Let $D$ be a tube domain (i.e. a bounded symmetric domain of tube type). In [D, §1], Deligne gives a description of $D$ as the moduli space of certain Hodge structures. Using these methods, we show that $D$ parametrizes a canonical variation $\mathcal{V}$ of polarized real Hodge structures, which is effective of weight $= \text{rank}(D)$ and enjoys several remarkable properties. We end with some speculation on how $\mathcal{V}$ might appear in algebraic geometry.

1.

Let $D$ be a simple tube domain, and let $G$ be (the real points of) the simply-connected, simple real algebraic group which acts transitively on $D$. Let $K$ be a maximal compact subgroup of $G$; then $K$ fixes a unique point of $D$ and $D \simeq G/K$. The integer $n = \text{rank}(D)$ is defined to be the real rank of $G$, and the integer $d = \dim(D)$ is defined to be the complex dimension of the domain, which is one-half the real dimension of $G/K$. The quotient $2d/n$ is always an integer [S, p. 37].

Since $D$ is tube, there is a self-adjoint homogeneous cone $C$ in a Euclidean space $N$ over $\mathbb{R}$ such that $D \simeq N + iC \subseteq N_C$ [S, p. 128].

We recall [D, 1.2.6] that the simple bounded symmetric domains are classified by pairs $(\Delta, v)$, where $\Delta$ is a connected Dynkin diagram and $v$ is a special vertex of $\Delta$ which is equivalent to the extended vertex $\mu$ under an automorphism of the affine diagram $\Delta' = \Delta \cup \{\mu\}$ [T, pp. 33-34, p. 53]. The domain is tube if $v$ is fixed by the opposition involution of $\Delta$.

We now tabulate the relevant pairs $(\Delta, v)$, where the special vertex is circled, and give the groups $G$ and $K$ associated to the tube domain $D$. We also describe the cone $C$, using the notation $S_n(F)^+$ for the cone of positive definite, $n \times n$ Hermitian symmetric matrices over the $\mathbb{R}$-algebras $F = \mathbb{R}, \mathbb{C}, \mathbb{H}$.

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\( A_{2n-1} \quad \begin{array}{c}
\cdots \\
\circ
\end{array} \quad n^{th}\text{vertex} \)

\[ n \geq 1 \]

\( G = SU(n,n) \quad \text{rank}(D) = n \)

\( K = S(U(n) \times U(n)) \quad \text{dim}(D) = n^2 \)

\( C = S_n(\mathbb{C})^+ \)

\( B_n \quad n \geq 2 \quad \begin{array}{c}
\circ \\
\cdots
\end{array} \)

\( G = \text{Spin}(2,2n-1) \quad \text{rank}(D) = 2 \)

\( K = \text{Spin}(2) \times \mu_2 \text{ Spin}(2n-1) \quad \text{dim}(D) = 2n-1 \)

\( C = \text{light cone in Minkowski space } \mathbb{R}^{1,2n-2} \)

\( C_n \quad n \geq 1 \quad \begin{array}{c}
\cdots \\
\circ
\end{array} \)

\( G = Sp(2n,\mathbb{R}) \quad \text{rank}(D) = n \)

\( K = U(n) \quad \text{dim}(D) = n(n+1)/2 \)

\( C = S_n(\mathbb{R})^+ \)

\( D^\mathbb{R}_n \quad n \geq 3 \quad \begin{array}{c}
\circ \\
\cdots
\end{array} \)

\( G = \text{Spin}(2,2n-2) \quad \text{rank}(D) = 2 \)

\( K = \text{Spin}(2) \times \mu_2 \text{ Spin}(2n-2) \quad \text{dim}(D) = 2n-2 \)

\( C = \text{light cone in Minkowski space } \mathbb{R}^{1,2n-3} \)

\( D^\mathbb{H}_n \quad n \geq 2 \quad \begin{array}{c}
\circ \\
\cdots
\end{array} \)

\( G = \text{Spin}^+(4n) \quad \text{rank}(D) = n \)

\( K = U(1) \times \mu_n SU(2n) \quad \text{dim}(D) = n(2n-1) \)

\( C = S_n(\mathbb{H})^+ \)

\( E_7 \quad \begin{array}{c}
\circ \\
\end{array} \)

\( G = E_{7,3} \quad \text{rank}(D) = 3 \)

\( K = U(1) \times \mu_3 E_6 \quad \text{dim}(D) = 27 \)

\( C = S_3(\mathbb{O})^+ = \text{exceptional cone} \)

\( \mathbb{O} = \text{octonions} = \text{Cayley numbers.} \)
2.

The vertex \( v \) of \( \Delta \) determines a conjugacy class of maximal parabolic subgroups \( P = L \cdot N \) in \( G \), with unipotent radical \( N \) abelian and Levi subgroup \( L \) a real form of \( K \). The cone \( C \) is \( L \)-homogeneous in the real vector space \( N \).

The vertex \( v \) also determines a fundamental, irreducible representation \( V \) of \( G \) over \( \mathbb{R} \). The orbit of a highest weight vector in \( \mathbb{P}(V) \) is the real projective variety \( G/P \). We describe the fundamental representation \( V \), in terms of the standard representation of \( G \), in the table below.

<table>
<thead>
<tr>
<th>Type</th>
<th>( G )</th>
<th>( V )</th>
<th>dim ( V )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td>( SU(n,n) )</td>
<td>( V_{\mathbb{C}} = \bigwedge^n \mathbb{C}^{2n} )</td>
<td>( \binom{2n}{n} )</td>
</tr>
<tr>
<td>( B, D^R )</td>
<td>( \text{Spin}(2,m) )</td>
<td>( \mathbb{R}^{2n} ) of ( SO(2,m) )</td>
<td>( 2 + m )</td>
</tr>
<tr>
<td>( C )</td>
<td>( Sp(2n,\mathbb{R}) )</td>
<td>( V \oplus \bigwedge^{n-2} \mathbb{R}^{2n} = \bigwedge^n \mathbb{R}^{2n} )</td>
<td>( \binom{2n}{n} - \binom{2n}{n-2} )</td>
</tr>
<tr>
<td>( D^H )</td>
<td>( \text{Spin}^*(4n) )</td>
<td>unique ( \frac{1}{2} )-spin which is real</td>
<td>( 2^{2n-1} )</td>
</tr>
<tr>
<td>( E )</td>
<td>( E_{7,3} )</td>
<td>unique miniscule representation</td>
<td>56</td>
</tr>
</tbody>
</table>

3.

We will see that the fundamental representation \( V \) gives rise to a canonical variation of polarized real Hodge structures on \( D \). To do this, following Deligne [D, §1], we must first realize \( V \) as a representation of a reductive group \( G_1 \), whose center contains \( \mathbb{G}_m \) and whose derived group is \( G \). We construct \( G_1 \) as follows.

Let \( \epsilon \) be the unique element of order 2 in the center of \( G \) which is contained in the connected component of the center of \( K \). We recall that \( Z(K)^+ \simeq U(1) \) in all cases. Let \( G_1 \) be the quotient of \( \mathbb{G}_m \times G \) by the central subgroup generated by the involution \(-1 \times \epsilon\). Since \( \epsilon \) acts as \((-1)^{n}

\text{on } V, \text{ with } n = \text{rank}(D), \text{ \( V \) extends uniquely to a representation of \( G_1 \) such that } \lambda \in \mathbb{G}_m \text{ acts by } \lambda^{-n}.

Let \( S = \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m = \mathbb{G}_m \times U(1)/(-1 \times -1) \). A point of \( D \) determines a homomorphism \( U(1) \hookrightarrow K \hookrightarrow G \), which is an oriented isomorphism from \( U(1) \) to the connected component of the center of \( K \). Since this maps the element \(-1 \) of \( U(1) \) to the element \( \epsilon \) of \( G \), it determines a homomorphism

\[
(3.1) \quad h : S \longrightarrow G_1
\]

which is the identity on \( \mathbb{G}_m \). The \( G_1 \)-conjugacy class \( X \) of \( h \) has two connected components, each of which is isomorphic to \( D \).

Finally, the representation \( V \) of \( G_1 \), when combined with the morphism \( h \) of (3.1), gives rise to a polarized variation of real Hodge structures on
X (and hence on D), by the results of Deligne [D, Prop. 1.1.14], [M, Ch. II, Prop. 3.2]. We put \( \mathcal{V} = V \otimes \mathcal{O}_D \); this is an equivariant holomorphic vector bundle on \( D \) with connection \( \nabla \). Since \( \lambda \in \mathbb{G}_m \) acts as \( \lambda^{-n} \) on \( V \), the Hodge structures associated to \( V \) are pure of weight \( n = \text{rank}(D) \).

4.

We now investigate the properties of \( V \) as a variation of Hodge structure. We recall that \( \mathcal{V} \) has a filtration by holomorphic sub-bundles \( \cdots \supset F^p V \supset \cdots \), and that the quotient bundles

\[
\mathcal{W}^{p,q} = \frac{F^p \mathcal{V}}{F^{p+1} \mathcal{V}} \quad p + q = n
\]

are equivariant vector bundles on \( X \). On \( D \cong G/K \), these bundles correspond to the \( K \)-submodules \( W^{p,q} \) of \( V_C \) on which elements \( z \in U(1) = Z(K)^+ \) act by the character \( z^{-p}z^{-q} = z^{q-p} \).

**Proposition 4.1.** The variation of Hodge structures \( V \) is effective of weight \( n \), so \( \mathcal{W}^{p,q} = 0 \) unless both \( p, q \geq 0 \).

If \( p, q \geq 0 \) the equivariant vector bundle \( \mathcal{W}^{p,q} \) is irreducible. Both \( F^0 \mathcal{V} = W^{n,0} \) and \( \mathcal{V}/F^1 \mathcal{V} = \mathcal{W}^{0,n} \) are holomorphic line bundles on \( X \).

**Proof.** This results from a determination of the eigenspaces \( W^{p,q} \) for the action of \( U(1) \cong Z(K)^+ \) on \( V_C \), as representations of \( K \). Only the characters \( z^n, z^{n-2}, \ldots, z^{2-n}, z^{-n} \) occur, and we tabulate these representations below.

<table>
<thead>
<tr>
<th>Type</th>
<th>( K )</th>
<th>( W^{p,q} )</th>
<th>( p, q \geq 0 )</th>
<th>dim ( W^{p,q} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>( S(U(n) \times U(n)) )</td>
<td>((\wedge^p \mathbb{C}^n)^* \otimes (\wedge^q \mathbb{C}^n)^*)</td>
<td>((n \choose p)(n \choose q))</td>
<td></td>
</tr>
<tr>
<td>B, D ( ^R )</td>
<td>( \text{Spin}(2) \times_{\mu^2} \text{Spin}(m) )</td>
<td>( W^{2,0} = \mathbb{C}(-2) \otimes \mathbb{C} ) ( W^{1,1} = \mathbb{C} \otimes \mathbb{C}^m ) ( W^{0,2} = \mathbb{C}(2) \otimes \mathbb{C} )</td>
<td>( 1 ) ( m ) ( 1 )</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>( U(n) )</td>
<td>Irreducible summand of ((\wedge^p \mathbb{C}^n)^* \otimes (\wedge^q \mathbb{C}^n)^*) with highest weight ( (= 2\omega_q) )</td>
<td>((n \choose p)(n \choose q) - (n \choose p-1)(n \choose q-1))</td>
<td></td>
</tr>
<tr>
<td>D ( ^H )</td>
<td>( U(1) \times_{\mu_n} SU(2n) )</td>
<td>( \mathbb{C}(q-p) \otimes (\wedge^{2p} \mathbb{C}^{2n})^* )</td>
<td>((2n \choose 2p))</td>
<td></td>
</tr>
<tr>
<td>E</td>
<td>( U(1) \times_{\mu_3} E_6 )</td>
<td>( W^{3,0} = \mathbb{C}(-3) \otimes \mathbb{C} ) ( W^{2,1} = \mathbb{C}(-1) \otimes W^2_{27} ) ( W^{1,2} = \mathbb{C}(1) \otimes W^2_{27} ) ( W^{0,3} = \mathbb{C}(3) \otimes \mathbb{C} )</td>
<td>( 1 ) ( 27 ) ( 27 ) ( 1 )</td>
<td></td>
</tr>
</tbody>
</table>
The connection $\nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega^1_X$ satisfies Griffiths transversality: $\nabla(F^p\mathcal{V}) \subset F^{p-1}\mathcal{V} \otimes \Omega^1_X$. Hence, if $\Theta_X$ is the holomorphic tangent bundle of $X$, differentiating $q$ times gives a morphism of equivariant vector bundles

\[(5.1) \quad \nabla^q : \text{Sym}^q \Theta_X \to \text{Hom}(F^n\mathcal{V}, F^p\mathcal{V}/F^{p+1}\mathcal{V}),\]

where $p + q = n$.

**Proposition 5.2.** The morphism $\nabla^q$ is surjective for all $0 \leq q \leq n$, and $\nabla : \Theta_X \to \text{Hom}(F^n\mathcal{V}, F^{n-1}\mathcal{V}/F^n\mathcal{V})$ is an isomorphism.

**Proof.** Since $\text{Hom}(F^n\mathcal{V}, F^p\mathcal{V}/F^{p+1}\mathcal{V}) \simeq \mathcal{W}^{p,q} \otimes (F^n\mathcal{V})^{-1}$ is an irreducible equivariant bundle, it suffices to check that $\nabla^q \neq 0$. This reduces to the study of the action of elements in $N_C \subset G_C$ on the eigenspace $W^{n,0}$ of $\mathcal{V}_C$ (cf. [D, Prop. 1.1.14]). We leave the details to the reader. Since $\dim(W^{n-1,1}) = \dim(X) = d$ and the map $\nabla$ is surjective, it is an isomorphism.

The maps $\nabla^q$ defined in (5.1) give a surjection $f = \bigoplus_{q \geq 0} \nabla^q$ of vector bundles on $X$:

\[(5.3) \quad \text{Sym}^\bullet(\Theta_X) = \bigoplus_{q \geq 0} \text{Sym}^q \Theta_X \xrightarrow{f} \bigoplus_{q=0}^n \mathcal{W}^{p,q} \otimes (\mathcal{W}^{m,0})^{-1} \to 0.\]

The kernel of $f$ is a graded ideal $I = \bigoplus_{q \geq 2} I^q$ of $\text{Sym}^\bullet(\Theta_X)$, probably generated by the irreducible equivariant bundle $I^2 = \ker(\nabla^2 : \text{Sym}^2 \Theta_X \to \mathcal{W}^{n-2,2} \otimes (\mathcal{W}^{m,0})^{-1})$.

6.

If $D$ is any tube domain, it admits a product decomposition $D = D_1 \times D_2 \times \cdots \times D_k$ into simple tube domains of the type classified in §1. The variation $\mathcal{V} = \mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \cdots \otimes \mathcal{V}_k$ of polarized real Hodge structures on $D$ is effective of weight $n = \text{rank}(D) = \sum_{i=1}^k n_i$, $F^n\mathcal{V} = F^{n_1}\mathcal{V}_1 \otimes F^{n_2}\mathcal{V}_2 \otimes \cdots \otimes F^{n_k}\mathcal{V}_k$ is a holomorphic line bundle on $D$, $\nabla^q : \text{Sym}^q \Theta_D \to \text{Hom}(F^n\mathcal{V}, F^p\mathcal{V}/F^{p+1}\mathcal{V})$ is surjective for all $0 \leq q \leq n$, and $\nabla = \nabla^1$ is an isomorphism. These properties characterize $\mathcal{V}$ on $D$.

We end with some remarks on the irreducible representation $V = V_1 \otimes V_2 \otimes \cdots \otimes V_k$ of $G = G_1 \times G_2 \times \cdots \times G_k$ for the general tube domain $D$. Let
Let \( n \) be the real rank of \( G \), and let \( \{ \gamma_1, \gamma_2, \cdots, \gamma_n \} \) be a system of strongly orthogonal, non-compact positive roots for a compact torus \( T \) in \( G \) [S, p. 60]. Then

\[
\lambda = \frac{1}{2}(\gamma_1 + \gamma_2 + \cdots + \gamma_n)
\]

is the highest weight for \( T \) on \( V_\mathbb{C} \). In particular, the restriction of \( V \) to the subgroup \( \text{SL}_2(\mathbb{R})^n \) of \( G \) given by the strongly orthogonal roots contains the irreducible representation \((\mathbb{R}^2)^{\otimes n}\), and the rank of \( W^{p,q} \) is \( \geq \binom{n}{p} \).

Let \( S^\bullet(V) = \bigoplus_{n \geq 0} S^n(V) \) be the symmetric algebra on \( V \), and let \( S^\bullet(G) \) denote the sub-algebra of \( G \)-invariants. We find [K, Tables II and III] that \( V \) is a polar representation of \( G \), and that \( S^\bullet(V)^G \) is free, if and only if \( n \leq 4 \). More precisely:

\[
S^\bullet(V)^G = \begin{cases} 
\mathbb{R} & \text{if } n = 1 \\
\mathbb{R}[f_2] & \text{if } n = 2 \\
\mathbb{R}[f_4] & \text{if } n = 3 \\
\mathbb{R}[f_2, \cdots, f_k] & \text{if } n = 4 
\end{cases}
\]

where \( \deg f_2 = 2, \deg f_4 = 4 \), and there are \( \geq 4 \) independent invariants \( f_d \) of degrees \( d = 2, \cdots, k \) when \( n = 4 \). For example, when \( G = \text{SL}_2(\mathbb{R})^4 \) and \( V = (\mathbb{R}^2)^{\otimes 4} \), the invariants are freely generated by polynomials \( f_2, f_4, f_4', f_6 \) of degrees \( d = 2, 4, 4, 6 \). For the three simple tube domains \( D \) of rank 4, the degrees of the generating invariants are given by the degrees of the generating invariants for the reflection representations of the Weyl groups of type \( E \):

\[
\begin{align*}
G &= \text{Sp}_8(\mathbb{R}) & V &= \Lambda_0^4 \mathbb{R}^8 & d &= 2, 5, 6, 8, 9, 12 & W(E_6) \\
G &= \text{SU}_{4,4} & V_\mathbb{C} &= \Lambda_1^4 \mathbb{C}^8 & d &= 2, 6, 8, 10, 12, 14, 18 & W(E_7) \\
G &= \text{Spin}_{16}^* & V &= \frac{1}{2} \text{spin} & d &= 2, 8, 12, 14, 18, 20, 24, 30 & W(E_8)
\end{align*}
\]

We return to the case when \( D \) is simple. Let \( \check{D} = G_\mathbb{C}/P_\mathbb{C} = (G_1)_\mathbb{C}/(P_1)_\mathbb{C} \) be the compact dual of \( D \), and let \( X = D^\pm \hookrightarrow \check{D} \) be the Borel embedding [S, pp. 58–59]. Then \( X \) is the unique open orbit of \( G_1 \) on \( \check{D} \).

The equivariant vector bundles \( W^{p,q} \) on \( X \) are all pull-backs of algebraic vector bundles on \( \check{D} \). The line bundles \( \omega = W^{0,0} \) and \( \mathcal{L} = W^{0,n} \) give the two generators of \( \text{Pic}(\check{D}) \cong \mathbb{Z} \). The line bundle \( \mathcal{L} \) is ample on \( \check{D} \), and the canonical bundle of \( \check{D} \) is isomorphic to \( \omega^{2d/n} \).

Let \( \Sigma \subset \partial D \) be the Shilov boundary of \( D \), which is the unique closed orbit \( G/P \) of \( G \) in \( D \). An interesting question is to study the behavior of
the variation $\mathcal{V}$ as one approaches a point $\sigma$ of the Shilov boundary $\Sigma$ of $X$. The resulting mixed Hodge structure $\mathcal{V}_\sigma$ “mirrors” that of $\mathcal{V}$. One has $W_{p,q}^p = 0$ unless $p = q$, and the dimension of $W_{p,p}^p$ is equal to the dimension of $W_{n-p}^n$, with $p + q = n$.

Let $\mathcal{A}^q = W_{p,q} \otimes (W_{n,0})^{-1} = W_{p,q} \otimes L$. The exact sequence (5.3) of holomorphic, equivariant, algebra bundles

$$0 \longrightarrow I \longrightarrow \text{Sym}^* (\Theta) \longrightarrow \bigoplus_{q=0}^n \mathcal{A}^q \longrightarrow 0$$

on $X$ extends, as a sequence of complex algebraic, equivariant, algebra bundles, to $\hat{\mathcal{D}} = G_C/P_C$. It then descends to a sequence of algebra bundles over the real algebraic variety $\Sigma = G/P$, with complex points $\hat{\mathcal{D}}$. This gives an algebra structure $\bigoplus_{q=0}^n \mathcal{A}^q_\mathbb{R}$ on the limit mixed Hodge structure $\mathcal{V}_\sigma$ at points $\sigma$ of $\Sigma$, with ample cone $C \subset N = (\mathcal{A}^1_\mathbb{R})_\sigma$.

8.

Another interesting question is whether the variation $\mathcal{V}$ occurs in nature (i.e. algebraic geometry). There one obtains local systems of $\mathbb{Q}$-vector spaces, so the first requirement is to specify a descent $(G_\mathbb{Q}, V_\mathbb{Q})$ of the pair $(G, V)$ from $\mathbb{R}$ to $\mathbb{Q}$. In the cases $C_n$ ($n$ odd) and $E_7$, $V$ is a faithful representation of $G$ and there is a unique descent of the pair $(G, V)$ to $\mathbb{Q}$. The resulting group $G_\mathbb{Q}$ is split over $\mathbb{Q}_p$ for all finite primes $p$. In the cases $C_n$ ($n$ even), and in certain of the cases $B$ and $D^R$, one can specify a descent by insisting that $G_\mathbb{Q}$ is split over $\mathbb{Q}_p$ for all finite primes $p$. In the other cases, a descent requires some choice — such as an imaginary quadratic field or a definite quaternion algebra over $\mathbb{Q}$.

Assume that a descent $(G_\mathbb{Q}, V_\mathbb{Q})$ of the pair $(G, V)$ has been specified. One can then ask if there is a family $f : Y \to S$ of smooth complex polarized projective varieties, where the base $S = \Gamma \setminus D$ is uniformized by $D$ and $\Gamma$ is an arithmetic subgroup of $G_\mathbb{Q}$, such that $V_\mathbb{Q}$ is the pull-back to $D$ of the variation $(R^n f_* \mathcal{O}_{\mathbb{Q}})_{\text{primitive}} = V_Y$ on $S$. The limit mixed Hodge structure $\bigoplus_{q=0}^n \mathcal{A}^q_\mathbb{Q}$ will then be associated to degenerations in the family $f : Y \to S$ over the 0-dimensional cusps $\Gamma \setminus G_\mathbb{Q}/P_\mathbb{Q}$ of the Satake compactification $\overline{S}$.

9.

The simplest geometric families $f : Y \to S$ to study are those where the fibres $Y_s$ have dimension $n = \text{rank}(D)$ and trivial canonical class $(c_1(Y_s) = 0$, or equivalently $\Omega^n_{Y_s} \simeq \mathcal{O}_{Y_s}$). Then $F^n V_Y = f_* \Omega^n_{Y/S}$ is a holomorphic
line bundle on $S$, and (when $S$ is a universal family) the Kodaira-Spencer map:

$$\nabla : \Theta_S \longrightarrow \text{Hom}(F^n\mathcal{V}_Y, F^{n-1}\mathcal{V}_Y / F^n\mathcal{V}_Y)$$

is an isomorphism (cf. [Tn]). Hence, several of the key properties of $\mathcal{V}_\mathbb{Q}$ established in §4 hold for $\mathcal{V}_Y$.

For example, if one takes the descent $G_\mathbb{Q} = Sp(2n, \mathbb{Q})$ of $G = Sp(2n, \mathbb{R})$ to $\mathbb{Q}$, which is the only choice when $n$ is odd and is split at all finite primes $p$ in general, then $\mathcal{V}_\mathbb{Q}$ is the pull-back of $\mathcal{V}_Y$ for a universal family $f : Y \to S$ of polarized abelian varieties of dimension $n$.

10. More generally, $\mathcal{V}_\mathbb{Q}$ might arise from a sub-Hodge structure $\mathcal{V}_{Y,p} \subset \mathcal{V}_Y$ on $S$, where $p$ is a projector in the primitive cohomology of dimension $n$. For example, if $G = \text{Spin}(2,10)$, the homogeneous cone $C$ associated to $D$ is isomorphic to $S_2(\mathbb{O})^+$, the cone of $2 \times 2$ positive definite symmetric matrices over the octonions. Since the octonions have a unique descent to $\mathbb{Q}$, one obtains a descent $G_\mathbb{Q}$ for $G$ which is split at all finite primes $p$. In this case $\mathcal{V}_\mathbb{Q}$ has type $(1,10,1)$ and arises as the pull-back of $\mathcal{V}_Y^{n=1}$, for a universal family $f : Y \to S$ of polarized K3 surfaces with an Enriques involution $\sigma$ (which is generically fixed point free on $Y_s$).

Can such a geometric realization be found in the case when $G = E_{7,3}$ and $C = S_3(\mathbb{O})^+$? Here there is a unique descent $(G_\mathbb{Q}, \mathcal{V}_\mathbb{Q})$ to $\mathbb{Q}$, and one is looking for a family of 3-folds $f : Y \to S$ with trivial canonical class, such that $\mathcal{V}_\mathbb{Q}$ is the pull-back of a sub-Hodge structure $\mathcal{V}_{Y,p} \subset \mathcal{V}_Y$.

The Hodge numbers of $\mathcal{V}_\mathbb{Q}$ are $(1,27,27,1)$. Hence we must have $h^{1,0}(Y_s) \geq 27$; the general theory of 3-folds with $c_1 = 0$ then shows that one must have $h^{1,0}(Y_s) = h^{2,0}(Y_s) = 0$. For reasons of mirror symmetry, it seems unlikely that $\mathcal{V}_\mathbb{Q}$ is the pull-back of the entire 3-cohomology $\mathcal{V}_Y$. The simplest guess, in analogy with the case where $C = S_2(\mathbb{O})^+$, is that $\mathcal{V}_\mathbb{Q}$ arises from a sub-Hodge structure $\mathcal{V}_Y^{n=1}$, where $\sigma$ is an involution which generically has 16 isolated fixed points on $Y_s$. Then $F^3(\mathcal{V}_Y^{n=-1})$ is a line bundle on $S$.

Any geometric realization $f : Y \to S$ of the variation $\mathcal{V}$ associated with $G = E_{7,3}$ must be fairly complicated, as Deligne has remarked that the associated Hodge structures have no Picard-Lefshetz degenerations. This excludes the many constructions of 3-folds with $c_1 = 0$ which are given as smooth complete intersections in weighted projective spaces. Perhaps the existence of small resolutions of nodes in dimension 3 can be profitably used in this context.
References


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