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# HYPERBOLIC DYNAMICAL SYSTEMS, INVARIANT GEOMETRIC STRUCTURES, AND RIGIDITY

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The problems and results described in this article are situated at a point of contact between differential geometry on the one hand and the theory of hyperbolic dynamical systems on the other. They are part of a varied and growing body of research that has generally been described as *Geometric Rigidity*. ([22] provides a fairly representative cross-section of the field.) We consider the following, for the moment vague, question: What can be said about the analytic and geometric properties (and the existence) of geometric structures preserved under group actions that exhibit some form of hyperbolic behavior?

By a differential geometric structure, I wish to understand that kind considered by M. Gromov in [6], which he calls *rigid geometric structure*. A basic example of such structures, and the one considered here for the most part, is an *affine connection*. These rigid structures have in common that their pseudo-group of local automorphisms is finite dimensional. (That is to be contrasted, say, with a symplectic structure. This notion generalizes Cartan's structures of finite type.) The present discussion can be regarded as a specialization of some of the general questions formulated in [6].

The theory of smooth dynamical systems with hyperbolic behavior has developed, since the seminal work of D. V. Anosov [1], into a vast and rich body of pure as well as applied mathematics. I refer to [15] and [20] for general information on this subject. More recently, the theory has been applied with great profit to the study of actions of higher rank groups (see [18], [8]).

The concept of *rigidity* is understood here as the property that certain classes of group actions possess that permit them to be classified, under some suitable notion of equivalence, by a small number of algebraically defined model actions. Usually, these model actions are given as automorphisms of double coset spaces of the form $\Gamma\backslash G/H$, where $H$ is a closed subgroup of the Lie group $G$ and $\Gamma$ is a discrete subgroup. Important examples of rigidity theories that are related to our discussion are the above

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mentioned theory of rigid transformation groups by M. Gromov, R. Zimmer's superrigidity theory for cocycles of actions of higher rank semisimple Lie groups and lattices ([23] and [24]), A. Katok and R. Spatzier's study of abelian Anosov actions [13], the works of S. Hurder, J. Lewis, A. Katok, and R. Zimmer ([10], [11], [7], [8], and [12]) concerning perturbations of linear actions of lattices in semisimple Lie groups of $\mathbb{R}$-rank $\geq 2$ and the classification of affine Anosov diffeomorphisms and contact Anosov flows with smooth expanding and contracting foliations, due to Y. Benoist, P. Foulon, and F. Labourie ([3] and [2]).

Before stating the main results of this article it will help to illustrate the subject with a rather elementary example which, surprisingly, already demonstrates some aspects of the general discussion. It consists of a 1-dimensional (expanding) dynamical system (and of 1-dimensional differential geometry)! In spite of its simplicity, the result still has some interest on its own right and is, to our knowledge, new. (We take the opportunity offered by this review article to state a number of new results. These are the theorems for which a proof is provided.)

## A toy rigidity theorem

Perhaps the simplest class of smooth dynamical systems that already exhibit the remarkable complexity of orbit behaviors studied in hyperbolic dynamics are *expanding maps* of the circle. Let $f$ be a $C^\infty$ transformation of the circle $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ whose derivative satisfies $f'(x) > 1$ for all $x \in \mathbb{T}^1$. As will be the case more generally, we are provided with an 'algebraic model' for such system, namely the linear degree $k$-map $f_k : x \mapsto kx \pmod 1$. It is a well known fact that any expanding map $f$ is $C^0$-conjugate to such a linear expanding map, that is, there exists a homeomorphism $h$ of the circle such that $h \circ f = f_k \circ h$ for some $k$. It is also well known that the homeomorphism cannot be differentiable in general. On the other hand, M. Shub and D. Sullivan have shown [19] that if $f$ and $g$ are $C^r$ expanding maps of the circle that are conjugate via an absolutely continuous homeomorphism, then the homeomorphism is a $C^r$ diffeomorphism ($r > 1$). This result also illustrates the important point, about which we shall have more to say later, that establishing rigidity of a group action entails a discussion of the degree of regularity or differentiablity of the conjugating map. The following result is similar in nature:

**Theorem 1.** *Let $f$ be a $C^\infty$ expanding map of the circle $\mathbb{T}^1$ of degree $k$. Assume that $f$ preserves an $L^1_\lambda$-connection on $T\mathbb{T}^1$, where $\lambda$ is the Lebesgue measure. Then the connection is $C^\infty$ and $f$ is $C^\infty$ conjugate to the linear degree $k$ map $x \mapsto kx$.*

Before starting the proof, a few words are needed about invariant affine connections. An $L^1_\lambda$-connection on the tangent bundle $T\mathbb{T}^1$ is a map

$$\nabla : \{C^1 \text{ sections of } T\mathbb{T}^1\} \to \{L^1_\lambda \text{ sections of } T^*\mathbb{T}^1 \otimes T\mathbb{T}^1\}$$

satisfying the property: $\nabla f X = df \otimes X + f \nabla X$, where $X$ is a $C^1$ vector field on $\mathbb{T}^1$ and $f$ is a $C^1$ function. If $Y$ is another vector field on $\mathbb{T}^1$, $\nabla_Y X$ denotes the covariant derivative of $X$ along $Y$. Let $X = \frac{d}{dx}$ denote the coordinate vector field on $\mathbb{T}^1$, so that $\nabla$ can be described by the Christoffel symbol $\Gamma$ such that $\nabla_X X = \Gamma X$. Then $\Gamma$ is an $L^1_\lambda$ function on the interval $[0, 1]$.

The map $f$ is locally invertible and acts on $\nabla$ in the following way: at each point $x \in \mathbb{T}^1$, let $U$ be an open set containing $x$ and $V$ an open set containing $f(x)$, such that $g = f|_U : U \to V$ is a diffeomorphism. Then,

$$\nabla^f_X X|_U = g_*^{-1} \nabla_{g_* X|_U} g_* X|_U.$$

The Christoffel symbol of $\nabla$ transforms as follows: $\Gamma^f = f' \Gamma \circ f + \frac{f''}{f'}$. The connection is $f$-invariant if $\nabla^f = \nabla$ ($\lambda$-a.e.), which is equivalent to

$$\Gamma = f' \Gamma \circ f + \frac{f''}{f'} \quad (\lambda \ a.e.).$$

**Lemma 2.** *Assume the conditions of the theorem and that* $\nabla$ *is* $C^\infty$. *Then the conclusion of the theorem holds.*

The proof of the lemma illustrates one important step in the proof of the other rigidity theorems. Having a geometric structure that satisfies the infinitesimal conditions characterizing a locally homogeneous manifold (for affine connections, these conditions are $\nabla R \equiv 0$ and $\nabla T \equiv 0$, where $R$ and $T$ are, respectively, the curvature and torsion tensors), in order to prove that the manifold has the form of a double coset space, it has to be established that the structure is (geodesically) *complete.* It is interesting to observe that, for invariant structures under a hyperbolic action (at least for the uniformly hyperbolic actions discussed in this article), completeness follows directly from the dynamical properties of the action. That can be seen very simply in the following proof.

*Proof.* Define a vector field on $\mathbb{R}$ in the following way: Let $p$ be a fixed point of $f$ (which exists since $x \mapsto kx$ has one, namely 0). Let $Y(q)$ denote the parallel transport of $X(p) = (\frac{d}{dx})_p$ from $p$ to $q$. It can be very easily shown that $Y(x) = Y(0)\exp\{-\int_0^x \Gamma(s)ds\}$. The connection is geodesically complete exactly when $\int_0^1 \Gamma(s)ds = 0$, but this condition is satisfied since

$$\begin{aligned}\int_0^1 \Gamma(s)ds &= \int_0^1 f'\Gamma(f(s))ds + \int_0^1 (\log f')'ds \\ &= \int_0^{f(1)} \Gamma(s)ds \\ &= \text{degree}(f)\int_0^1 \Gamma(s)ds.\end{aligned}$$

Therefore $\int_0^1 \Gamma(s)ds = 0$. This means that $\nabla$ has trivial *holonomy*, so that $Y$ defines a smooth vector field on $\mathbb{T}^1$. This vector field satisfies $f_*Y = \text{degree}(f)Y$. Denoting by $\phi_t$ the flow that integrates $Y$, we obtain a smooth map $\mathbb{R} \to \mathbb{T}^1$ defined as $t \mapsto \phi_t(p)$. If $\tau$ is the least positive number such that $\phi_\tau(p) = p$, we obtain from $\phi$ a diffeomorphism between $\mathbb{R}/\tau\mathbb{Z}$ and $\mathbb{T}^1$ with the desired properties.

The following lemma shows how to 'bootstrap' the regularity of the connection.

**Lemma 3.** *An $f$-invariant, $L^1_\lambda$-connection on $T\mathbb{T}^1$ is $C^\infty$.*

*Proof.* Let $U_0,\ U_1,\ U_2,\ \dots$ be any sequence of open sets such that $f_i = f|_{U_{i+1}} : U_{i+1} \to U_i$ is a diffeomorphism. Denote $g_i = f_i^{-1} : U_i \to U_{i+1}$ $(i = 0, 1, 2, \dots)$. Then $\Gamma = g_i'\Gamma \circ g_i + \frac{g_i''}{g_i'}$ ($\lambda$-a.e. in $U_i$). Denote $g_{(j)} = g_{j-1} \circ g_{j-2} \circ \cdots \circ g_1 \circ g_0 : U_0 \to U_j$ and $H_j = g_j''/g_j' : U_j \to \mathbb{R}$. Then, one obtains by induction that

$$\Gamma = g'_{(n+1)}\Gamma \circ g_{(n+1)} + \sum_{l=0}^{n} g'_{(l)} H_l \circ g_{(l)} \ (\lambda\text{-a.e. in } U_0).$$

We claim that the sequence of continuous functions $S_{(n)} = \sum_{l=0}^n g'_{(l)} H_l \circ g_{(l)}$ on $U_0$ converges uniformly to a function $S = \sum_{l=0}^\infty g'_{(l)} H_l \circ g_{(l)}$. To see that, consider $a = \text{minimum}(f')$, a number greater that 1. Then $|g_i'| \leq \frac{1}{a}$ and

$$|S_{(n)} - S_{(m)}| = |\sum_{l=m}^{n-1} g'_{(l)} H_l \circ g_{(l)}| \leq C \sum_{l=m}^{n-1} \frac{1}{a^l} \leq C' \frac{1}{a^m} \ (m \leq n).$$

Moreover, $S$ is determined uniquely, independently of the choice of a sequence of inverses $g_i$. In fact, $S$ and $\Gamma$ must agree a.e. in $U_0$. This is so because:

$$\begin{aligned}|\Gamma - S| &\leq |\Gamma - S_{(n)}| + |S_{(n)} - S| \\ &= |g'_{(n)}\Gamma \circ g_{(n)}| + |S_{(n)} - S|,\end{aligned}$$

so that

$$\begin{aligned}\int_{U_0} |\Gamma - S| &\leq \int_{U_0} |\Gamma - S_{(n)}| + \int_{U_0} |S_{(n)} - S| \\ &= \int_{g_{(n)}(U_0)} |\Gamma| + \int_{U_0} |S_{(n)} - S|.\end{aligned}$$

But if $|\Gamma|$ is an $L^1$-function, $\int |\Gamma|$ is a function of bounded variation and absolutely continuous. Therefore $\int_{g_{(n)}(U_0)} |\Gamma| \to 0$, so that $\int_{U_0} |\Gamma - S| = 0$. It follows that $\|\Gamma - S\|_1 = 0$, hence $\Gamma = S$ a.e. Once we have established continuity of $\Gamma$, differentiability follows by differentianting $S$ term by term and showing that the series of derivatives, as was verified for $S$ itself, converges uniformly. The details are straightforward.

The theorem now follows from the above two lemmas.

## Existence of invariant connections

One way by which the notion of (non-uniform) hyperbolicity can be defined is by means of the so called *(Lyapunov) characteristic exponents*. Let $f$ be a smooth diffeomorphism of a compact smooth manifold $M$, preserving a finite Borel measure $\mu$. For a nonzero vector $v \in TM_x$, one defines

$$\chi^+(v) = \limsup_{n\to\infty} \frac{1}{n} \|Tf^n_x v\|,$$

where $Tf_x$ denotes the derivative map of $f$ at $x$. Then $f$ is said to be *non-uniformly hyperbolic* if for $\mu$-a.e. $x \in M$ and every nonzero $v \in TM_x$, $\chi^+(v) \neq 0$. 'Typically', as justified in the following theorem, dynamical systems with such property must preserve (a unique) measurable $f$-invariant affine connection on the tangent bundle of $M$ (in the $\mu$ a.e. sense). In the following theorem, $f$ will be said to *avoid a* $2:1$-*resonance* if for all $x \in M$ there do not exist vectors $v_1, v_2, v_3 \in TM_x \backslash \{0\}$ such that

$$\chi^+(v_1) = \chi^+(v_2) + \chi^+(v_3).$$

**Theorem 4.** *Let $f$ be a (non-uniformly) hyperbolic diffeomorphism of a compact manifold $M$ with invariant Borel probability measure $\mu$. Assume that $f$ avoids a $2:1$-resonance $\mu$-a.e. Then $f$ preserves a unique Borel affine connection ($\mu$-a.e.).*

This result does not apply to the 1-dimensional system discussed before since $f$ was not a diffeomorphism in that case. Nevertheless, it is interesting to note what happens in that case when the measure $\mu$ is supported on a finite orbit: the Taylor series of the Christoffel symbol $\Gamma$ at a periodic point $p$ of period $l$ is completely determined by the jets of $f^l$ at $p$, as can be easily checked.

Before starting the proof of the theorem, it will be necessary to recall a fundamental theorem in ergodic theory: the *multiplicative ergodic theorem*, due to V. I. Oseledec. One way to state it is as follows. Let $p : E \to M$ be a smooth vector bundle over $M$ and $(\bar{f}, f)$ a bundle automorphism, so that $\bar{f} \circ p = p \circ f$. Let $\mathcal{B}$ denote the $\sigma$-algebra of Borel subsets of $M$ and $\mathcal{M}(M, f)$ the set of all $f$-invariant probability measures on $\mathcal{B}$, for a diffeomorphism $f$ of $M$. (The references for the following theorem are [21] and [18].)

**Theorem 5 [Oseledec's multiplicative ergodic theorem].** *Let $E \to M$ be a smooth, real vector bundle of rank $q$ over a compact manifold $M$ and $(\bar{f}, f)$ a bundle automorphism. Then there exist: 1. a set $\Lambda \in \mathcal{B}$ such that $\mu(\Lambda) = 1$ for all $\mu \in \mathcal{M}(M, f)$ and $f(\Lambda) = \Lambda$, 2. measurable, $f$-invariant functions $s : \Lambda \to \{1, \cdots, q\}$, $\chi_i : \Lambda \to \mathbb{R}$ $(1 \le i \le s)$ satisfying $\chi_1 < \cdots < \chi_s$, and 3. a measurable, $\bar{f}$-invariant decomposition $E|_\Lambda = E_1 \oplus \cdots \oplus E_s$ (the Oseledec decomposition for $\bar{f}$) such that for all $v \in E_i(x) \backslash \{0\}$,*

$$\lim_{n \to \pm\infty} \frac{1}{n} \log \|\bar{f}^n v\| = \chi_i(x)$$

*exists and the convergence is uniform over the vectors of unit norm in $E_i(x)$. For all $x \in \Lambda$ and all $v \in E(x) \backslash \{0\}$, the limits*

$$\chi^+(\bar{f}, v) \stackrel{\text{def}}{=} \lim_{n \to +\infty} \frac{1}{n} \log \|\bar{f}^n v\|,$$
$$\chi^-(\bar{f}, v) \stackrel{\text{def}}{=} \lim_{n \to -\infty} \frac{1}{n} \log \|\bar{f}^n v\|$$

*also exist. $\chi^+(\bar{f}, v) \ge \chi^-(\bar{f}, v)$ and we have $\chi^+(\bar{f}, v) = \chi^-(\bar{f}, v)$ if and only if $v \in E_i(x) \backslash \{0\}$ for some $i$.*

*If $L$ is a smooth, $\bar{f}$-invariant (vector) subbundle of $E$ with projection $\pi : E \to F = E/L$, let $F = F_1 \oplus \cdots \oplus F_{s'}$ be the decomposition of $F$ associated with characteristic exponents ${\chi'}_1 < \cdots < {\chi'}_{s'}$ ($s'$ and ${\chi'}_j$ being* *measurable functions on* $\Lambda$.*) Then, for each* $j$ $(1 \le j \le s')$ *there is* $i(j)$, $1 \le i(j) \le s$, *such that* ${\chi'}_j = \chi_{i(j)}$ *and* $F_j = \pi(E_{i(j)})$ *on* $\Lambda$. *Moreover* $L|_\Lambda = \bigoplus_{i=1}^{s}(L \cap E_i)$.

We also need a few basic facts about invariant connnections.

Let $p : F \to M$ be a $C^r$ vector bundle over $M$. Let $J_1(F)$ denote the vector bundle over $M$ consisting of first jets of germs of differentiable sections of $F$. The following short sequence of vector bundles is exact [14]:

$$0 \to T^*M \otimes F \xrightarrow{i} J_1(F) \xrightarrow{\pi} F \to 0.$$

A connection on $F$ can be described as a splitting of this exact sequence: $\sigma : F \to J_1(F)$, $\pi \circ \sigma = Id_F$. Such splitting defines a covariant derivative map $\nabla : \Gamma^1(F) \to \Gamma^0(T^*M \otimes F)$, where $\Gamma^r(F)$ denotes the space of $C^r$-sections of $F$, as follows: For $X \in \Gamma^1(F)$, and denoting $j_1X \in \Gamma^0(J_1(F))$ the first jet of $X$, set $\nabla X = (Id - \sigma \circ \pi)j_1X$.

We now assume that $(\bar{f}, f)$ is a $C^r$ automorphism of $p : F \to M$ (so that $p \circ \bar{f} = f \circ p$), from which we obtain automorphisms of $J_1(F)$ and $T^*M \otimes F$ as follows: For $j_1X(x) \in J_1(F)_x$, $\bar{f} \cdot j_1X(x) = j_1(\bar{f}X \circ f^{-1})(f(x))$ and for $\alpha \otimes X \in T^*M_x \otimes F_x$, $\bar{f}(\alpha \otimes X) = \bar{f}\alpha \otimes \bar{f}X$, where $\bar{f}\alpha = \alpha \circ Tf^{-1}|_{TM_{f(x)}}$.

An $\bar{f}$-invariant $C^r$-connection $\nabla : \Gamma^{r+1}(F) \to \Gamma^r(T^*M \otimes F)$ can be described as a $C^r$-splitting $\sigma$ of the above diagram, for which $\sigma \circ \bar{f}|_F = \bar{f}|_{J_1(F)} \circ \sigma$. Equivalently, it can be described as an $\bar{f}$-equivariant $C^r$ subbundle in $J_1(F)$, complementary to $i(T^*M \otimes F)$.

We can proceed with the proof of Theorem 4. Using the multiplicative ergodic theorem, we conclude that the characteristic exponents for the action of $\bar{f}$ on $J_1(TM)$ are the union of the set of exponents for the action on $TM$ and the set of exponents for the action on $T^*M \otimes TM$. But the assumption that $f$ avoids $2:1$-resonances implies (via Lemma 13 of [4]) that these sets must be disjoint. Therefore the sequence defined above must split. That the connection is unique follows from Lemma 14 of [4] and the fact that the difference of two invariant connections is an invariant tensor field of type $(2, 1)$, i.e. a section of $T^*M \otimes T^*M \otimes TM$. This concludes the proof.

If hyperbolicity is uniform, it is natural to expect that the connection just defined is continuous. This is the case if the diffeomorphism satisfies the *pinching* condition of the following definition.

An *Anosov diffeomorphism* is a diffeomorphism for which hyperbolicity holds in a uniform sense. More precisely, let $M$ be a compact $C^\infty$ manifold and $f$ a $C^\infty$ diffeomorphism of $M$. The diffeomorphism is *Anosov* if $TM$ decomposes continuously as a direct sum $TM = E^+ \oplus E^-$ of invariant subbundles $E^+$ and $E^-$, so that the following estimate applies: For some (in fact, any) Riemannian metric $\|\cdot\|$, there exist positive constants $C > 1$, $\epsilon < a < A$, such that for all $x \in M$, for all positive integers $n$, and for all $v \in E^{\pm}(x)$,

$$\frac{1}{C}\|v\|e^{-nA} \leq \|Tf_x^{\mp n}v\| \leq C\|v\|e^{-na}.$$

The subbundles $E^{\pm}$ are the tangent bundles of $C^0$ foliations $\mathcal{E}^{\pm}$, the *Anosov foliations* of $f$, whose leaves are smooth (if $f$ is smooth).

We say that $f$ satisfies the $\frac{1}{2}$*-pinching condition* if $A < 2a$.

For Anosov diffeomorphisms satifying the $\frac{1}{2}$-pinching, there exists a unique continuous invariant connection. The following theorem is taken from [4].

**Theorem 6.** *Let $M$ be a $C^\infty$ compact manifold and $f$ a $C^\infty$ Anosov diffeomorphism that satisfies the $\frac{1}{2}$-pinching assumption. Then $f$ preserves a continuous affine connection. This connection is unique among the $f$-invariant affine connections, it is torsion-free and, with respect to it, the leaves of the stable and unstable Anosov foliations are totally geodesic, complete and flat. (The restriction of this connection to the leaves of the stable and unstable foliations is differentiable, so it makes sense to define its curvature tensor there.) If this connection is $C^r$-differentiable, $r \geq 2$, then $f$ is $C^{r+2}$-conjugate to a hyperbolic automorphism of a complete flat manifold.*

In a way similar to what was done in the 'toy theorem', one shows that the restriction of the connection to leaves of the stable and unstable foliations is differentiable, which allows one to define the tangential curvature and prove that it vanishes. Existence could not be asserted in Theorem 1 due to the fact that an expanding map is not invertible. Otherwise, the convergent series that appears in the proof of Lemma 3 could have been used to characterize the invariant connection. This difficulty does not arise in Theorem 6.

In [9], M. Kanai defined an affine connection on the unit tangent bundle of a negatively curved Riemannian manifold invariant under the geodesic flow. Essentially the same construction produces an affine connection that is invariant under symplectic Anosov diffeomorphisms (or Anosov diffeomorphisms preserving any other nondegenerate bilinear form) with differentiable stable and unstable foliations: one uses the Bott connection to define the covariant derivative transversely to the Anosov foliations and the duality obtained from the symplectic form to define a covariant derivative tangentially (here one uses that the Anosov foliations must be Lagrangian foliations). In this symplectic case, $C^{r+1}$-regularity of the Anosov foliations is equivalent to the $C^r$ regularity of the invariant connection.

It shoud be pointed out that an affine connection preserved by a $C^\infty$ Anosov diffeomorphism (which, as noted before, is often uniquely determined by the diffeomorphism) may be $C^r$ without being $C^{r+1}$, for any $r$. (In fact, the connection may even be assumed to be torsion-free, flat, and complete.) This follows from a remarkable observation due to R. de la Llave concerning the regularity of conjugacies between Anosov diffeomorphisms [16]. The situation is quite different for systems preserving a smooth nondegenerate bilinear form, such as a symplectic form. In this case, the following theorem taken from [4] holds. In the symplectic case, the invariant connection is referred to as the *Kanai connection*. (Corollary 1.1 in [17] shows similar 'bootstrap' of regularity for conjugacies of symplectic diffeomorphisms when the conjugacy is part of a 1-parameter family of $C^1$-diffeomorphisms starting at the identity.)

**Theorem 7.** *Let $M$ be a $C^\infty$ compact manifold and $f$ a $C^\infty$ Anosov diffeomorphism preserving a $C^\infty$ symplectic form, whose Anosov foliations are $C^3$. Assume moreover that $f$ satifies the $2:1$-nonresonance condition. Then the Kanai connection of $f$ is $C^\infty$ and $f$ is $C^\infty$-conjugate (via an affine diffeomorphism) to a hyperbolic automorphism of a complete flat manifold.*

If one drops the pinching or the nonresonance assumption, it is expected that results similar to those above should still hold, with *flat* manifold replaced with *infranil*manifold. The best result to date in that direction is the following [3]:

**Theorem 8 [Benoist, Labourie].** *Let $M$ be a compact $C^\infty$ manifold with a $C^\infty$ affine connection. Let $f$ be a topologically transitive Anosov diffeomorphism preserving the connection and such that the stable and unstable distributions $E^+$ and $E^-$ are $C^\infty$. Then $f$ is $C^\infty$ conjugate to a hyperbolic automorphism of an infranilmanifold.*

We point out that the above theorem relies on Gromov's result [6] that asserts that the pseudo-group of local automorphisms of a rigid geometric structure admitting a topologically transitive group of automorphisms (that is, having a dense orbit) must be transitive on an *open* dense set. That result, in turn, requires a great deal of differentiability of the invariant structures and that is the reason for the $C^\infty$ assumption in the theorem. We believe that the use of Gromov's theorem can be avoided, and the differentiability conditions of the above theorem greatly improved.

### Higher order structures

Some of what is discussed above can also be applied to rigid structures of higher order. An example of a higher order structure is a connection on the principal bundle of $n$-order frames

$$F_rM = \{r\text{-jets at 0 of (germs of) diffeomorphisms from } \mathbb{R}^n \text{ into } M\}$$

(or a linear connection on the vector bundle of $r-1$-jets of (germs of) vector fields on $M$). The 'toy' rigidity theorem also holds if one assumes that a higher order connection is preserved.

It is tempting to suppose that some of the results discussed before would have counterparts for more general (rigid) structures and [6] is strong evidence for that. It also seems reasonable to conjecture that if a (sufficiently differentiable) rigid structure of higher order is preserved by an Anosov diffeomorphism, then it also preserves an affine connection (of related degree of differentiability). The following simple result is but a hint that something like that could be true. Recall that a *projective structure*, in the sense of differential geometry, is an equivalence class of affine connections with the same family of geodesic lines, without regard to parameter. It is a rigid structure of second order in the sense that an automorphism for such structure is determined by its second jet at one point.

**Proposition 9.** *Let an Anosov diffeomorphism $f$ of a compact manifold $M$ preserve a continuous projective structure. Then it also preserves a unique continuous affine connection compatible with the projective structure.*

*Proof.* Let $\nabla$ be a representative connection for the projective structure. Then invariance of $[\nabla]$ means that

$$\nabla^f - \nabla = \theta(f) \otimes I + I \otimes \theta(f),$$

where $\theta(f) \in \Gamma^0(T^*M)$. It can be easily shown that $f \mapsto \theta(f)$ is a cocycle over the $\mathbb{Z}$-action generated by the diffeomorphism, i.e. if $f_1$ and $f_2$ are powers of $f$, then

$$\theta(f_1 \circ f_2) = f_2^* \theta(f_1) + \theta(f_2).$$

To prove the existence of an invariant connection it suffices to show that the cocycle $\theta$ is a coboundary, i.e. there exists a continuous section $\alpha \in \Gamma^0(T^*M)$ such that $\theta(f) = \alpha - f^*\alpha$. In that case $\nabla' = \nabla + \alpha \otimes I + I \otimes \alpha$ will be invariant.

The Anosov decomposition $TM = E^+ \oplus E^-$ yields a decomposition of the cotangent bundle $T^*M = E^{+*} \oplus E^{-*}$, so that we can write $\theta(f) = \theta^+(f) + \theta^-(f)$. The series

$$\alpha^- = \sum_{i=0}^{\infty} f^{i*}\theta^-(f) \text{ and } \alpha^+ = -\sum_{i=1}^{\infty} f^{-i*}\theta^+(f)$$

are absolutely convergent and the form $\alpha = \alpha^+ + \alpha^-$ is the solution of the above cocycle equation.

### Actions of lattices of semisimple Lie groups

In this section, we would like to consider measure preserving actions by lattices of semisimple Lie groups of $\mathbb{R}$-rank greater than or equal to two.

Here, the main question was posed by Robert Zimmer in his address to the International Congress of Mathematicians in Berkeley, in 1986. He asked then [26] whether all ergodic, volume preserving actions of lattices in higher rank semisimple Lie groups on compact manifolds can be derived from essentially algebraic building-block actions obtained from homomorphisms of $G$ into other groups. The model actions given in [26] are of three types: (1) a special kind of isometric action; (2) affine actions on compact nilmanifolds and (3) left translations on compact quotients $L/\Lambda$ where $L$ is a connected Lie group, $\Lambda$ a cocompact lattice and $\Gamma$ acts on the quotient via a homomorphism $\pi : \Gamma \to L$.

A new class of actions, not previously considered by Zimmer, was discovered by Katok and Lewis in [10]. Their examples, obtained by a kind of blow-up construction at finite orbits of examples of type (2) above, show that $M$ cannot always be expected to possess an invariant complete locally homogeneous structure. Similar exceptional examples in which more geometric structure is preserved than only a volume form seems, however, very hard to construct and possibly do not exist. Their examples also leave open the possibility that the volume preserving actions of $\Gamma$ still correspond to those built out of the original examples described by Zimmer in [26] at least on an open invariant set (of full measure).

A significant part of the work to date concerning Zimmer's conjecture has been focused on proving that the examples of type (2) above (the nilmanifold case) are rigid under $C^r$ ($r \geq 1$) perturbations of the action when the group $\Gamma$ contains Anosov diffeomorphisms. The main approach, initiated by S. Hurder [7], consists of proving rigidity in essentially two steps. The first one is to show that such higher rank actions with Anosov elements are topologically rigid under perturbations. One makes use, here, of the classical structural stability theorem for Anosov diffeomorphisms and the fact that it suffices to control the action on a dense set of periodic points in order to obtain a common $C^0$-conjugacy that works for the entire group. The second step consists of using the action of a maximal abelian subgroup of diagonalizable elements of $\Gamma$ to prove that a continuous conjugacy is as smooth as the action itself.

The following nonperturbative theorem was achieved in [12]. It uses dynamical ideas from the theory of hyperbolic systems together with Zimmer's cocycle superrigidity.

**Theorem 10 [Katok, Lewis, Zimmer].** *Suppose* $\Gamma$ *is a subgroup of finite index in* $SL_n\mathbb{Z}$, $n \geq 3$, $M = \mathbb{T}^n$, *and* $\rho : \Gamma \to \mathrm{Diff}(M)$ *is a smooth action such that (i)* $\rho$ *preserves an absolutely continuous probability measure and (ii) there exists and element* $\gamma_0 \in \Gamma$ *such that the diffeomorphism* $\rho(\gamma_0)$ *is Anosov. Let* $\rho_* : \Gamma \to GL_n\mathbb{Z}$ *denote the homomorphism corresponding to the action on* $H_1(M) \simeq \mathbb{Z}^n$. *Then there exists a* 1*-cocycle* $\alpha : \Gamma \to \mathbb{Q}^n/\mathbb{Z}^n$ *(where* $\Gamma$ *acts on* $\mathbb{Q}^n/\mathbb{Z}^n$ *via* $\rho_*$*) and a diffeomorphism* $h$ *of* $M$ *conjugating* $\rho$ *to the affine action given by* $\rho_*$ *and* $\alpha$, *i.e.,* $\rho(\gamma) = h(\rho_*(\gamma) + \alpha(\gamma))h^{-1}$ *for every* $\gamma \in \Gamma$. *In particular,* $\rho$ *is smoothly conjugate to* $\rho_*$ *on a subgroup of finite index.*

We would like to suggest a more geometric approach to the problem of rigidity of lattice actions than the one used in Theorem 10: One first tries to obtain a rigid geometric sructure preserved by the action (say, a connection), possibly with little regularity. One then tries to recognize its geometry as corresponding to one of the model actions of Zimmer's conjecture. The main role played by hyperbolicity would be to yield better regularity properties for the invariant geometric structures, as in the 'toy' theorem of the first section. The main difference between this philosophy and that behind the proof of the above theorem is that we would like to shift the focus from the conjugating diffeomorphism to a formally infinitesimal object, prove that this object is locally homogeneous and obtain the conjugacy from a developing map. Theorem 12 below illustrates this.

The following observation concerning the existence of invariant connections was made by Zimmer [27].

**Theorem 11 [Zimmer].** *Let* $\Gamma$ *be a lattice in a higher rank semisimple Lie group* $G$. *Assume that* $\Gamma$ *acts ergodically on* $M$ *by diffeomorphisms preserving a Borel probability measure. Then there exists a* $\Gamma$*-invariant Borel affine connection on* $TM$. *As a special case, let* $\Gamma$ *be a lattice in* $SL_d\mathbb{R}$, $d \geq 3$, *and* $\dim M = d$. *If the action is not measurably isometric, it must preserve* a unique *Borel connection.*

If the group $\Gamma$ contains a $\frac{1}{2}$-pinched Anosov diffeomorphism, it follows from the above and Theorem 6 that $\Gamma$ actually preserves a *continuous* connection. With that in hand, one can prove the following theorem [5].

**Theorem 12.** *Let* $\Gamma$ *be a lattice in a higher rank semisimple Lie group* $G$. *Assume that* $\Gamma$ *acts on a compact,* $d$*-dimensional smooth manifold* $M$ *so as to preserve a volume form and so that some element of* $\Gamma$ *is a* $\frac{1}{2}$*-pinched Anosov diffeomorphism. Also assume that the dimension of the first nontrivial representation of the universal cover of* $G$ *is* $d$. *Then the action is* $C^2$ *isomorphic to an affine action on a flat manifold. (See example E1 given below.)*

Most of the work to establish the above result goes into the study of the holonomy group of $\Gamma$-invariant connections. The key fact concerning the geometry of such connections is the theorem stated below. We first make the following assumptions.

Let $\Gamma$ be a lattice subgroup of a higher rank semisimple Lie group $G$. More precisely, $G = \mathbf{G}(\mathbb{R})^\circ$ is the identity component of the set of real points of $\mathbf{G}$ and $\mathbf{G}$ is a semisimple $\mathbb{R}$-group whose almost simple factors have $\mathbb{R}$-rank at least 2. We assume that the elements of $\Gamma$ are automorphisms of a principal $H$-bundle $P$ that is an $H$-reduction of the frame bundle $F(M)$ of a compact $d$-dimensional smooth manifold $M$ and $H \subset SL_d\mathbb{R}$ is the set of real points of a linear algebraic $\mathbb{R}$-group $\mathbf{H}$. $\Gamma$ acts, in this way, as a group of diffeomorphisms of $M$ preserving a volume form. Assume, moreover:

- **H1.** *Given any nontrivial homomorphism* $\pi : \tilde{\mathbf{G}} \to \mathbf{H}$, *where* $\tilde{\mathbf{G}}$ *is the algebraic universal cover of* $\mathbf{G}$, *the representation of* $\tilde{\mathbf{G}}$ *that* $\pi$ *defines on* $\mathbb{C}^d$ *is irreducible.*
- **H2.** *The action of* $\Gamma$ *on* $M$ *preserves a* $C^r$ *connection* ($r \geq 1$) *on* $P$. *If the associated connection on* $TM$ *is torsion-free, it suffices that* $r \geq 0$.
- **H3.** *The action does not preserve a Riemannian metric.*
- **H4.** *The smooth measure associated to the* $\Gamma$*-invariant volume form has countably many ergodic components.*

Under the assumptions made above, we are left with the following subset of the examples considered in [26]:

- **E1.** *Affine actions on flat manifolds and tori.* $\Gamma$ acts on $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ via a homomorphism $\pi : \Gamma \to SL_d\mathbb{Z}$. We also allow for a translational component given by a finite homomorphism of $\Gamma$ into $\mathbb{R}^d$. More generally, a smooth manifold $M$ may admit $\mathbb{T}^d$ as a finite cover and the action of $\Gamma$ on $M$ lifts to an affine action (of some finite extension of $\Gamma$) on $\mathbb{T}^d$.
- **E2.** *Left translations on compact quotients of semisimple Lie groups.* Let $L$ be a connected, semisimple Lie group and $\Lambda$ a cocompact lattice in $L$. Also consider a homomorphism $\pi : \Gamma \to L$. Then $\Gamma$ acts on $L/\Lambda$ by left multiplication through $\pi$. More generally, $M$ may admit $L/\Lambda$ as a finite cover such that the action of $\Gamma$ on $M$ lifts to an action by left translation (by a finite extension of $\Gamma$) on that cover.

We can now state the main geometric lemma:

**Theorem 13.** *Let a lattice group* $\Gamma$ *of a higher rank semisimple Lie group* $G$ *act by bundle automorphisms on a principal* $H$*-bundle* $P$ *over a smooth* *compact manifold $M$, so that hypothesis H1 through H4 are satisfied. Then, either the Zariski closure of the connection's full holonomy group contains a nontrivial homomorphic image of $G$, or the action is $C^{r+2}$ isomorphic to either E1 or E2. Here $r \geq 1$ is the same as in H2, where the connection is assumed of class $C^r$. If the connection is torsion-free and $C^r$ for $r \geq 0$, the action is $C^{r+2}$ isomorphic to an example of type E1.*

**Corollary 14.** *Assume the same conditions of the theorem. Assume moreover that the connection in H2 is either Lorentzian or has amenable holonomy (e.g. a Riemannian or conformal connection), not necessarily without torsion, and is $C^r$. Then the action is $C^{r+2}$ isomorphic to either E1 or E2.*

The following comes from the proof of the theorem.

**Corollary 15.** *Assume the conditions of the theorem. Assume moreover that the invariant connection's restricted holonomy does not contain a nontrivial homomorphic image of $G$ and that the fundamental group of $M$ is virtually solvable. Then the action is $C^{r+2}$isomorphic to an affine action on a flat manifold.*

The fundamental fact concerning measure preserving actions of higher rank groups that is used in all of the above results is Zimmer's *cocycle superrigidity theorem.* It says, essentially, that the ergodic theory of such actions is known to a great extent from the knowledge of the linear representations of **G**. The following result is a special case of the theorem. It is the main tool used in the proof of the theorems discussed here.

For an algebraic $\mathbb{R}$-group **H** with real points $H$, the *algebraic hull* of an ergodic measure preserving action on a principal $H$-bundle is (the conjugacy class of) the algebraic subgroup $L$ such that there exists a measurable invariant reduction of the bundle to an $L$-subbundle, but not to an $L'$-subbundle, where $L'$ is a proper algebraic subgroup.

**Theorem 16 [Zimmer].** *Let $\mathbf{G}$ be a connected almost $\mathbb{R}$-simple $\mathbb{R}$-group with $\mathbb{R}$-rank at least two, $\mathbf{G}(\mathbb{R})^0$ the connected component of the set of real points of $\mathbf{G}$ and $\Gamma \subset \mathbf{G}(\mathbb{R})^0$ a lattice. Let $\mathbf{H}$ be a connected $\mathbb{R}$-simple Lie group (with trivial center) such that $\mathbf{H}(\mathbb{R})$ is not compact. Let $p : P \to M$ be an $\mathbf{H}(\mathbb{R})$-principal bundle over a manifold $M$ and assume that $\Gamma$ acts on $P$ via bundle automorphisms so that the action on $M$ leaves invariant an ergodic Borel probability measure $\mu$. If the algebraic hull of the action is $\mathbf{H}(\mathbb{R})$, there exists a Borel section $s : M \to P$ and an $\mathbb{R}$-rational homomorphism $\pi : \mathbf{G} \to \mathbf{H}$ such that for all $\gamma \in \Gamma$ and $\mu$-a.e. $p \in M$, $\gamma s(p) = s(\gamma(p))\pi(\gamma)$.*

It follows from the above that if $\Gamma$ acts as a group of automorphims of a vector bundle associated to $P$, the Lyapunov exponents of an element $\gamma \in \Gamma$ are determined by the eigenvalues of $\pi(\gamma)$. If the algebraic hull of the action is compact, then [25] $\Gamma$ must preserve a measurable Riemannian metric on any associated vector bundle.

## Geodesic flows on manifolds of negative curvature

The geodesic flow on the unit tangent bundle of a compact manifold of negative sectional curvature constitutes perhaps the earliest and most interesting example of a uniformly hyperbolic, or Anosov, flow. In this case, the regularity of invariant geometric structures on the unit tangent bundle of the manifold is related to the regularity of the *horospheric foliations*. If these foliations are $C^1$, there exists a canonically defined continuous connection on the unit tangent bundle, invariant under the geodesic flow. This connection, first defined by M. Kanai [9] was used in several papers concerning the rigidity of geodesic flows on negatively curved manifolds with smooth horospheric foliations. We refer the reader to [2] and the references therein.

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