A Sobolev Inequality and Neumann Heat Kernel Estimate for Unbounded Domains

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Abstract. Suppose $D$ is an unbounded domain in $\mathbb{R}^d$ ($d \geq 2$) with compact boundary and that $D$ satisfies a uniform interior cone property. We show that for $1 \leq p < d$, there exists a constant $c = c(D, p)$ such that for each $f \in W^{1,p}(D)$ the following Sobolev inequality holds:

$$\|f\|_q \leq c \|\nabla f\|_p,$$

where $1/q = 1/p - 1/d$ and for $r = p, q$, $\|\cdot\|_r$ denotes the norm in $L^r(D)$. As an application of this Sobolev inequality, assuming in addition that $D$ is a Lipschitz domain in $\mathbb{R}^d$ with $d \geq 3$, we obtain a Gaussian upper bound estimate for the heat kernel on $D$ with zero Neumann boundary condition.

1. Introduction

For a domain $U \subset \mathbb{R}^d$ and $p \in [1, \infty)$, we define $L^p(U)$ to be the space of real-valued functions defined on $U$ that are $L^p$-integrable relative to Lebesgue measure on $U$. The norm on $L^p(U)$ is given by

$$\|f\|_p = \left( \int_U |f(x)|^p \, dx \right)^{1/p}.$$

We further define

$$W^{1,p}(U) = \left\{ f \in L^p(U) : \frac{\partial f}{\partial x_i} \in L^p(U) \text{ for } i = 1, \ldots, d \right\},$$

with norm $\|f\|_{1,p} \equiv \|f\|_p + \|\nabla f\|_p$, where $\nabla f = \left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_d} \right)$. Here the partial derivatives $\frac{\partial f}{\partial x_i}$ are understood in the distributional sense. Note that in the above we do not indicate the dependence of $\|\cdot\|_p$ and $\|\cdot\|_{1,p}$ on $U$, since usually there will only be one relevant domain $U$. If there is

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any chance of ambiguity, we shall explicitly indicate the domain $U$ in the norm, for example, $\| \cdot \|_{U,p}$ for $\| \cdot \|_p$.

In the sequel, we assume that $D$ is an unbounded domain in $\mathbb{R}^d$ with compact boundary and that $D$ has the following uniform interior cone property, henceforth referred to simply as the cone property.

**Cone Property.** The domain $D$ is said to have the cone property if there exists a finite cone

$$V = \left\{ x = (x_1, \ldots, x_d) \in \mathbb{R}^d : x_d > \alpha (x_1^2 + \cdots + x_{d-1}^2)^{1/2} \text{ and } \| x \| < \beta \right\}$$

for some $\alpha, \beta > 0$ such that each point $x \in D$ is the vertex of a finite cone $V_x$ contained in $D$ which is congruent to $V$. Here $\| \cdot \|$ denotes the Euclidean norm on $\mathbb{R}^d$.

Our main result is the following.

**Theorem 1.** Suppose that $D$ is an unbounded domain with compact boundary and that $D$ has the cone property. Then for $1 \leq p < d$, there exists a constant $c = c(D,p)$ such that the following Sobolev inequality holds:

$$\| f \|_q \leq c \| \nabla f \|_p$$

for all $f \in W^{1,p}(D)$, where $1/q = 1/p - 1/d$.

It is well known (cf. [1], [5]) that the above Sobolev inequality holds with $W^{1,p}_0(D)$ in place of $W^{1,p}(D)$ for arbitrary domains $D$, where $W^{1,p}_0(D)$ is the subspace of $W^{1,p}(D)$ obtained by completing the space of $C^\infty$ functions having compact support in $D$ with respect to the norm $\| \cdot \|_{1,p}$. However, the Sobolev inequality (2) on $W^{1,p}(D)$ cannot hold without any restrictions on $D$. For example, (2) cannot be true for a domain $D$ with finite Lebesgue measure since in this case $1 \in W^{1,p}(D)$ and the right hand side of (2) vanishes.

In [9], using a form of capacity, Maz’ja characterizes the class $\mathcal{J}$ of open sets $D$ for which the Sobolev inequality (2) holds. He also gives the best constant $c$ in the Sobolev inequality (2) (see Theorem 4.7.4 of [9]). However we found Maz’ja’s condition difficult to check in practice, despite the fact that the class $\mathcal{J}$ is closed under the operation of taking finite unions (by Theorem 4.7.4 and Proposition 4.3.1/1 in [9]). This motivated us to prove the Sobolev inequality (2) directly under the assumptions in Theorem 1. In particular, by Theorem 1 and Theorem 4 below, unbounded domains with compact boundary having the cone property and exteriors of closed convex sets are in $\mathcal{J}$.

As an application of the Sobolev inequality (2) for $p = 2$ and $d \geq 3$, we shall prove Theorem 2 below. This has been applied in [3] to the study of
semilinear elliptic equations with Neumann boundary conditions. Before we can state Theorem 2, several notions need to be introduced.

A domain $D$ is said to be Lipschitz (or $C^{0,1}$) if locally near $\partial D$, $D$ can be represented as the region lying above the graph of a Lipschitz function (see, e.g., p.244 of [5]). For such a domain $D$, denote by $E$ the quadratic form defined on $W^{1,2}(D)$ by:

$$E(f, g) = \frac{1}{2} \int_D f(x) g(x) dx, \quad \text{for } f, g \in W^{1,2}(D).$$

There is a unique self-adjoint non-positive operator $A$, with domain $D(A)$, associated with $(W^{1,2}(D), E)$. In particular,

$$D(A) = \left\{ f \in W^{1,2}(D) : \exists g \in L^2(D) \text{ s.t. } E(f, h) = -\int_D gh dx \text{ for all } h \in W^{1,2}(D) \right\},$$

and for $f$ and $g$ as in the description of $D(A)$, $Af = g$ (see [7]). The symmetric strongly continuous contraction semigroup $\{P_t\}_{t>0}$ associated with $(A, D(A))$ has a symmetric integral kernel $p(t, x, y)$ which is smooth on $(0, \infty) \times D \times D$ and such that $P_t f(x) = \int_D p(t, x, y) f(y) dy$ a.e. on $D$ for $f \in L^2(D)$. See Lemma 2.11 of [6] for details on the existence of $p(t, x, y)$ (note that although in [6] domains are assumed to be bounded, the proof of the above fact works for unbounded domains as well). When $\partial D$ is smooth, $p(t, x, y)$ can be shown to be the fundamental solution for the heat equation with zero Neumann boundary condition (see [12], for example). By analogy, when $D$ is Lipschitz, we call $p(t, x, y)$ the heat kernel for $\frac{1}{2} \Delta$ on $D$ with zero Neumann boundary condition.

**Theorem 2.** Suppose $D$ is an unbounded domain with compact Lipschitz boundary in $\mathbb{R}^d$ where $d \geq 3$. Then the heat kernel $p(t, x, y)$ of $\frac{1}{2} \Delta$ on $D$ with zero Neumann boundary condition can be extended continuously to $(0, \infty) \times \overline{D} \times \overline{D}$; we still denote the extension by $p(t, x, y)$. Then there exist constants $c_1 > 0$ and $M > 1$ such that

$$p(t, x, y) \leq \frac{c_1 t^{d/2}}{M^2} \exp \left( -\frac{|x-y|^2}{Mt} \right), \quad \text{for all } t > 0, \quad x, y \in \overline{D}. \quad (3)$$

For $x, y \in \overline{D}$, let $G(x, y) = \int_0^\infty p(t, x, y) dt$. When $G(x, y)$ is finite for all $x, y \in \overline{D}$ with $x \neq y$, it is called the Green’s function for $\frac{1}{2} \Delta$ on $D$ with zero Neumann boundary condition. Integrating both sides of (3) gives:
Corollary 3. The Green’s function $G(x,y)$ for $\frac{1}{2}\Delta$ on $D$ with zero Neumann boundary condition exists and is continuous on $\bar{D} \times \bar{D}$, except on the diagonal. Furthermore, there exists a constant $c_2 = c_2(D) > 0$ such that
\begin{equation}
G(x,y) \leq \frac{c_2}{|x-y|^{d-2}} \quad \text{for all } x,y \in \bar{D}.
\end{equation}

2. Proof of Theorem 1

We begin by proving the Sobolev inequality (2) for the exterior of a closed convex set.

Theorem 4. Suppose that $U$ is the exterior of a closed convex set in $\mathbb{R}^d$. Then for $1 \leq p < d$ there exists a constant $c = c(U,p)$ such that for $f \in W^{1,p}(U)$,
\begin{equation}
\|f\|_q \leq c \|
abla f\|_p,
\end{equation}
where $1/q = 1/p - 1/d$. In particular, the above inequality holds for the exterior of a bounded closed ball.

Remark. In the above theorem, $U$ may have non-compact boundary.

Proof. Since $U$ is the exterior of a closed convex set, $U$ is Lipschitz (see, for example, Theorem 4.2 of Ch.V in [5]). Therefore by Theorem 4.7 of Chapter V in [5], the set of restrictions to $U$ of all $C^\infty$ functions with compact support in $\mathbb{R}^d$ is $\|\cdot\|_{1,p}$-dense in $W^{1,p}(U)$ for $p \geq 1$. Hence it suffices to prove (5) for all functions $f$ in $W^{1,p}(D)$ that are smooth in $D$ and such that $f(x)$ vanishes when $|x|$ is sufficiently large. Since $\mathbb{R}^d \setminus U$ is convex, for $x \in U$ and each $i \in \{1, \cdots, d\}$, there is a half-line in $U$ which is parallel to the $i$th coordinate axis and has $x$ as its initial point. Thus one has, for all $x \in U$,
\begin{equation}
|f(x)| \leq \int_{-\infty}^{\infty} 1_D(\xi) \left| \frac{\partial f}{\partial \xi_i}(\xi) \right| \, d\xi_i, \quad i = 1, 2, \ldots, d.
\end{equation}

Inequality (5) then follows from the standard argument for proving the corresponding Sobolev inequality in $W^{1,p}_0(D)$ (see the proof of Theorem 3.6 in Ch.V of [5], for example). \hfill \Box

For $r > 0$, denote by $B_r$ the ball $\{x \in \mathbb{R}^d : |x| < r\}$. Let $D_r = D \cap B_r$. Define $W^{1,p}_r(D_r)$ to be the closure in $W^{1,p}(D_r)$ of the set of restrictions to $D_r$ of all $C^\infty(\mathbb{R}^d)$ functions having compact support in $B_r$. Intuitively, $W^{1,p}_r(D_r)$ contains those functions in $W^{1,p}(D_r)$ that vanish on $\partial B_r$.

We have the following Poincaré inequality on $W^{1,p}_r(D_r)$ for $r > 0$ such that $B_r \supset \partial D$. 

[Continued text]
Lemma 5. Suppose that \(1 \leq p < d\) and \(r > 0\) such that \(B_r \supset \partial D\). There exists a constant \(c = c(D, r, p) > 0\) such that for each \(f \in W^{1, p}_r(D_r)\),

\[
\|f\|_p \leq c \|\nabla f\|_p.
\]

(7)

Proof. Let

\[
\lambda = \inf \left\{ \frac{\|\nabla f\|_p}{\|f\|_p} : f \in W^{1, p}_r(D_r) \right\}.
\]

There exists a sequence \(\{f_n\}_{n \geq 1} \subset W^{1, p}_r(D_r)\) such that \(\|f_n\|_p = 1\) for all \(n\) and \(\|\nabla f\|_p\) decreases to \(\lambda\) as \(n \to \infty\). Since \(D_r\) is a bounded domain with the cone property, by the Rellich-Kondrachov compactness theorem (see [1], p. 144), the imbedding \(W^{1, p}_r(D_r) \hookrightarrow L^p(D_r)\) is compact for \(1 \leq p < d\) (note that for this one uses the fact that \(p < q \equiv pd/(d - p)\)). Therefore, without loss of generality, we may assume that \(\{f_n\}_{n \geq 1}\) converges in \(L^p(D_r)\) to some \(f\) with \(\|f\|_p = 1\). Now suppose that \(\lambda = 0\). Then \(\|\nabla f_n\|_p\) decreases to zero as \(n \to \infty\), and for any smooth function \(\psi\) with compact support in \(D_r\) and \(i = 1, 2, \ldots, d\), using integration by parts we have

\[
\int_{D_r} f(x) \frac{\partial \psi}{\partial x_i}(x) \, dx = \lim_{n \to \infty} \int_{D_r} f_n(x) \frac{\partial \psi}{\partial x_i}(x) \, dx
\]

\[
= - \lim_{n \to \infty} \int_{D_r} \frac{\partial f_n}{\partial x_i}(x) \psi(x) \, dx
\]

\[
= 0.
\]

Thus \(\nabla f = 0\) and \(f\) is a constant function on \(D_r\). But \(f\) is a limit in \(W^{1, p}(D_r)\) of functions in \(W^{1, p}_r(D_r)\) and hence \(f \in W^{1, p}_r(D_r)\). The only constant function in this space is identically zero, which contradicts the fact that \(\|f\|_p = 1\). Therefore \(\lambda > 0\) and (7) is established with \(c = 1/\lambda\). \(\Box\)

Proof of Theorem 1. In this proof we have functions defined on different domains, \(U_1, U_2, D\). To clearly indicate which domain applies for integration, for this proof only we shall use the notation

\[
\|g\|_{U, p} = \left( \int_U |g(x)|^p \, dx \right)^{1/p}
\]

for a domain \(U\) and \(g \in L^p(U)\). In this proof, \(c\) will denote various constants.

Let \(r > 1\) such that \(B_{r-1} \supset \partial D\). Let \(\phi\) be a \(C^\infty\) function on \(\mathbb{R}^d\) with compact support in \(B_r\) such that \(0 \leq \phi \leq 1\) and \(\phi = 1\) on \(B_{r-\frac{1}{2}}\). Let \(U_1 = D \cap B_r\) and \(U_2 = \{x \in \mathbb{R}^d : \|x\| > r - 1\}\). For \(f \in W^{1, p}(D)\), set
\( f_1 = f \phi \) and \( f_2 = f (1 - \phi) \) on \( D \). Then \( f = f_1 + f_2 \). We next consider \( f_1 \) as an element of \( W^{1,p}_r(U_1) \) and \( f_2 \) as an element of \( W^{1,p}_0(U_2) \). Since \( U_1 \) is bounded and has the cone property, by the Sobolev imbedding theorem for \( W^{1,p}_r(U_1) \) (cf. Theorem 5.4 of [1]),
\[
\|f_1\|_{U_1,q} \leq c (\|\nabla f_1\|_{U_1,p} + \|f_1\|_{U_1,p}),
\]
which by Lemma 5 implies
\[
\|f_1\|_{U_1,q} \leq c \|\nabla f_1\|_{U_1,p} \leq c \|\nabla f\|_{D,p} + \|f \nabla \phi\|_{D,p}.
\]
By Theorem 4,
\[
\|f_2\|_{U_2,q} \leq c \|\nabla f_2\|_{U_2,p} \leq c \|\nabla f\|_{D,p} + \|f \nabla \phi\|_{D,p}.
\]
Since \( \nabla \phi \) is supported in \( B_r \setminus B_{r-1} \), by Hölder’s inequality,
\[
\|f \nabla \phi\|_{D,p} = \|f \nabla \phi\|_{U_2,p} \leq \|\nabla \phi\|_{D,d} \|f\|_{U_2,q}.
\]
Note that the restriction of \( f \) to \( U_2 \) is in \( W^{1,p}(U_2) \) and so by Theorem 4,
\[
\|f\|_{U_2,q} \leq c \|\nabla f\|_{U_2,p} \leq c \|\nabla f\|_{D,p}.
\]
Since \( f = f_1 + f_2 \) where \( f_1 \) has support in \( U_1 \) and \( f_2 \) has support in \( U_2 \), combining (8)-(11) proves the Sobolev inequality (2).

3. Proof of Theorem 2

In this section we assume that \( D \) is an unbounded domain in \( \mathbb{R}^d \) with compact Lipschitz boundary and \( d \geq 3 \). Thus, the Sobolev inequality (2) yields
\[
\|f\|_p \leq c \|\nabla f\|_2 \quad \text{for all } f \in W^{1,2}(D),
\]
with \( p = 2d/(d-2) \). Let \( p(t, x, y) \) be the heat kernel for \( \frac{1}{2} \Delta \) on \( D \) with zero Neumann boundary condition as described in Section 1. Recall that \( p(t, x, y) \) is symmetric in \( x, y \) and is smooth on \((0, \infty) \times D \times D \). By using the standard method in [4] (more specifically, Theorems 2.4.6, 2.2.3, and a straightforward adaptation of Section 3.2 to the case of zero Neumann boundary conditions), one can use (12) to show that
\[
p(t, x, y) \leq \frac{c_3}{t^{d/2}} \exp \left( -\frac{|x - y|^2}{Mt} \right), \quad \text{for all } (t, x, y) \in (0, \infty) \times D \times D,
\]
for some constants \( c_3 > 0 \) and \( M > 1 \).

We are going to show now that \( p(t, x, y) \) can be extended continuously to \((0, \infty) \times \overline{D} \times \overline{D} \). Let \( r > 0 \) such that \( B_r \supset \partial D \). Denote by \( h(t, x, y) \)
the (symmetric) heat kernel for \( \frac{1}{2}\Delta \) on \( D_r = D \cap B_r \) with zero Neumann boundary condition. From [2] we have that \( h \) can be extended continuously to \( (0, \infty) \times \overline{D}_r \times \overline{D}_r \) and that for each \( T > 0 \) there exist constants \( c_r = c_r(D, r, T) > 0 \) and \( M_r = M_r(D, r, T) > 1 \) such that

\[
(14) \quad h(t, x, y) \leq \frac{c_r}{t^{d/2}} \exp \left( -\frac{|x - y|^2}{M_r t} \right) \quad \text{for all} \quad (t, x, y) \in (0, T] \times \overline{D}_r \times \overline{D}_r.
\]

Let \( (Y, \{Q_x, x \in \overline{D}_r\}) \) be the continuous strong Markov process that is (normally) reflecting Brownian motion on the bounded Lipschitz domain \( D_r \) (see [2]). It follows from [2] that this process has \( h(t, x, y) \) as its transition density function. Denote by \( p_r(t, x, y) \) the symmetric integral kernel on \( \overline{D} \cap B_r \) for the semigroup of \( Y \) killed on hitting \( \partial B_r \). The existence of \( p_r \) follows from the strong Markov property of \( Y \) in a similar manner to that in [10], p.33; in particular we have

\[
(15) \quad p_r(t, x, y) = h(t, x, y) - E^{Q_x}[h(t - \tau_r, Y_{\tau_r}, y); \tau > \tau_r]
\]

for \( (t, x, y) \in (0, \infty) \times (\overline{D} \cap B_r) \times (\overline{D} \cap B_r) \), where \( \tau_r = \inf\{t \geq 0 : Y_t \in \partial B_r\} \).

Intuitively, \( p_r(t, x, y) \) is the heat kernel for \( \frac{1}{2}\Delta \) on \( D_r \) with zero Neumann boundary condition on \( \partial D \) and zero Dirichlet boundary condition on \( \partial B_r \).

For \( \epsilon \in (0, r) \), by (14),

\[
|h(t, x, y)| \leq \frac{c_r}{t^{d/2}} \exp \left( -\frac{\epsilon^2}{M_r t} \right)
\]

for all \( (t, x, y) \in (0, T] \times \partial B_r \times (\overline{D} \cap \overline{B}_{r-\epsilon}) \).

Thus as \( t \to 0 \), \( h(t, x, y) \) converges to zero uniformly for \( (x, y) \in \partial B_r \times (\overline{D} \cap \overline{B}_{r-\epsilon}) \). Therefore \( \{h(t, x, y)\}_{x \in \partial B_r} \) as a family of functions of \( (t, y) \) is equi-continuous on \( (0, T] \times (\overline{D} \cap \overline{B}_{r-\epsilon}) \). Since \( Y_{\tau_r} \in \partial B_r \),

\[
\{E^{Q_x}[h(t - \tau_r, Y_{\tau_r}, y); \tau > \tau_r]\}_{x \in \overline{D} \cap \overline{B}_r}
\]

is equi-continuous for \( (t, y) \in (0, T] \times (\overline{D} \cap \overline{B}_{r-\epsilon}) \). It follows from (15) that \( \{p_r(t, x, y)\}_{x \in \overline{D} \cap \overline{B}_{r-\epsilon}} \) is equi-continuous for \( (t, y) \in [\epsilon, T] \times (\overline{D} \cap \overline{B}_{r-\epsilon}) \) for any \( \epsilon \in (0, T) \). Since \( p_r(t, x, y) \) is symmetric in \( (x, y) \), \( p_r(t, x, y) \) is uniformly continuous on \( [\epsilon, T] \times (\overline{D} \cap \overline{B}_{r-\epsilon}) \times (\overline{D} \cap \overline{B}_{r-\epsilon}) \). Therefore \( p_r(t, x, y) \) is continuous on \( (0, T] \times (\overline{D} \cap \overline{B}_{r}) \times (\overline{D} \cap \overline{B}_{r}) \) for each \( T > 0 \).

On the other hand, if \( (X, \{P_x, x \in \overline{D}\}) \) denotes the continuous strong Markov process that is (normally) reflecting Brownian motion on \( \overline{D} \), then similar to (15) we have for \( (t, x, y) \in (0, \infty) \times D_r \times D_r \),

\[
(16) \quad p_r(t, x, y) = p(t, x, y) - E^{P_x}[p(t - \tau_r, X_{\tau_r}, y); \tau > \tau_r],
\]
where \( \tau_r = \inf\{t \geq 0 : X_t \in \partial B_r \} \). Since \( X_{\tau_r} \in \partial B_r \), it follows from the inequality (13) that for each positive integer \( k \), as a function of \((t,x,y)\) in \((0,\infty) \times D_k \times D_k\), \( E_{X_{\tau_r}}^P[p(t-\tau_r, X_{\tau_r}, y); t > \tau_r] \) converges to zero uniformly as \( r \to \infty \), where \( D_k = D \cap B_k \). Therefore for each \( k > 0 \), \( p_r(t,x,y) \) converges to \( p(t,x,y) \) uniformly on \((0,\infty) \times D_k \times D_k \) as \( r \to \infty \). Hence \( p(t,x,y) \) can be extended continuously to \((0,\infty) \times \overline{D_k} \times \overline{D_k} \) and therefore on \((0,\infty) \times \overline{D} \times \overline{D} \). Thus inequality (13) holds for \((t,x,y) \in (0,\infty) \times \overline{D} \times \overline{D} \). Theorem 2 is now proved. □

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