable that we will use the same idea in mathematical studies of bifluid systems. The problem with dissipative terms $\mathbb{P} : \mathbb{D}$ is worthy of special notice. These terms are frequently omitted in mathematical studies; on the other hand, a useful observation is that one can avoid this difficulty using the total energy balance [5], [6], [7], [20], [23]. In contrast to mono-fluid systems, in some models of multifluids omitting dissipative terms is not only a mathematical trick, but it may be important via modeling arguments (the details are presented in Subsection 3.3).

2. Mathematical model of bifluids.

Let us now specify the ideas written above and formulate them as mathematical equations and inequalities.

2.1. Dynamics. The mathematical model contains two continuity equations:

$$\frac{\partial \rho_i}{\partial t} + \operatorname{div}(\rho_i \boldsymbol{u}^{(i)}) = 0, \quad i = 1, 2,$$

(ρ_1, ρ_2 are the densities, $\boldsymbol{u}^{(1)}$, $\boldsymbol{u}^{(2)}$ are the velocities), two momentum equations:

$$\frac{\partial(\rho_i \boldsymbol{u}^{(i)})}{\partial t} + \operatorname{div}(\rho_i \boldsymbol{u}^{(i)} \otimes \boldsymbol{u}^{(i)}) + \nabla p_i - \operatorname{div} \mathbb{P}^{(i)} = \rho_i \boldsymbol{f}^{(i)} + \boldsymbol{J}^{(i)},\ i = 1, 2, \tag{1}$$

and the energy equations (written in Subsection 2.3). Here p_1, p_2 are the pressures, $\mathbb{P}^{(i)}$ are the viscous parts of the stress tensors $\widetilde{\mathbb{P}}^{(i)} = -p_i\mathbb{I} + \mathbb{P}^{(i)}$ of each constituent:

$$\mathbb{P}^{(i)} = \sum_{j=1}^{2} \Big(\lambda_{ij} \operatorname{div} \boldsymbol{u}^{(j)}\mathbb{I} + 2\mu_{ij}\mathbb{D}(\boldsymbol{u}^{(j)})\Big), \quad i = 1, 2,$$

where λ_{ij}, μ_{ij} are viscosity coefficients, $\mathbb{D}$ means the rate of deformation (strain) tensor, i. e. $\mathbb{D}(\boldsymbol{v}) = \dfrac{1}{2}((\nabla \otimes \boldsymbol{v}) + (\nabla \otimes \boldsymbol{v})^*)$, and $\mathbb{I}$ is the identity tensor.

Hence, the viscosities λ_{ij}, μ_{ij} and $\nu_{ij} = \lambda_{ij} + 2\mu_{ij}$ ("total" viscosities) form the matrices

$$\Lambda = \begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix}, \quad M = \begin{bmatrix} \mu_{11} & \mu_{12} \\ \mu_{21} & \mu_{22} \end{bmatrix}, \quad N = \begin{bmatrix} \nu_{11} & \nu_{12} \\ \nu_{21} & \nu_{22} \end{bmatrix}.$$

Finally, $\boldsymbol{f}^{(i)} = (f_1^{(i)}, f_2^{(i)}, f_3^{(i)})$ are external forces, and

$$\boldsymbol{J}^{(i)} = (-1)^{i+1} a(\boldsymbol{u}^{(2)} - \boldsymbol{u}^{(1)}), \quad i = 1, 2, \quad a = \text{const} > 0$$

stands for the momentum supply for each constituent.

As a result, (1) means that beside external mass forces $\boldsymbol{f}^{(i)}$, there exist internal mass forces between the constituents, and internal surface forces arise not only inside each constituent but also between them. Diagonal entries of viscosity matrices are responsible for internal friction inside each constituent, and non-diagonal entries are responsible for friction between constituents of the bifluid.

2.2. Properties of viscosity matrices. Let us make several observations concerning properties of matrices Λ and M.

- Onsager principle (see e. g. [22]) implies that viscosity matrices must be symmetric, but it causes strong mathematical difficulties, and we do not suppose that (however, then the problem is of mathematical interest).
- It is very important (physically and mathematically) to validate Second law of thermodynamics which means (more detailed analysis is presented in Subsection 3.3)

$$\sum_{i=1}^{2} \mathbb{P}^{(i)} : \mathbb{D}(\boldsymbol{u}^{(i)}) \geqslant 0 \tag{2}$$

  or even (for mathematical purposes) some coercivity

$$\sum_{i=1}^{2} \mathbb{P}^{(i)} : \mathbb{D}(\boldsymbol{u}^{(i)}) \geqslant C \sum_{i=1}^{2} |\nabla \otimes \boldsymbol{u}^{(i)}|^2;$$

  or at least some weaker versions of presented inequalities:

$$\sum_{i=1}^{2} \int_{\Omega} \mathbb{P}^{(i)} : \mathbb{D}(\boldsymbol{u}^{(i)}) d\boldsymbol{x} \geqslant 0 \tag{3}$$

  (here and hereafter Ω is the flow domain), or again (here we suppose $\boldsymbol{u}^{(i)}|_{\partial\Omega} = 0$)

$$\sum_{i=1}^{2} \int_{\Omega} \mathbb{P}^{(i)} : \mathbb{D}(\boldsymbol{u}^{(i)}) d\boldsymbol{x} \geqslant C \sum_{i=1}^{2} \int_{\Omega} |\nabla \otimes \boldsymbol{u}^{(i)}|^2 d\boldsymbol{x}. \tag{4}$$

  The formulated positiveness or coercivity can be provided by the following properties of viscosity matrices: the properties

$$n\Lambda + 2M \geqslant 0, \qquad M \geqslant 0 \tag{5}$$

  provide (2), and the properties

$$N = \Lambda + 2M > 0, \qquad M > 0 \tag{6}$$

  provide (4), etc. (n is the dimension of the flow).
  Physically we must provide local inequalities (however, maybe for the sum of two constituents), but mathematically it may suffice to provide integral estimates (however, always for each constituent).
- The very important observation is that viscosity matrices should not be diagonal. Momentum supply $\boldsymbol{J}^{(i)} = (-1)^{i+1} a(\boldsymbol{u}^{(2)} - \boldsymbol{u}^{(1)})$ gives lower order terms (physically important, but mathematically causing no difficulties), and if the matrices are diagonal then $\boldsymbol{J}^{(i)}$ is the only connection between two constituents, so we have two NSF systems connected only via lower order terms. Earlier such problems were relevant (even in 1D), but nowadays such results almost automatically come from the theory of mono-fluid systems (compressible NS or NSF).
  If viscosity matrices are "complete" then we have interesting mathematical problems.

Summary: mathematical statement implies (5) or (6) etc. but without symmetry of matrices which are preferred to be non-diagonal.

2.3. Temperature. To complete the model, it is left to formulate energy equations (and constitutive equations which are to be presented further, in this Subsection):

$$\frac{\partial(\rho_i E_i)}{\partial t} + \mathrm{div}(\rho_i E_i \boldsymbol{u}^{(i)}) + \mathrm{div}(p_i \boldsymbol{u}^{(i)}) - \mathrm{div}(\mathbb{P}^{(i)} \boldsymbol{u}^{(i)}) =$$

$$= \rho_i \boldsymbol{f}^{(i)} \cdot \boldsymbol{u}^{(i)} + K_i - \mathrm{div}\boldsymbol{q}^{(i)}, \quad i = 1, 2,$$

here E_i is total specific energy, i. e.

$$E_i = \frac{1}{2}|\boldsymbol{u}^{(i)}|^2 + U_i, \quad i = 1, 2,$$

U_i is specific internal energy; K_i is energy supply rate for the i-th constituent:

$$K_i = \Gamma_i + \boldsymbol{J}^{(i)} \cdot \boldsymbol{u}^{(i)}, \quad i = 1, 2,$$

where

$$\Gamma_i = (-1)^{i+1} b(\theta_2 - \theta_1) + \frac{a}{2}|\boldsymbol{u}^{(1)} - \boldsymbol{u}^{(2)}|^2, \quad i = 1, 2, \quad b = \mathrm{const} > 0$$

(θ_i is the temperature of the i-th constituent); $\boldsymbol{q}^{(i)}$ is the heat flux for the i-th constituent:

$$\boldsymbol{q}^{(i)} = -k_i \nabla \theta_i, \quad i = 1, 2,$$

where $k_i = k_i(\theta_i)$ is the thermal conductivity coefficient of the i-th constituent.

Another form of the energy equations is

$$\frac{\partial(\rho_i U_i)}{\partial t} + \mathrm{div}(\rho_i U_i \boldsymbol{u}^{(i)}) + p_i \mathrm{div}\, \boldsymbol{u}^{(i)} - \mathbb{P}^{(i)} : (\nabla \otimes \boldsymbol{u}^{(i)}) = \Gamma_i - \mathrm{div}\boldsymbol{q}^{(i)}, \quad i = 1, 2.$$

For instance, if we specify the constitutive equations as follows

$$p_i = \rho_i^\gamma + \rho_i \theta_i, \quad i = 1, 2, \quad \gamma = \mathrm{const} > 1,$$

$$U_i = \frac{1}{\gamma - 1} \rho_i^{\gamma - 1} + \theta_i, \quad i = 1, 2,$$

$$k_i = 1 + \theta_i^m, \quad i = 1, 2, \quad m = \mathrm{const} > 1,$$

then the energy equations (steady version) take the form

$$\mathrm{div}(\rho_i \theta_i \boldsymbol{u}^{(i)}) - \mathbb{P}^{(i)} : (\nabla \otimes \boldsymbol{u}^{(i)}) =$$

$$= \mathrm{div}((1 + \theta_i^m) \nabla \theta_i) - \rho_i \theta_i \mathrm{div} \boldsymbol{u}^{(i)} + \Gamma_i, \quad i = 1, 2.$$

These constitutive equations can be generalized, e. g. γ and m may differ for each constituent, the form of $\boldsymbol{J}^{(i)}$, Γ_i and $\boldsymbol{q}^{(i)}$ $(i = 1, 2)$ may be more general, and finally the coefficients a, b, λ_{ij}, μ_{ij}, γ and m may depend on variables ρ_i, θ_i and $\boldsymbol{u}^{(i)}$, $i = 1, 2$.

2.4. Mathematical results for multifluids and mixtures. Known mathematical results for the model formulated above are not so numerous and concern mainly approximate models. In the papers [10] and [11] stationary Stokes system without convective terms is studied (solvability in 3D space, uniqueness under additional restrictions are proved). Quasi-stationary model (in 3D bounded domain, with special boundary conditions) is investigated in [12] (classic solutions are constructed). Complete model (no terms are omitted) in barotropic case is considered in [15] (in 3D bounded domain).

For models with temperature we have almost nothing. We can mention only 1D results [25] and [26] that deal with diagonal viscosity matrix in approximate models.

Other models of multifluids and mixtures were previously investigated in [8], [13], [16] (mixtures) and [1], [2] (heterogeneous multifluids).

As a summary, we conclude that the problem formulated in Subsections 2.1–2.3 is open.

3. Problems in modeling and mathematics. General model of bifluids formulated above contains some difficulties.

3.1. Problem 1. Even for barotropic case, for arbitrary $p_i(\rho_1,\rho_2)$ first energy estimate fails (and some other problems arise such as entropy production, cf. Subsection 3.3). Simple way out is to suppose that $p_1 = p_1(\rho_1)$ and $p_2 = p_2(\rho_2)$. We extend this idea to the model with temperature:

$$p_1 = p_1(\rho_1,\theta_1), \qquad p_2 = p_2(\rho_2,\theta_2).$$

3.2. Problem 2. Automatic extension of the theory of compressible NS to the theory of bifluids requires

$$\operatorname{div}\operatorname{div}\mathbb{P}^{(i)} = \mathrm{const}\cdot\Delta\operatorname{div}\boldsymbol{u}^{(i)},$$

but we have

$$\begin{bmatrix}\operatorname{div}\operatorname{div}\mathbb{P}^{(1)}\\ \operatorname{div}\operatorname{div}\mathbb{P}^{(2)}\end{bmatrix} = \begin{bmatrix}\nu_{11} & \nu_{12}\\ \nu_{21} & \nu_{22}\end{bmatrix}\begin{bmatrix}\Delta\operatorname{div}\boldsymbol{u}^{(1)}\\ \Delta\operatorname{div}\boldsymbol{u}^{(2)}\end{bmatrix}.$$

Hence, the method is to be developed, if we wish to consider not only diagonal matrices.

3.3. Problem 3. We need appropriate sign of entropy production

$$\sum_{i=1}^{2}\frac{\mathbb{P}^{(i)}:\mathbb{D}(\boldsymbol{u}^{(i)})}{\theta_i} + \sum_{i=1}^{2}\frac{2k_i|\nabla\theta_i|^2}{\theta_i^2} + a\sum_{i=1}^{2}\frac{|\boldsymbol{u}^{(2)}-\boldsymbol{u}^{(1)}|^2}{\theta_i} + b\frac{(\theta_1-\theta_2)^2}{\theta_1\theta_2} -$$

$$-\sum_{i=1}^{2}\operatorname{div}\frac{\boldsymbol{q}^{(i)}}{\theta_i} = \sum_{i=1}^{2}\rho_i\frac{d_i s_i}{dt} \geqslant -\sum_{i=1}^{2}\operatorname{div}\frac{\boldsymbol{q}^{(i)}}{\theta_i},$$

so we must provide

$$\sum_{i=1}^{2}\frac{\mathbb{P}^{(i)}:\mathbb{D}(\boldsymbol{u}^{(i)})}{\theta_i} + \sum_{i=1}^{2}\frac{2k_i|\nabla\theta_i|^2}{\theta_i^2} + a\sum_{i=1}^{2}\frac{|\boldsymbol{u}^{(2)}-\boldsymbol{u}^{(1)}|^2}{\theta_i} + b\frac{(\theta_1-\theta_2)^2}{\theta_1\theta_2} \geqslant 0,$$

and we obtain obvious requirements $k_i \geqslant 0$, $a \geqslant 0$, $b \geqslant 0$ and not so evident inequality

$$\sum_{i=1}^{2} \frac{\mathbb{P}^{(i)} : \mathbb{D}(\boldsymbol{u}^{(i)})}{\theta_i} \geqslant 0 \tag{7}$$

in order to (a priori) ensure Second law of thermodynamics. It is easy to provide positiveness of the sum

$$\mathbb{P}^{(1)} : \mathbb{D}(\boldsymbol{u}^{(1)}) + \mathbb{P}^{(2)} : \mathbb{D}(\boldsymbol{u}^{(2)}) \tag{8}$$

via positiveness of viscosity matrices (see Subsection 2.2), but (7) and positiveness of (8) are not the same. Hence, in general, the model needs verification. As regards classic monographs, this problem did not arise there: [28] deals with models without temperatures (however, very generally), and [22] considers models with temperatures, but details are given only for the model with one temperature.

We consider the model that compiles their ideas, and to avoid difficulties we see two obvious ways:

1. *Model with one temperature.* Then (7) immediately follows from positiveness of (8).
2. *Two-temperature model, but omitting terms* $\mathbb{P}^{(i)} : \mathbb{D}(\boldsymbol{u}^{(i)})$ *in the energy equations.* This is important physically (firstly) and mathematically (secondly). Otherwise serious study of modeling (firstly) and mathematics (secondly) is needed.

3.4. Announcement. In two formulated cases we face with only one difficulty (mentioned in Subsection 3.2 as *Problem 2*): how to generalize the technique of effective viscous flux to the tensorial case.

We study both versions (however, for steady equations, that is not essential, as we believe). The result is that the theory of one-constituent gas generalizes to bifluids, but with difficulties for arbitrary viscosity matrices. Now we will explain how the result is proved mathematically. It does not matter here which model to choose in order to show the matrix version of effective viscous flux method. Let us select the second (two-temperature) model.

4. Bifluid with two temperatures, steady 3D flows: mathematics. Precise mathematical formulation of the problem is as follows.

4.1. Statement of the problem and the result. A bounded domain $\Omega \subset \mathbb{R}^3$ with $\partial\Omega \in C^2$ is given. We are trying to find

- vector fields of velocities $\boldsymbol{u}^{(i)}$, $i = 1, 2$,
- scalar fields of densities $\rho_i \geqslant 0$, $i = 1, 2$,
- scalar fields of temperatures $\theta_i > 0$, $i = 1, 2$,

which obey the following equations, boundary conditions, and mass assignments:

$$\operatorname{div}(\rho_i \boldsymbol{u}^{(i)}) = 0 \quad \text{in}\,\Omega, \quad i = 1, 2,$$

$$\sum_{j=1}^{2} L_{ij}\boldsymbol{u}^{(j)} + \operatorname{div}(\rho_i \boldsymbol{u}^{(i)} \otimes \boldsymbol{u}^{(i)}) + \nabla p_i = \boldsymbol{J}^{(i)} + \rho_i \boldsymbol{f}^{(i)} \quad \text{in}\,\Omega, \quad i = 1, 2,$$

$$\operatorname{div}(\rho_i\theta_i \boldsymbol{u}^{(i)}) + \operatorname{div}\boldsymbol{q}^{(i)} = -\rho_i\theta_i \operatorname{div}\boldsymbol{u}^{(i)} + \Gamma_i \quad \text{in}\,\Omega, \quad i = 1, 2,$$

$$\boldsymbol{u}^{(i)} = 0 \quad \text{at}\,\partial\Omega, \quad i = 1,2,$$

$$k_i(\theta_i)\nabla\theta_i \cdot \boldsymbol{n} + L(\theta_i)(\theta_i - \widehat{\theta}) = 0 \quad \text{at}\,\partial\Omega, \quad i = 1,2,$$

$$\int\limits_\Omega \rho_i\,d\boldsymbol{x} = M_i > 0, \quad i = 1,2.$$

Here

$$L_{ij} = -\mu_{ij}\Delta - (\lambda_{ij} + \mu_{ij})\nabla\mathrm{div}, \quad i,j = 1,2,$$

so that $\mathrm{div}\,\mathbb{P}^{(i)} = -\sum\limits_{j=1}^{2} L_{ij}\boldsymbol{u}^{(j)}$, $i = 1,2$;

$$p_i = \rho_i^\gamma + \rho_i\theta_i, \quad i = 1,2,$$

$$\boldsymbol{J}^{(i)} = (-1)^{i+1}a(\boldsymbol{u}^{(2)} - \boldsymbol{u}^{(1)}), \quad i = 1,2,$$

$$\boldsymbol{q}^{(i)} = -k_i\nabla\theta_i, \quad i = 1,2,$$

$$\Gamma_i = (-1)^{i+1}b(\theta_2 - \theta_1) + \frac{a}{2}|\boldsymbol{u}^{(1)} - \boldsymbol{u}^{(2)}|^2, \quad i = 1,2,$$

$$k_i(\theta_i) = 1 + \theta_i^m,$$

$$L(\theta_i) = 1 + \theta_i^{m-1},$$

$\widehat{\theta} > 0$ is a given function, $\boldsymbol{n}$ is the unit outer normal to the boundary.

Viscosity matrices are supposed to obey (6), that implies (see (4)) for some $C(\Lambda, M) > 0$:

$$\sum_{i,j=1}^{2}\int\limits_\Omega L_{ij}\boldsymbol{u}^{(j)} \cdot \boldsymbol{u}^{(i)}\,d\boldsymbol{x} \geqslant C\sum_{i=1}^{2}\int\limits_\Omega |\nabla \otimes \boldsymbol{u}^{(i)}|^2\,d\boldsymbol{x}.$$

Definition. *Generalized solution of the problem is a group of functions:*

- *nonnegative $\rho_i \in L_{2\gamma}(\Omega)$, $i = 1,2$,*
- *positive $\theta_i \in W_2^1(\Omega)\bigcap L_{3m}(\Omega)\bigcap L_{2m}(\partial\Omega)$, $i = 1,2$,*
- *and vectorial $\boldsymbol{u}^{(i)} \in \overset{\circ}{W}{}_2^1(\Omega)$, $i = 1,2$,*

such that

- *for all differentiable functions G_i with bounded derivatives $G_i' \in C(\mathbb{R})$, $i = 1,2$ and arbitrary test functions $\psi_i \in C^\infty(\overline{\Omega})$, $i = 1,2$ the following identities hold $(i = 1,2)$*

$$\int\limits_\Omega \Big(G_i(\rho_i)\boldsymbol{u}^{(i)} \cdot \nabla\psi_i + (G_i(\rho_i) - G_i'(\rho_i)\rho_i)\,\psi_i\,\mathrm{div}\,\boldsymbol{u}^{(i)}\Big)\,d\boldsymbol{x} = 0,$$

i. e. the densities are renormalized solutions of the continuity equations;

- *for arbitrary test vector fields* $\boldsymbol{\varphi}^{(i)} \in C_0^\infty(\Omega)$, $i = 1, 2$ *the following identities hold*

$$\sum_{j=1}^{2} \left[\mu_{ij} \int_\Omega (\nabla \otimes \boldsymbol{u}^{(j)}) : (\nabla \otimes \boldsymbol{\varphi}^{(i)})\, d\boldsymbol{x} + (\lambda_{ij} + \mu_{ij}) \int_\Omega \operatorname{div} \boldsymbol{u}^{(j)} \operatorname{div} \boldsymbol{\varphi}^{(i)}\, d\boldsymbol{x} \right] -$$

$$- \int_\Omega \left(\rho_i \boldsymbol{u}^{(i)} \otimes \boldsymbol{u}^{(i)} \right) : (\nabla \otimes \boldsymbol{\varphi}^{(i)})\, d\boldsymbol{x} = \int_\Omega \rho_i^\gamma \operatorname{div} \boldsymbol{\varphi}^{(i)}\, d\boldsymbol{x} +$$

$$+ \int_\Omega \rho_i \theta_i \operatorname{div} \boldsymbol{\varphi}^{(i)}\, d\boldsymbol{x} + \int_\Omega (\boldsymbol{J}^{(i)} + \rho_i \boldsymbol{f}^{(i)}) \cdot \boldsymbol{\varphi}^{(i)}\, d\boldsymbol{x}, \quad i = 1, 2;$$

- *for arbitrary test functions* $\eta_i \in C^\infty(\overline{\Omega})$, $i = 1, 2$ *the following identities hold*

$$- \int_\Omega \rho_i \theta_i \boldsymbol{u}^{(i)} \cdot \nabla \eta_i\, d\boldsymbol{x} + \int_{\partial\Omega} L(\theta_i)(\theta_i - \widehat{\theta}) \eta_i\, d\sigma + \int_\Omega k_i(\theta_i) \nabla \theta_i \cdot \nabla \eta_i\, d\boldsymbol{x} =$$

$$= - \int_\Omega \rho_i \theta_i \operatorname{div} \boldsymbol{u}^{(i)} \eta_i\, d\boldsymbol{x} + \int_\Omega \Gamma_i \eta_i\, d\boldsymbol{x}, \quad i = 1, 2.$$

THEOREM 1. *Let* $\gamma > 3$ *and*

$$m > \frac{6(\gamma - 1)(2\gamma - 1)}{(\gamma - 3)(6\gamma - 1)}.$$

Then for any $M_i > 0$, $\boldsymbol{f}^{(i)} \in C(\overline{\Omega})$, $i = 1, 2$, $\widehat{\theta} \in C^1(\partial\Omega)$, $\widehat{\theta} > 0$ *the problem formulated in Subsection 4.1 admits at least one generalized solution.*

REMARK 1. $\gamma > 3$ is not realistic physically but $\gamma > 7/3$ is not hard to reach without radical change of the methods, using the same ideas as in [20]. Moreover, further decrease of lower bound (e. g. $\gamma = 5/3$) is also possible, using the same arguments as in [23]. Now we do not concentrate on this aspect.

4.2. Regularization. Solution is constructed via elliptic regularization (similar to [20] for NSF):

$$-\varepsilon \Delta \rho_i^\varepsilon + \operatorname{div}(\rho_i^\varepsilon \boldsymbol{u}_\varepsilon^{(i)}) + \varepsilon \rho_i^\varepsilon = \varepsilon \frac{M_i}{|\Omega|} \quad \text{in}\, \Omega, \quad i = 1, 2,$$

$$\sum_{j=1}^{2} L_{ij} \boldsymbol{u}_\varepsilon^{(j)} + \frac{\varepsilon}{2} \rho_i^\varepsilon \boldsymbol{u}_\varepsilon^{(i)} + \frac{\varepsilon}{2} \frac{M_i}{|\Omega|} \boldsymbol{u}_\varepsilon^{(i)} + \frac{1}{2} \rho_i^\varepsilon (\boldsymbol{u}_\varepsilon^{(i)} \cdot \nabla) \boldsymbol{u}_\varepsilon^{(i)} +$$

$$+ \frac{1}{2} \operatorname{div}(\rho_i^\varepsilon \boldsymbol{u}_\varepsilon^{(i)} \otimes \boldsymbol{u}_\varepsilon^{(i)}) + \nabla p_i^\varepsilon = \boldsymbol{J}_\varepsilon^{(i)} + \rho_i^\varepsilon \boldsymbol{f}^{(i)} \quad \text{in}\, \Omega, \quad i = 1, 2,$$

$$\operatorname{div}(\rho_i^\varepsilon \theta_i^\varepsilon \boldsymbol{u}_\varepsilon^{(i)}) - \operatorname{div}\left(k_i(\theta_i^\varepsilon)\frac{\varepsilon+\theta_i^\varepsilon}{\theta_i^\varepsilon}\nabla\theta_i^\varepsilon\right) = -\rho_i^\varepsilon\theta_i^\varepsilon \operatorname{div}\boldsymbol{u}_\varepsilon^{(i)} + \Gamma_i^\varepsilon \quad \text{in}\,\Omega, \quad i=1,2,$$

$$\boldsymbol{u}_\varepsilon^{(i)} = 0, \quad \nabla\rho_i^\varepsilon\cdot\boldsymbol{n} = 0 \quad \text{at}\,\partial\Omega, \quad i=1,2,$$

$$k_i(\theta_i^\varepsilon)\frac{\varepsilon+\theta_i^\varepsilon}{\theta_i^\varepsilon}\nabla\theta_i^\varepsilon\cdot\boldsymbol{n} + \varepsilon\ln\theta_i^\varepsilon + L(\theta_i^\varepsilon)(\theta_i^\varepsilon - \widehat{\theta}) = 0 \quad \text{at}\,\partial\Omega, \quad i=1,2,$$

$$\int\limits_\Omega \rho_i^\varepsilon\, d\boldsymbol{x} = M_i, \quad i=1,2,$$

where $p_i^\varepsilon = (\rho_i^\varepsilon)^\gamma + \rho_i^\varepsilon\theta_i^\varepsilon$, $\boldsymbol{J}_\varepsilon^{(i)} = (-1)^{i+1}a(\boldsymbol{u}_\varepsilon^{(2)} - \boldsymbol{u}_\varepsilon^{(1)})$,

$$\Gamma_i^\varepsilon = (-1)^{i+1}b(\theta_2^\varepsilon - \theta_1^\varepsilon) + \frac{a}{2}|\boldsymbol{u}_\varepsilon^{(1)} - \boldsymbol{u}_\varepsilon^{(2)}|^2, \quad i=1,2, \quad \varepsilon\in(0,1].$$

Initial steps, namely,

- performing estimates for initial and regularized problems,
- solving the regularized problem (strong solution via fixed point argument)

are rather standard (however, laborious).

Properties of regularized solution are written compactly as follows

$$\sum_{i=1}^{2}\Big(\|\rho_i^\varepsilon\|_{L_{2\gamma}(\Omega)} + \|\boldsymbol{u}_\varepsilon^{(i)}\|_{\overset{\circ}{W}{}_2^1(\Omega)} + \|\varepsilon\nabla\rho_i^\varepsilon\|_{L_{\frac{6\gamma}{\gamma+3}}(\Omega)} + \|\theta_i^\varepsilon\|_{L_{3m}(\Omega)} +$$

$$+ \|\nabla\theta_i^\varepsilon\|_{L_2(\Omega)} + \int\limits_{\partial\Omega}(e^{s_i^\varepsilon} + e^{-s_i^\varepsilon})\,d\sigma + \|\nabla s_i^\varepsilon\|_{L_2(\Omega)}\Big) \leqslant C,$$

where $s_i^\varepsilon = \ln\theta_i^\varepsilon$, $i=1,2$, and $C>0$ depends only upon $\|\boldsymbol{f}^{(i)}\|_{C(\Omega)}$, $\|\widehat{\theta}\|_{C(\partial\Omega)}$, $\min\widehat{\theta}$, λ_{ij}, μ_{ij}, m, γ, Ω, a and M_i (and does not depend on ε).

4.3. Limit. While passing to the limit as $\varepsilon\to 0$ we face with only one grave difficulty, namely to show that

$$\overline{\rho_i^\gamma} = \rho_i^\gamma \quad \text{(for the momentum equations)}$$

and

$$\overline{\rho_i \operatorname{div}\boldsymbol{u}^{(i)}} = \rho_i\operatorname{div}\boldsymbol{u}^{(i)} \quad \text{(for the energy equations)},$$

for what it suffices to show strong convergence of ρ_i.

5. Tensorial version of effective viscous flux argument. We attract our attention to new aspects (arising for bifluids) in the technique of effective viscous flux.

5.1. General scheme. It is easy to show the scheme step by step:

Step 1. In contrast to mono-fluid systems, we have not one but four communicative relations (effective viscous flux identities)

$$\overline{\rho_j(\rho_i^\gamma + \rho_i\theta_i - \nu_{i1}\mathrm{div}\boldsymbol{u}^{(1)} - \nu_{i2}\mathrm{div}\boldsymbol{u}^{(2)})} = \rho_j(\overline{\rho_i^\gamma} + \rho_i\theta_i - \nu_{i1}\mathrm{div}\boldsymbol{u}^{(1)} - \nu_{i2}\mathrm{div}\boldsymbol{u}^{(2)}),$$

$i, j = 1, 2$, but with inconvenient mixed products such as $\rho_1\mathrm{div}\boldsymbol{u}^{(2)}$ etc., even if $i = j$. Proof of Step 1 is given in Subsection 5.2.

Step 2. Renormalization of the continuity equations entails

$$\int\limits_\Omega \rho_i\mathrm{div}\boldsymbol{u}^{(i)}d\boldsymbol{x} = 0, \quad i = 1, 2,$$

while approximate continuity equations provide appropriate inequalities

$$\int\limits_\Omega \overline{\rho_i\mathrm{div}\boldsymbol{u}^{(i)}}d\boldsymbol{x} \leqslant 0, \quad i = 1, 2.$$

Step 3. Assume now that the matrix N is triangular, for instance, $\nu_{12} = 0$. Setting $i = j = 1$ in Step 1, we get a relation without mixed products

$$\overline{\rho_1(\rho_1^\gamma + \rho_1\theta_1 - \nu_{11}\mathrm{div}\boldsymbol{u}^{(1)})} = \rho_1(\overline{\rho_1^\gamma} + \rho_1\theta_1 - \nu_{11}\mathrm{div}\boldsymbol{u}^{(1)}),$$

and after integration over Ω via Step 2 we obtain

$$\int\limits_\Omega \overline{\rho_1(\rho_1^\gamma + \rho_1\theta_1)}d\boldsymbol{x} \leqslant \int\limits_\Omega \rho_1(\overline{\rho_1^\gamma} + \rho_1\theta_1)d\boldsymbol{x}.$$

Step 4. Exactly as in one-constituent case, for all $v \in L_{2\gamma}(\Omega)$, $v \geqslant 0$ we have due to the monotonicity

$$(\rho_1^\gamma + \rho_1\theta_1 - v^\gamma - v\theta_1)(\rho_1 - v) \geqslant 0,$$

and consequently

$$\int\limits_\Omega (\rho_1^\gamma + \rho_1\theta_1)\rho_1 d\boldsymbol{x} \geqslant \int\limits_\Omega (\rho_1^\gamma + \rho_1\theta_1)v d\boldsymbol{x} + \int\limits_\Omega (v^\gamma + v\theta_1)(\rho_1 - v)d\boldsymbol{x}$$

in notations before the limit.

Step 5. Now we combine all facts:

$$\int\limits_\Omega \rho_1(\overline{\rho_1^\gamma} + \rho_1\theta_1)d\boldsymbol{x} \overset{\text{Step 3}}{\geqslant} \int\limits_\Omega \overline{\rho_1(\rho_1^\gamma + \rho_1\theta_1)}d\boldsymbol{x} \overset{\text{Step 4}}{\geqslant}$$

$$\overset{\text{Step 4}}{\geqslant} \int\limits_\Omega v(\overline{\rho_1^\gamma} + \rho_1\theta_1)d\boldsymbol{x} + \int\limits_\Omega (v^\gamma + v\theta_1)(\rho_1 - v)d\boldsymbol{x},$$

i. e.

$$\int_\Omega (\rho_1 - v)(\overline{\rho_1^\gamma} + \rho_1\theta_1) d\boldsymbol{x} \geqslant \int_\Omega (\rho_1 - v)(v^\gamma + v\theta_1) d\boldsymbol{x}.$$

Here we set $v = \rho_1 + \alpha\psi$ with $\psi \in L_{2\gamma}(\Omega)$, $\psi \geqslant 0$, $\alpha \in \mathbb{R}^+$, and obtain

$$-\int_\Omega (\overline{\rho_1^\gamma} + \rho_1\theta_1)\psi d\boldsymbol{x} \geqslant -\int_\Omega [(\rho_1 + \alpha\psi)^\gamma + (\rho_1 + \alpha\psi)\theta_1]\psi d\boldsymbol{x}.$$

Tending $\alpha \to 0$ we get

$$\int_\Omega (\overline{\rho_1^\gamma} + \rho_1\theta_1)\psi d\boldsymbol{x} \leqslant \int_\Omega (\rho_1^\gamma + \rho_1\theta_1)\psi d\boldsymbol{x},$$

$$\text{i. e.} \qquad \int_\Omega (\overline{\rho_1^\gamma} - \rho_1^\gamma)\psi d\boldsymbol{x} \leqslant 0.$$

Since $\overline{\rho_1^\gamma} \geqslant \rho_1^\gamma$ and $\psi \geqslant 0$, we obtain $(\overline{\rho_1^\gamma} - \rho_1^\gamma)\psi = 0$ and (due to free choice of ψ) $\overline{\rho_1^\gamma} = \rho_1^\gamma$.

Step 6. Setting $i = 1$, $j = 2$ in Step 1 we get

$$\overline{\rho_2(\rho_1^\gamma + \rho_1\theta_1 - \nu_{11}\mathrm{div}\boldsymbol{u}^{(1)})} = \rho_2(\overline{\rho_1^\gamma} + \rho_1\theta_1 - \nu_{11}\mathrm{div}\boldsymbol{u}^{(1)}),$$

and due to strong convergence of ρ_1 and good convergence of θ_i (weekly in $W_2^1(\Omega)$ and strongly in $L_{3m}(\Omega)$, $m > 2$) we deduce

$$\overline{\rho_2\mathrm{div}\boldsymbol{u}^{(1)}} = \rho_2\mathrm{div}\boldsymbol{u}^{(1)}.$$

Step 7. Setting $i = 2$, $j = 2$ in Step 1 we come to the relation

$$\overline{\rho_2(\rho_2^\gamma + \rho_2\theta_2 - \nu_{21}\mathrm{div}\boldsymbol{u}^{(1)} - \nu_{22}\mathrm{div}\boldsymbol{u}^{(2)})} = \rho_2(\overline{\rho_2^\gamma} + \rho_2\theta_2 - \nu_{21}\mathrm{div}\boldsymbol{u}^{(1)} - \nu_{22}\mathrm{div}\boldsymbol{u}^{(2)}),$$

and due to Step 6 this implies

$$\overline{\rho_2(\rho_2^\gamma + \rho_2\theta_2 - \nu_{22}\mathrm{div}\boldsymbol{u}^{(2)})} = \rho_2(\overline{\rho_2^\gamma} + \rho_2\theta_2 - \nu_{22}\mathrm{div}\boldsymbol{u}^{(2)}),$$

and then we repeat Steps 3–5 and conclude $\overline{\rho_2^\gamma} = \rho_2^\gamma$.

5.2. Proof of communicative relations (Step 1 in Subsection 5.1). We are about to prove that

$$\overline{\rho_j^\beta(\rho_i^\gamma + \rho_i\theta_i - \nu_{i1}\mathrm{div}\boldsymbol{u}^{(1)} - \nu_{i2}\mathrm{div}\boldsymbol{u}^{(2)})} =$$

$$= \overline{\rho_j^\beta} \cdot (\overline{\rho_i^\gamma} + \rho_i\theta_i - \nu_{i1}\mathrm{div}\boldsymbol{u}^{(1)} - \nu_{i2}\mathrm{div}\boldsymbol{u}^{(2)}), \quad i = 1, 2$$

(in Subsection 5.1, Step 1, we used only the case $\beta = 1$). This fact is not a surprise due to the theory of mono-fluid systems, we are just to deal with upper indices which mark corresponding constituents.

Substep 1.1. Denote

$$\mathbb{P}^{(ik)} = \lambda_{ik} \mathrm{div}\, \boldsymbol{u}^{(k)} \mathbb{I} + 2\mu_{ik} \mathbb{D}(\boldsymbol{u}^{(k)}),$$

so that

$$\mathbb{P}^{(i)} = \sum_{k=1}^{2} \mathbb{P}^{(ik)}, \quad i = 1, 2.$$

For all functions α we have

$$\mathbb{P}^{(ik)} : (\nabla \otimes \nabla \alpha) \simeq \nu_{ik} \left(\mathrm{div}\, \boldsymbol{u}^{(k)} \right) \Delta \alpha,$$

where $\simeq$ stands for equality modulo terms vanishing after integration over Ω.

Let us take $\alpha = \Delta^{-1}(\rho_j^\beta - \overline{\rho_j^\beta})$, so we get

$$-\mathrm{div}\mathbb{P}^{(i)} \cdot \nabla \Delta^{-1}(\rho_j^\beta - \overline{\rho_j^\beta}) \simeq \sum_{k=1}^{2} \nu_{ik} (\mathrm{div}\, \boldsymbol{u}^{(k)})(\rho_j^\beta - \overline{\rho_j^\beta}).$$

Remark 2. More exactly, we should take $\alpha = \tau \Delta^{-1}(\tau(\rho_j^\beta - \overline{\rho_j^\beta}))$ with a cut-off function τ in order to provide vanish near the boundary, and Δ^{-1} acts in the whole space, but this comment gives only lower order (negligible) terms, and we roughly set here $\tau = 1$ for more simple explanations.

Remark 3. There might be any weakly converging sequence instead of $\rho_j^\beta - \overline{\rho_j^\beta}$.

Substep 1.2. Due to the continuity equations we have

$$-\frac{\varepsilon}{2} \boldsymbol{u}^{(i)} \Delta \rho_i + \frac{\boldsymbol{u}^{(i)}}{2} \mathrm{div}(\rho_i \boldsymbol{u}^{(i)}) + \frac{\varepsilon}{2} \rho_i \boldsymbol{u}^{(i)} = \frac{\varepsilon}{2} \frac{M_i}{|\Omega|} \boldsymbol{u}^{(i)},$$

and the momentum equations read

$$-\mathrm{div}\mathbb{P}^{(i)} = \rho_i \boldsymbol{f}^{(i)} + \boldsymbol{J}^{(i)} - \nabla p_i - \mathrm{div}(\rho_i \boldsymbol{u}^{(i)} \otimes \boldsymbol{u}^{(i)}) + \frac{\varepsilon}{2} \boldsymbol{u}^{(i)} \Delta \rho_i - \varepsilon \rho_i \boldsymbol{u}^{(i)}.$$

Substep 1.3. Denoting $\mathrm{Comm(a, b)} = (\nabla \otimes \nabla \Delta^{-1} \mathrm{a})\mathrm{b} - \mathrm{a}(\nabla \otimes \nabla \Delta^{-1} \mathrm{b})$ and using one more time the continuity equations:

$$\nabla \Delta^{-1} \mathrm{div}(\rho_i \boldsymbol{u}^{(i)}) = \varepsilon \nabla \Delta^{-1} \left(\frac{M_i}{|\Omega|} - \rho_i \right) + \varepsilon \nabla \rho_i,$$

we compile from Substeps 1.1 and 1.2:

$$\left(\sum_{k=1}^{2}\nu_{ik}\mathrm{div}\,\boldsymbol{u}^{(k)} - p_i\right)(\rho_j^\beta - \overline{\rho_j^\beta}) \simeq$$

$$\simeq (\rho_i\boldsymbol{u}^{(i)}\otimes\boldsymbol{u}^{(i)}) : \left[\nabla\otimes\nabla\Delta^{-1}(\rho_j^\beta - \overline{\rho_j^\beta})\right] +$$

$$+\left[\nabla\Delta^{-1}(\rho_j^\beta - \overline{\rho_j^\beta})\right]\cdot[\rho_i\boldsymbol{f}^{(i)} + \boldsymbol{J}^{(i)} + \frac{\varepsilon}{2}\boldsymbol{u}^{(i)}\Delta\rho_i - \varepsilon\rho_i\boldsymbol{u}^{(i)}] \simeq$$

$$\simeq \boldsymbol{u}^{(i)}\mathrm{Comm}(\rho_j^\beta - \overline{\rho_j^\beta}, \rho_i\boldsymbol{u}^{(i)}) + \left[\nabla\Delta^{-1}(\rho_j^\beta - \overline{\rho_j^\beta})\right]\cdot(\rho_i\boldsymbol{f}^{(i)} + \boldsymbol{J}^{(i)}) - \tag{9}$$

$$-\left[\frac{1}{2}(\nabla\otimes\boldsymbol{u}^{(i)})\varepsilon\nabla\rho_i + \varepsilon\rho_i\boldsymbol{u}^{(i)}\right]\cdot\nabla\Delta^{-1}(\rho_j^\beta - \overline{\rho_j^\beta}) -$$

$$-\frac{1}{2}\left((\nabla\otimes\nabla\Delta^{-1}(\rho_j^\beta - \overline{\rho_j^\beta}))\boldsymbol{u}^{(i)}\right)\cdot\varepsilon\nabla\rho_i +$$

$$+(\rho_j^\beta - \overline{\rho_j^\beta})\boldsymbol{u}^{(i)}\cdot\left(\varepsilon\nabla\Delta^{-1}\left(\frac{M_i}{|\Omega|} - \rho_i\right) + \varepsilon\nabla\rho_i\right).$$

We have at our disposal good enough boundedness and convergence:

$$\rho_i \in L_{2\gamma}(\Omega), \quad \boldsymbol{u}^{(i)} \in L_6(\Omega), \quad \boldsymbol{f}^{(i)} \in L_\infty(\Omega),$$

$$\rho_i \overset{w}{\to} \quad \text{in } L_{2\gamma}(\Omega), \quad \boldsymbol{u}^{(i)} \overset{s}{\to} \quad \text{in } L_{6-\delta}(\Omega), \quad \varepsilon\nabla\rho_i \overset{s}{\to} 0 \quad \text{in } L_{\frac{6\gamma}{\gamma+3}-\delta}(\Omega),$$

and we have well-known communicative property

$$a_k \overset{w}{\to} a \quad \text{in } L_p(\Omega), \quad b_k \overset{w}{\to} 0 \quad \text{in } L_q(\Omega) \quad \Longrightarrow \quad \mathrm{Comm}(a_k, b_k) \overset{w}{\to} 0 \quad \text{in } L_{\frac{pq}{p+q}}(\Omega).$$

This suffices to conclude that the right-hand side in (9) tends to zero as soon as $\beta < \dfrac{4\gamma}{3} - 1$.

6. Arbitrary viscosity matrix. Let us present several observations concerning this open problem, e. g. for the model without temperature.

Denote $\boldsymbol{\rho}^\beta = \begin{bmatrix}\rho_1^\beta\\ \rho_2^\beta\end{bmatrix}$, $\mathbf{div}\boldsymbol{u} = \begin{bmatrix}\mathrm{div}\,\boldsymbol{u}^{(1)}\\ \mathrm{div}\,\boldsymbol{u}^{(2)}\end{bmatrix}$, then proved communicative relations take the form

$$\overline{\boldsymbol{\rho}^\beta\otimes(\boldsymbol{\rho}^\gamma - N\mathbf{div}\boldsymbol{u})} = \overline{\boldsymbol{\rho}^\beta}\otimes(\overline{\boldsymbol{\rho}^\gamma} - N\mathbf{div}\boldsymbol{u}).$$

For the tensor $\mathbb{R} := \overline{\boldsymbol{\rho}^\beta\otimes\boldsymbol{\rho}^\gamma} - \overline{\boldsymbol{\rho}^\beta}\otimes\overline{\boldsymbol{\rho}^\gamma}$ we have the equality

$$\mathbb{R} = \overline{\boldsymbol{\rho}^\beta\otimes N\mathbf{div}\boldsymbol{u}} - \overline{\boldsymbol{\rho}^\beta}\otimes N\mathbf{div}\boldsymbol{u},$$

and we are about to deduce strong convergence of the densities which is equivalent to $\mathbb{R} = 0$ or at least

$$R_{ii} \equiv \overline{\rho_i^{\beta+\gamma}} - \overline{\rho_i^\beta}\cdot\overline{\rho_i^\gamma} = 0. \tag{10}$$

Up to now we know only that $R_{ii} \geqslant 0$.

We obviously get (* stands for matrix transposition)

$$\int_\Omega \mathbb{R} d\boldsymbol{x} = AN^*, \quad \text{i. e.} \quad \int_\Omega \mathbb{R} Z d\boldsymbol{x} = A, \tag{11}$$

where $Z = (N^*)^{-1} > 0$ and

$$A_{sr} = \int_\Omega \left(\overline{\rho_s^\beta \operatorname{div} \boldsymbol{u}^{(r)}} - \overline{\rho_s^\beta} \operatorname{div} \boldsymbol{u}^{(r)} \right) d\boldsymbol{x}. \tag{12}$$

The question is what to do next. Specific β's have their own merits and demerits:

- If $\beta = \gamma$ then

$$\mathbb{R}^* = \mathbb{R} \tag{13}$$

  but A is hard to analyze (namely, the second term in the integral (12) presents difficulties).
- If $\beta = 1$ then

$$A_{ii} \leqslant 0, \quad i = 1, 2, \tag{14}$$

  but (13) is questionable.

Problem is to conclude from (11) that (10) holds.

Remark 4. It is easy to show that [(13), (14), $\nu_{12}\nu_{21} \leqslant 0$] $\Longrightarrow$ (10).

7. Bifluid with one temperature, steady 3D flows: mathematics. The second model implies coincidence of temperatures $\theta_1 = \theta_2 = \theta$, but the energy equation (now it is one equation) is complete:

$$\frac{\partial}{\partial t}\left(\sum_{i=1}^{2} \rho_i E_i\right) + \operatorname{div}\Big(\sum_{i=1}^{2} \rho_i E_i \boldsymbol{u}^{(i)}\Big) + \operatorname{div}\Big(\sum_{i=1}^{2} p_i \boldsymbol{u}^{(i)}\Big) - \operatorname{div}\Big(\sum_{i=1}^{2} \mathbb{P}^{(i)} \boldsymbol{u}^{(i)}\Big) =$$

$$= \sum_{i=1}^{2} \rho_i \boldsymbol{f}^{(i)} \cdot \boldsymbol{u}^{(i)} - 2\operatorname{div}\boldsymbol{q}, \quad i = 1, 2,$$

where $\boldsymbol{q} = \boldsymbol{q}^{(1)} = \boldsymbol{q}^{(2)} = -k\nabla\theta$, and k is the thermal conductivity coefficient.

We seek for

- vector fields of velocities $\boldsymbol{u}^{(i)}$, $i = 1, 2$,
- scalar fields of densities $\rho_i \geqslant 0$, $i = 1, 2$
- and scalar field of temperature $\theta > 0$

which verify the following equations, boundary conditions and mass assignments:

$$\operatorname{div}(\rho_i \boldsymbol{u}^{(i)}) = 0 \quad \text{in}\,\Omega, \quad i = 1, 2,$$

$$\sum_{j=1}^{2} L_{ij} \boldsymbol{u}^{(j)} + \operatorname{div}(\rho_i \boldsymbol{u}^{(i)} \otimes \boldsymbol{u}^{(i)}) + \nabla p_i = \rho_i \boldsymbol{f}^{(i)} + \boldsymbol{J}^{(i)} \quad \text{in}\,\Omega, \quad i = 1, 2,$$

$$\operatorname{div}\Big(\sum_{i=1}^{2}\rho_i E_i \boldsymbol{u}^{(i)}\Big) + \operatorname{div}\Big(\sum_{i=1}^{2} p_i \boldsymbol{u}^{(i)}\Big) - \operatorname{div}\Big(\sum_{i=1}^{2}\mathbb{P}^{(i)}\boldsymbol{u}^{(i)}\Big) =$$

$$= \sum_{i=1}^{2}\rho_i \boldsymbol{f}^{(i)}\cdot\boldsymbol{u}^{(i)} - 2\operatorname{div}\boldsymbol{q} \quad \text{in}\,\Omega, \quad i=1,2,$$

$$\boldsymbol{u}^{(i)} = 0 \quad \text{at}\,\partial\Omega, \quad i=1,2,$$

$$2k(\theta)\nabla\theta\cdot\boldsymbol{n} + L(\theta)(\theta-\widehat{\theta}) = 0 \quad \text{at}\,\partial\Omega,$$

$$\int_\Omega \rho_i\, d\boldsymbol{x} = M_i = \text{const} > 0, \quad i=1,2.$$

Here $E_i = \frac{1}{2}|\boldsymbol{u}^{(i)}|^2 + U_i$, $p_i = \rho_i^\gamma + \rho_i\theta$, $U_i = \frac{1}{\gamma-1}\rho_i^{\gamma-1} + \theta$, $i=1,2$, $\boldsymbol{q} = -k(\theta)\nabla\theta$, $k(\theta) = 1+\theta^m$, $L(\theta) = 1+\theta^{m-1}$, all other terms are defined as in Subsection 4.1.

Viscosity matrices are supposed to satisfy

$$M > 0, \qquad 3\Lambda + 2M \geqslant 0$$

(cf. (5)), that implies (cf. (3) and (4))

$$\sum_{i=1}^{2}\mathbb{P}^{(i)} : (\nabla\otimes\boldsymbol{u}^{(i)}) \geqslant 0,$$

$$\sum_{i,j=1}^{2}\int_\Omega L_{ij}\boldsymbol{u}^{(j)}\cdot\boldsymbol{u}^{(i)}\, d\boldsymbol{x} \geqslant C\sum_{i=1}^{2}\int_\Omega |\nabla\otimes\boldsymbol{u}^{(i)}|^2\, d\boldsymbol{x}$$

with a constant $C > 0$, i. e. now we require pointwise positiveness of this form in order to provide positive entropy production.

Definition of generalized solution of the problem is similar to the first problem (see Subsection 4.1), and the result is as follows.

Theorem 2. *Let* $\gamma > 3$ *and*

$$m > \frac{2}{3}\cdot\frac{6\gamma^2 - 7\gamma + 3}{2\gamma^2 - 5\gamma + 1}.$$

Then for any $M_i > 0$, $\boldsymbol{f}^{(i)} \in C(\overline{\Omega})$, $i=1,2$, $\widehat{\theta} \in C^1(\partial\Omega)$, $\widehat{\theta} > 0$ *the problem formulated in Section 7 admits at least one generalized solution.*

8. Conclusions.

8.1. Summary.

- At least two models of viscous compressible bifluids with complete or almost complete form of equations were found such that the results for mono-fluid systems are to be repeated (except technical details such as value of γ and essential difficulties in viscosity matrices).
- However, for heat-conductive gases (with one constituent) γ is a problem as well.
- When we omit dissipative terms, it is forced by problems in modeling. Beside that, the models are complete.

8.2. Open problems.

- Non-stationary versions of results (not difficult, as we believe).
- Arbitrary viscosity matrices, included physical situation of symmetric (but not diagonal) matrices. Possibly, it is a nontrivial mathematical problem, because the method of effective viscous fluxes is very specific and may not work in a general tensorial case.
- Two-temperature model with complete form of the energy equations, at least for convenient viscosity matrix. The main problem here is (cf. Subsection 3.3) to provide the relation

$$\sum_{i=1}^{2} \int_{\Omega} \frac{\mathbb{P}^{(i)} : (\nabla \otimes \boldsymbol{u}^{(i)})}{\theta_i} \, d\boldsymbol{x} \geqslant 0, \quad i = 1, 2,$$

  which is obvious only for diagonal viscosity matrices. This problem is not only mathematical but also physical (and maybe: physical rather than mathematical).

8.3. Detailed formulations and proofs. Purposes of the paper were to make a review of the area and to give a sketch of authors' results in it. Further details can be found in our papers [14] and [18].

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