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A REMARK ON TUBE DOMAINS

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Let D be a tube domain (i.e. a bounded symmetric domain of tube type). In [D, §1], Deligne gives a description of D as the moduli space of certain Hodge structures. Using these methods, we show that D parametrizes a *canonical* variation $\mathcal{V}$ of polarized real Hodge structures, which is effective of weight $=$ rank(D) and enjoys several remarkable properties. We end with some speculation on how $\mathcal{V}$ might appear in algebraic geometry.

1.

Let D be a simple tube domain, and let G be (the real points of) the simply-connected, simple real algebraic group which acts transitively on D. Let K be a maximal compact subgroup of G; then K fixes a unique point of D and $D \simeq G/K$. The integer $n = \mathrm{rank}(D)$ is defined to be the real rank of G, and the integer $d = \dim(D)$ is defined to be the complex dimension of the domain, which is one-half the real dimension of G/K. The quotient $2d/n$ is always an integer [S, p. 37].

Since D is tube, there is a self-adjoint homogeneous cone C in a Euclidean space N over $\mathbb{R}$ such that $D \simeq N + iC \subseteq N_{\mathbb{C}}$ [S, p. 128].

We recall [D, 1.2.6] that the simple bounded symmetric domains are classified by pairs (Δ, v), where Δ is a connected Dynkin diagram and v is a special vertex of Δ which is equivalent to the extended vertex μ under an automorphism of the affine diagram $\Delta' = \Delta \cup \{\mu\}$ [T, pp. 33-34, p. 53]. The domain is tube if v is fixed by the opposition involution of Δ.

We now tabulate the relevant pairs (Δ, v), where the special vertex is circled, and give the groups G and K associated to the tube domain D. We also describe the cone C, using the notation $S_n(F)^+$ for the cone of positive definite, $n \times n$ Hermitian symmetric matrices over the $\mathbb{R}$-algebras $F = \mathbb{R}, \mathbb{C}, \mathbb{H}$.

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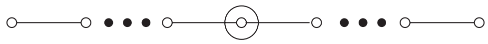
A_{2n-1}
$n \geq 1$



n^{th}vertex

$G = SU(n,n)$ — $\operatorname{rank}(D) = n$
$K = S(U(n) \times U(n))$ — $\dim(D) = n^2$
$C = S_n(\mathbb{C})^+$

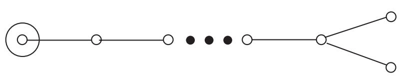
B_n
$n \geq 2$

$G = \operatorname{Spin}(2, 2n-1)$ — $\operatorname{rank}(D) = 2$
$K = \operatorname{Spin}(2) \times_{\mu_2} \operatorname{Spin}(2n-1)$ — $\dim(D) = 2n-1$
$C =$ light cone in Minkowski space $\mathbb{R}^{1,2n-2}$

C_n
$n \geq 1$

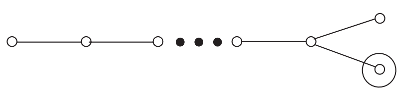
$G = Sp(2n, \mathbb{R})$ — $\operatorname{rank}(D) = n$
$K = U(n)$ — $\dim(D) = n(n+1)/2$
$C = S_n(\mathbb{R})^+$

$D_n^{\mathbb{R}}$
$n \geq 3$



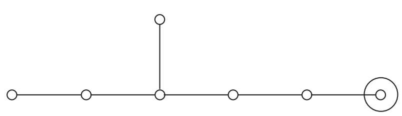
$G = \operatorname{Spin}(2, 2n-2)$ — $\operatorname{rank}(D) = 2$
$K = \operatorname{Spin}(2) \times_{\mu_2} \operatorname{Spin}(2n-2)$ — $\dim(D) = 2n-2$
$C =$ light cone in Minkowski space $\mathbb{R}^{1,2n-3}$

$D_{2n}^{\mathbb{H}}$
$n \geq 2$



$G = \operatorname{Spin}^*(4n)$ — $\operatorname{rank}(D) = n$
$K = U(1) \times_{\mu_n} SU(2n)$ — $\dim(D) = n(2n-1)$
$C = S_n(\mathbb{H})^+$

E_7



$G = E_{7,3}$ — $\operatorname{rank}(D) = 3$
$K = U(1) \times_{\mu_3} E_6$ — $\dim(D) = 27$
$C = S_3(\mathbb{O})^+ =$ exceptional cone
$\mathbb{O} =$ octonions $=$ Cayley numbers.

2.

The vertex v of Δ determines a conjugacy class of maximal parabolic subgroups $P = L \cdot N$ in G, with unipotent radical N abelian and Levi subgroup L a real form of K. The cone C is L-homogeneous in the real vector space N.

The vertex v also determines a fundamental, irreducible representation V of G over $\mathbb{R}$. The orbit of a highest weight vector in $\mathbb{P}(V)$ is the real projective variety G/P. We describe the fundamental representation V, in terms of the standard representation of G, in the table below.

Type	G	V	$\dim\ V$
A	$SU(n,n)$	$V_{\mathbb{C}} = \bigwedge^n \mathbb{C}^{2n}$	$\binom{2n}{n}$
B, D^R	$\mathrm{Spin}(2,m)$	$\mathbb{R}^{2,m}$ of $SO(2,m)$	$2+m$
C	$Sp(2n,\mathbb{R})$	$V \oplus \bigwedge^{n-2}\mathbb{R}^{2n} = \bigwedge^n \mathbb{R}^{2n}$	$\binom{2n}{n} - \binom{2n}{n-2}$
D^H	$\mathrm{Spin}^*(4n)$	unique $\frac{1}{2}$-spin which is real	2^{2n-1}
E	$E_{7,3}$	unique miniscule representation	56

3.

We will see that the fundamental representation V gives rise to a canonical variation of polarized real Hodge structures on D. To do this, following Deligne [D, §1], we must first realize V as a representation of a reductive group G_1, whose center contains $\mathbb{G}_m$ and whose derived group is G. We construct G_1 as follows.

Let ϵ be the unique element of order 2 in the center of G which is contained in the connected component of the center of K. We recall that $Z(K)^+ \simeq U(1)$ in all cases. Let G_1 be the quotient of $\mathbb{G}_m \times G$ by the central subgroup generated by the involution $-1 \times \epsilon$. Since ϵ acts as $(-1)^n$ on V, with $n = \mathrm{rank}(D)$, V extends uniquely to a representation of G_1 such that $\lambda \in \mathbb{G}_m$ acts by λ^{-n}.

Let $S = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m = \mathbb{G}_m \times U(1)/\langle -1 \times -1 \rangle$. A point of D determines a homomorphism $U(1) \hookrightarrow K \hookrightarrow G$, which is an oriented isomorphism from $U(1)$ to the connected component of the center of K. Since this maps the element -1 of $U(1)$ to the element ϵ of G, it determines a homomorphism

$$h : S \longrightarrow G_1 \tag{3.1}$$

which is the identity on $\mathbb{G}_m$. The G_1-conjugacy class X of h has two connected components, each of which is isomorphic to D.

Finally, the representation V of G_1, when combined with the morphism h of (3.1), gives rise to a polarized variation of real Hodge structures on

X (and hence on D), by the results of Deligne [D, Prop. 1.1.14], [M, Ch. II, Prop. 3.2]. We put $\mathcal{V} = V \otimes \mathcal{O}_D$; this is an equivariant holomorphic vector bundle on D with connection ∇. Since $\lambda \in \mathbb{G}_m$ acts as λ^{-n} on V, the Hodge structures associated to V are pure of weight $n = \operatorname{rank}(D)$.

4.

We now investigate the properties of $\mathcal{V}$ as a variation of Hodge structure. We recall that $\mathcal{V}$ has a filtration by holomorphic sub-bundles $\cdots \supset F^p\mathcal{V} \supset F^{p+1}\mathcal{V} \supset \cdots$, and that the quotient bundles

$$\mathcal{W}^{p,q} = F^p\mathcal{V}/F^{p+1}\mathcal{V} \qquad\qquad p+q=n$$

are equivariant vector bundles on X. On $D \simeq G/K$, these bundles correspond to the K-submodules $W^{p,q}$ of $V_{\mathbb{C}}$ on which elements $z \in U(1) = Z(K)^+$ act by the character $z^{-p}\overline{z}^{-q} = z^{q-p}$.

Proposition 4.1. *The variation of Hodge structures $\mathcal{V}$ is* effective *of weight n, so $\mathcal{W}^{p,q} = 0$ unless both $p, q \geq 0$.*

If $p, q \geq 0$ the equivariant vector bundle $\mathcal{W}^{p,q}$ is irreducible. Both $F^n\mathcal{V} = W^{n,0}$ and $\mathcal{V}/F^1\mathcal{V} = \mathcal{W}^{0,n}$ are holomorphic line bundles on X.

Proof. This results from a determination of the eigenspaces $W^{p,q}$ for the action of $U(1) \simeq Z(K)^+$ on $V_{\mathbb{C}}$, as representations of K. Only the characters $z^n, z^{n-2}, \cdots, z^{2-n}, z^{-n}$ occur, and we tabulate these representations below.

Type	K	$W^{p,q} \quad p,q \geq 0$	$\dim W^{p,q}$
A	$S(U(n) \times U(n))$	$(\bigwedge^p \mathbb{C}^n)^* \otimes (\bigwedge^q \mathbb{C}^n)^*$	$\binom{n}{p}\binom{n}{q}$
$B, D^{\mathbb{R}}$	$\mathrm{Spin}(2) \times_{\mu_2} \mathrm{Spin}(m)$	$W^{2,0} = \mathbb{C}(-2) \otimes \mathbb{C}$	1
		$W^{1,1} = \mathbb{C} \otimes \mathbb{C}^m$	m
		$W^{0,2} = \mathbb{C}(2) \otimes \mathbb{C}$	1
C	$U(n)$	Irreducible summand of $(\bigwedge^p \mathbb{C}^n)^* \otimes \bigwedge^q \mathbb{C}^n$ with highest weight $(= 2\omega_q)$	$\binom{n}{p}\binom{n}{q} - \binom{n}{p-1}\binom{n}{q-1}$
$D^{\mathbb{H}}$	$U(1) \times_{\mu_n} SU(2n)$	$\mathbb{C}(q-p) \otimes (\bigwedge^{2p} \mathbb{C}^{2n})^*$	$\binom{2n}{2p}$
E	$U(1) \times_{\mu_3} E_6$	$W^{3,0} = \mathbb{C}(-3) \otimes \mathbb{C}$	1
		$W^{2,1} = \mathbb{C}(-1) \otimes W_{27}$	27
		$W^{1,2} = \mathbb{C}(1) \otimes W_{27}^*$	27
		$W^{0,3} = \mathbb{C}(3) \otimes \mathbb{C}$	1

5.

The connection $\nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega^1_X$ satisfies Griffiths transversality: $\nabla(F^p\mathcal{V}) \subset F^{p-1}\mathcal{V} \otimes \Omega^1_X$. Hence, if Θ_X is the holomorphic tangent bundle of X, differentiating q times gives a morphism of equivariant vector bundles

$$\nabla^q : \mathrm{Sym}^q\Theta_X \longrightarrow \mathrm{Hom}(F^n\mathcal{V}, F^p\mathcal{V}/F^{p+1}\mathcal{V}), \tag{5.1}$$

where $p + q = n$.

Proposition 5.2. *The morphism ∇^q is surjective for all $0 \le q \le n$, and*

$$\nabla : \Theta_X \longrightarrow \mathit{Hom}(F^n\mathcal{V}, F^{n-1}\mathcal{V}/F^n\mathcal{V})$$

is an isomorphism.

Proof. Since $\mathrm{Hom}(F^n\mathcal{V}, F^p\mathcal{V}/F^{p+1}\mathcal{V}) \simeq \mathcal{W}^{p,q} \otimes (F^n\mathcal{V})^{-1}$ is an irreducible equivariant bundle, it suffices to check that $\nabla^q \neq 0$. This reduces to the study of the action of elements in $N^-_{\mathbb{C}} \subset G_{\mathbb{C}}$ on the eigenspace $W^{n,0}$ of $V_{\mathbb{C}}$ (cf. [D, Prop. 1.1.14]). We leave the details to the reader. Since $\dim(W^{n-1,1}) = \dim(X) = d$ and the map ∇ is surjective, it is an isomorphism.

The maps ∇^q defined in (5.1) give a surjection $f = \bigoplus_{q\ge 0}\nabla^q$ of vector bundles on X:

$$\mathrm{Sym}^\bullet(\Theta_X) = \bigoplus_{q\ge 0}\mathrm{Sym}^q\Theta_X \xrightarrow{f} \bigoplus_{q=0}^{n}\mathcal{W}^{p,q} \otimes (\mathcal{W}^{n,0})^{-1} \longrightarrow 0. \tag{5.3}$$

The kernel of f is a graded ideal $I = \bigoplus_{q\ge 2} I^q$ of $\mathrm{Sym}^\bullet(\Theta_X)$, probably generated by the irreducible equivariant bundle $I^2 = \ker(\nabla^2 : \mathrm{Sym}^2\Theta_X \longrightarrow \mathcal{W}^{n-2,2} \otimes (\mathcal{W}^{n,0})^{-1})$.

6.

If D is any tube domain, it admits a product decomposition $D = D_1 \times D_2 \times \cdots \times D_k$ into simple tube domains of the type classified in §1. The variation $\mathcal{V} = \mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \cdots \otimes \mathcal{V}_k$ of polarized real Hodge structures on D is effective of weight $n = \mathrm{rank}(D) = \sum_{i=1}^k n_i$, $F^n\mathcal{V} = F^{n_1}\mathcal{V}_1 \otimes F^{n_2}\mathcal{V}_2 \otimes \cdots \otimes F^{n_k}\mathcal{V}_k$ is a holomorphic line bundle on D, $\nabla^q : \mathrm{Sym}^q\Theta_D \longrightarrow \mathrm{Hom}(F^n\mathcal{V}, F^p\mathcal{V}/F^{p+1}\mathcal{V})$ is surjective for all $0 \le q \le n$, and $\nabla = \nabla^1$ is an isomorphism. These properties characterize $\mathcal{V}$ on D.

We end with some remarks on the irreducible representation $V = V_1 \otimes V_2 \otimes \cdots \otimes V_k$ of $G = G_1 \times G_2 \times \cdots \times G_k$ for the general tube domain D. Let

n be the real rank of G, and let $\{\gamma_1, \gamma_2, \cdots, \gamma_n\}$ be a system of strongly orthogonal, non-compact positive roots for a compact torus T in G [S, p. 60]. Then

$$\lambda = \frac{1}{2}(\gamma_1 + \gamma_2 + \cdots + \gamma_n)$$

is the highest weight for T on $V_{\mathbb{C}}$. In particular, the restriction of V to the subgroup $\mathrm{SL}_2(\mathbb{R})^n$ of G given by the strongly orthogonal roots contains the irreducible representation $(\mathbb{R}^2)^{\otimes n}$, and the rank of $\mathcal{W}^{p,q}$ is $\geq \binom{n}{p}$.

Let $S^\bullet(V) = \bigoplus_{n\geq 0} S^n(V)$ be the symmetric algebra on V, and let $S^\bullet(V)^G$ denote the sub-algebra of G-invariants. We find [K, Tables II and III] that V is a polar representation of G, and that $S^\bullet(V)^G$ is free, if and only if $n \leq 4$. More precisely:

$$S^\bullet(V)^G = \begin{cases} \mathbb{R} & \text{if} \quad n=1 \\ \mathbb{R}[f_2] & \text{if} \quad n=2 \\ \mathbb{R}[f_4] & \text{if} \quad n=3 \\ \mathbb{R}[f_2, \cdots, f_k] & \text{if} \quad n=4 \end{cases}$$

where $\deg f_2 = 2, \deg f_4 = 4$, and there are ≥ 4 independent invariants f_d of degrees $d = 2, \cdots, k$ when $n = 4$. For example, when $G = \mathrm{SL}_2(\mathbb{R})^4$ and $V = (\mathbb{R}^2)^{\otimes 4}$, the invariants are freely generated by polynomials f_2, f_4, f_4', f_6 of degrees $d = 2, 4, 4, 6$. For the three simple tube domains D of rank 4, the degrees of the generating invariants are given by the degrees of the generating invariants for the reflection representations of the Weyl groups of type E:

$$\begin{cases} G = Sp_8(\mathbb{R}) & V = \Lambda_0^4\mathbb{R}^8 & d = 2,5,6,8,9,12 & W(E_6) \\ G = SU_{4,4} & V_{\mathbb{C}} = \Lambda^4\mathbb{C}^8 & d = 2,6,8,10,12,14,18 & W(E_7) \\ G = \mathrm{Spin}_{16}^* & V = \frac{1}{2}\ \text{spin} & d = 2,8,12,14,18,20,24,30 & W(E_8) \end{cases}$$

7.

We return to the case when D is simple. Let $\check{D} = G_{\mathbb{C}}/P_{\mathbb{C}} = (G_1)_{\mathbb{C}}/(P_1)_{\mathbb{C}}$ be the compact dual of D, and let $X = D^{\pm} \hookrightarrow \check{D}$ be the Borel embedding [S, pp. 58–59]. Then X is the unique open orbit of G_1 on $\check{D}$.

The equivariant vector bundles $\mathcal{W}^{p,q}$ on X are all pull-backs of algebraic vector bundles on $\check{D}$. The line bundles $\omega = \mathcal{W}^{n,0}$ and $\mathcal{L} = \mathcal{W}^{0,n}$ give the two generators of $\mathrm{Pic}(\check{D}) \simeq \mathbb{Z}$. The line bundle $\mathcal{L}$ is ample on $\check{D}$, and the canonical bundle of $\check{D}$ is isomorphic to $\omega^{2d/n}$.

Let $\Sigma \subset \partial D$ be the Shilov boundary of D, which is the unique closed orbit G/P of G in $\check{D}$. An interesting question is to study the behavior of the variation $\mathcal{V}$ as one approaches a point σ of the Shilov boundary Σ of X. The resulting mixed Hodge structure $\mathcal{V}_\sigma$ "mirrors" that of $\mathcal{V}$. One has $\mathcal{W}_\sigma^{p,q} = 0$ unless $p = q$, and the dimension of $\mathcal{W}_\sigma^{p,p}$ is equal to the dimension of $\mathcal{W}^{q,p}$, with $p + q = n$.

Let $\mathcal{A}^q = \mathcal{W}^{p,q} \otimes (\mathcal{W}^{n,0})^{-1} = \mathcal{W}^{p,q} \otimes \mathcal{L}$. The exact sequence (5.3) of holomorphic, equivariant, algebra bundles

$$0 \longrightarrow I \longrightarrow \mathrm{Sym}^\bullet(\Theta) \longrightarrow \bigoplus_{q=0}^{n} \mathcal{A}^q \longrightarrow 0$$

on X extends, as a sequence of complex algebraic, equivariant, algebra bundles, to $\check{D} = G_\mathbb{C}/P_\mathbb{C}$. It then descends to a sequence of algebra bundles over the real algebraic variety $\Sigma = G/P$, with complex points $\check{D}$. This gives an algebra structure $\bigoplus_{q=0}^n \mathcal{A}_\mathbb{R}^q$ on the limit mixed Hodge structure $\mathcal{V}_\sigma$ at points σ of Σ, with ample cone $C \subset N = (\mathcal{A}_\mathbb{R}^1)_\sigma$.

8.

Another interesting question is whether the variation $\mathcal{V}$ occurs in nature (i.e. algebraic geometry). There one obtains local systems of $\mathbb{Q}$-vector spaces, so the first requirement is to specify a descent $(G_\mathbb{Q}, V_\mathbb{Q})$ of the pair (G, V) from $\mathbb{R}$ to $\mathbb{Q}$. In the cases C_n (n odd) and E_7, V is a faithful representation of G and there is a *unique* descent of the pair (G, V) to $\mathbb{Q}$. The resulting group $G_\mathbb{Q}$ is split over $\mathbb{Q}_p$ for all finite primes p. In the cases C_n (n even), and in certain of the cases B and $D^\mathbb{R}$, one can specify a descent by insisting that $G_\mathbb{Q}$ is split over $\mathbb{Q}_p$ for all finite primes p. In the other cases, a descent requires some choice — such as an imaginary quadratic field or a definite quaternion algebra over $\mathbb{Q}$.

Assume that a descent $(G_\mathbb{Q}, V_\mathbb{Q})$ of the pair (G, V) has been specified. One can then ask if there is a family $f : Y \to S$ of smooth complex polarized projective varieties, where the base $S = \Gamma \setminus D$ is uniformized by D and Γ is an arithmetic subgroup of $G_\mathbb{Q}$, such that $V_\mathbb{Q}$ is the pull-back to D of the variation $(R^n f_* \mathbb{Q})_{\text{primitive}} = V_Y$ on S. The limit mixed Hodge structure $\bigoplus_{q=0}^n \mathcal{A}_\mathbb{Q}^q$ will then be associated to degenerations in the family $f : Y \to S$ over the 0-dimensional cusps $\Gamma \setminus G_\mathbb{Q}/P_\mathbb{Q}$ of the Satake compactification $\overline{S}$.

9.

The simplest geometric families $f : Y \to S$ to study are those where the fibres Y_s have dimension $n = \mathrm{rank}(D)$ and trivial canonical class ($c_1(Y_s) = 0$, or equivalently $\Omega_{Y_s}^n \simeq \mathcal{O}_{Y_s}$). Then $F^n \mathcal{V}_Y = f_* \Omega_{Y/S}^n$ is a holomorphic line bundle on S, and (when S is a universal family) the Kodaira-Spencer map:

$$\nabla : \Theta_S \longrightarrow \mathrm{Hom}(F^n \mathcal{V}_Y, F^{n-1}\mathcal{V}_Y / F^n \mathcal{V}_Y)$$

is an isomorphism (cf. [Tn]). Hence, several of the key properties of $\mathcal{V}_{\mathbb{Q}}$ established in §4 hold for $\mathcal{V}_Y$.

For example, if one takes the descent $G_{\mathbb{Q}} = Sp(2n, \mathbb{Q})$ of $G = Sp(2n, \mathbb{R})$ to $\mathbb{Q}$, which is the only choice when n is odd and is split at all finite primes p in general, then $\mathcal{V}_{\mathbb{Q}}$ is the pull-back of $\mathcal{V}_Y$ for a universal family $f : Y \to S$ of polarized abelian varieties of dimension $= n$.

10.

More generally, $\mathcal{V}_{\mathbb{Q}}$ might arise from a sub-Hodge structure $\mathcal{V}_{Y,p} \subset \mathcal{V}_Y$ on S, where p is a projector in the primitive cohomology of dimension n. For example, if $G = \mathrm{Spin}(2, 10)$, the homogeneous cone C associated to D is isomorphic to $S_2(\mathbb{O})^+$, the cone of 2×2 positive definite symmetric matrices over the octonions. Since the octonions have a unique descent to $\mathbb{Q}$, one obtains a descent $G_{\mathbb{Q}}$ for G which is split at all finite primes p. In this case $\mathcal{V}_{\mathbb{Q}}$ has type (1,10,1) and arises as the pull-back of $\mathcal{V}_Y^{\sigma=-1}$, for a universal family $f : Y \to S$ of polarized K3 surfaces with an Enriques involution σ (which is generically fixed point free on Y_s).

Can such a geometric realization be found in the case when $G = E_{7,3}$ and $C = S_3(\mathbb{O})^+$? Here there is a unique descent $(G_{\mathbb{Q}}, V_{\mathbb{Q}})$ to $\mathbb{Q}$, and one is looking for a family of 3-folds $f : Y \to S$ with trivial canonical class, such that $\mathcal{V}_{\mathbb{Q}}$ is the pull-back of a sub-Hodge structure $\mathcal{V}_{Y,p} \subset \mathcal{V}_Y$.

The Hodge numbers of $\mathcal{V}_{\mathbb{Q}}$ are $(1, 27, 27, 1)$. Hence we must have $h^{2,1}(Y_s) \geq 27$; the general theory of 3-folds with $c_1 = 0$ then shows that one must have $h^{1,0}(Y_s) = h^{2,0}(Y_s) = 0$. For reasons of mirror symmetry, it seems unlikely that $\mathcal{V}_{\mathbb{Q}}$ is the pull-back of the *entire* 3-cohomology $\mathcal{V}_Y$. The simplest guess, in analogy with the case where $C = S_2(\mathbb{O})^+$, is that $\mathcal{V}_{\mathbb{Q}}$ arises from a sub-Hodge structure $\mathcal{V}_Y^{\sigma=-1}$, where σ is an involution which generically has 16 isolated fixed points on Y_s. Then $F^3(\mathcal{V}_Y^{\sigma=-1})$ is a line bundle on S.

Any geometric realization $f : Y \to S$ of the variation $\mathcal{V}$ associated with $G = E_{7,3}$ must be fairly complicated, as Deligne has remarked that the associated Hodge structures have no Picard-Lefshetz degenerations. This excludes the many constructions of 3-folds with $c_1 = 0$ which are given as smooth complete intersections in weighted projective spaces. Perhaps the existence of small resolutions of nodes in dimension 3 can be profitably used in this context.

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