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INFINITE GENERATION FOR RINGS OF SYMMETRIC TENSORS

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If X is a smooth projective complex variety, $\pi : \mathbb{P} = \mathbb{P}(\Omega^1_X) \to X$ is the canonical projection and L is the tautological divisor class on $\mathbb{P}$, then the linear systems $\mid nL \mid$ (or equivalently the spaces $H^0(\mathrm{Symm}^n \Omega^1_X)$) are important in algebraic geometry. For example, Bogomolov [1] used them to prove the boundedness of the set of curves of given genus on X if X is a surface of general type with $c_1^2 > c_2$.

These tensors also appear in the context of complex differential geometry. For example, Gromov [2] has defined a Kähler manifold (X, ω) to be *Kähler hyperbolic* if on the universal cover $\widetilde{X}$ of X, the pull-back of ω can be written as $d\alpha$, where α is a 1-form that is bounded in the L_∞-norm with respect to the induced Kähler metric on $\widetilde{X}$, and has asked whether, if X is compact, Ω^1_X is ample (or equivalently, whether L is ample on $\mathbb{P}$). Then it is clear that if X is the product of two non-hyperelliptic curves of genus at least 3, the linear system $\mid nL \mid$ has no base points and defines a morphism ϕ_n that is birational onto its image for all $n \geq 1$, while $\mathbb{P}$ contains disjoint sections X_1 and X_2 corresponding to the two direct summands of Ω^1_X that are mapped to curves by every ϕ_n, so that Ω^1_X is not ample. On the other hand, X is Kähler hyperbolic, since it is the quotient of the Hermitian bounded symmetric domain $\Delta \times \Delta$, where Δ is the unit disc.

However, the cotangent bundle can have worse behaviour, even for Kähler hyperbolic varieties.

Theorem. *Suppose that X is an irreducible compact quotient of $\Delta \times \Delta$. Then X is Kähler hyperbolic and the algebra $\bigoplus_{n \geq 0} H^0(X, \mathrm{Symm}^n \Omega^1_X)$ of holomorphic symmetric tensors on X is not finitely generated.*

Perhaps the surprising thing about this is not that such surfaces should exist, but that they should be so accessible and that it should be so easy (as the argument below is intended to show) to prove this apparently pathological behavior.

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Proof. By assumption, $X = \Delta \times \Delta/\Gamma$, where Γ is an irreducible, torsion-free and co-compact lattice in $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$. (For example, take Γ to be of sufficiently high finite index in the orthogonal group of a lattice of rank 4 and signature $(2,2)$ that does not represent 0.)

Note first that X is certainly Kähler hyperbolic, as is any quotient of a Hermitian bounded symmetric domain.

Put $\Delta \times \Delta = \widetilde{X}$, and let $p, q : \widetilde{X} \to \Delta$ be the two projections; then $\Omega^1_{\widetilde{X}} \cong p^*\Omega^1_\Delta \oplus q^*\Omega^1_\Delta$ as Γ-linearized bundles, so that by descent $\Omega^1_X \cong \mathcal{O}(A) \oplus \mathcal{O}(B)$ for some divisor classes A, B on X.

Lemma 1. *$h^0(\mathcal{O}(nA)) = h^0(\mathcal{O}(nB)) = 0$ for all $n \geq 1$.*

Proof. Let Γ_1, Γ_2 be the images of Γ in $SL_2(\mathbb{R})$ via the projections p, q. Then $H^0(\mathcal{O}(nA))$, resp. $H^0(\mathcal{O}(nB))$, is the space of automorphic forms on Δ of weight $2n$ with respect to Γ_1, resp. Γ_2. However, Γ_1 and Γ_2 are dense in $SL_2(\mathbb{R})$, so that these spaces of forms are zero.

Let X_1, resp. X_2, be the section of $\mathbb{P}$ corresponding to the surjection $\Omega^1_X \to \mathcal{O}(A)$, resp. $\Omega^1_X \to \mathcal{O}(B)$. Note that $X_1 \sim L - \pi^*B$ and $X_2 \sim L - \pi^*A$, so that $X_1 + X_2 \sim 2L - \pi^*K_X$.

Lemma 2. *$K_X^2 \geq 8$.*

Proof. By Hirzebruch proportionality, $K_X^2 = 2c_2(X) > 0$, and by Noether's formula $K_X^2 + c_2(X) \equiv 0 \pmod{12}$. The result follows.

Lemma 3. *(i) $A^2 = B^2 = 0$.*
(ii) $A.C > 0$ and $B.C > 0$ for all curves C on X.

Proof. Suppose that C is a reduced and irreducible curve on X, and that $\widetilde{C}$ is a component of its inverse image on $\Delta \times \Delta$. Consider the projections $p,\, q : \widetilde{C} \to \Delta$. Since X is irreducible, these maps are nonconstant, so that the first Chern class of $\mathcal{O}_C(A)$, resp. $\mathcal{O}_C(B)$, is represented by a form whose pull-back to $\widetilde{C}$ is the pull-back from Δ of a positive form, by construction of $\mathcal{O}(A)$, resp. $\mathcal{O}(B)$. Hence $A.C > 0$ and $B.C > 0$. It follows that $A^2 \geq 0$ and $B^2 \geq 0$; since $\mathcal{O}(A) \hookrightarrow \Omega^1_X$ and $\mathcal{O}(B) \hookrightarrow \Omega^1_X$ it follows from the lemma of Castelnuovo, de Franchis and Bogomolov [1] that $A^2 = 0$.

Lemma 4. *For all $n \geq 4$, the codimension 1 part of the base locus* $\mathrm{Bs} \mid nL \mid$ *is $X_1 + X_2$.*

Proof. Note that

$$\pi_*\mathcal{O}(nL) = \mathcal{O}(nA) \oplus \mathcal{O}((n-2)A + K) \oplus \ldots \oplus \mathcal{O}(K + (n-2)B) \oplus \mathcal{O}(nB),$$

while

$$\begin{aligned}\pi_*\mathcal{O}(nL - X_1 - X_2) &= \pi_*\mathcal{O}((n-2)L + \pi^*K)) \\ &= \mathcal{O}((n-2)A + K) \oplus \ldots \oplus \mathcal{O}(K + (n-2)B),\end{aligned}$$

so that by Lemma 1 the inclusion $\mathcal{O}(nL - X_1 - X_2) \hookrightarrow \mathcal{O}(nL)$ is isomorphic on spaces of global sections. By Riemann-Roch and Serre duality, $H^0(\mathcal{O}(rK + sA))$ and $H^0(\mathcal{O}(rK + sB))$ are non-zero if either $r \geq 2$ and $s \geq 0$ or $r \geq 1$ and $s \geq 1$, so that if

$$\Sigma = \mathrm{Bs} \mid (n-2)A + K \mid \cap \mathrm{Bs} \mid (n-4)A + 2K \mid \cap \ldots \cap \mathrm{Bs} \mid K + (n-2)B \mid,$$

then the sheaf $\pi_*\mathcal{O}(nL - X_1 - X_2)$ is generated over $X - \Sigma$ by its sections over X. Hence the base locus of $\mid nL - X_1 - X_2 \mid$ is contained in $\pi^{-1}(\Sigma)$. However, by Reider's results [3], if $\mid (n-4)A + 2K \mid$ has a base point, then there is a curve E on X such that either $((n-4)A + K).E = 1$ and $E^2 = 0$, or $((n-4)A + K).E = 0$ and $E^2 = -1$. By Lemma 2 and the adjunction formula, this is impossible, so that $\mid (n-4)A + K \mid$ has no base points. Hence $\mid nL - X_1 - X_2 \mid$ has no fixed surface, and the lemma is proved.

To prove the theorem, recall an observation by Zariski [4]: if L is a divisor class on a smooth projective variety such that the codimension 1 part of $\mathrm{Bs} \mid nL \mid$ is non-empty and bounded for all $n \geq 0$, then the graded ring $\bigoplus_{n\geq 0} H^0(\mathcal{O}(nL))$ is not finitely generated. $\square$

Remark. Since L is not ample on $\mathbb{P}$, it must fail the Nakai-Moishezon numerical criterion. In fact L has the following numerical properties:

(i) $L^3 > 0$;

(ii) $L.C > 0$ for all curves C on $\mathbb{P}$, since Ω^1_X is a direct sum of line bundles $\mathcal{O}(A) \oplus \mathcal{O}(B)$ where $A.C > 0$ and $B.C > 0$ for all curves C on X;

(iii) $L^2.S \geq 0$ for all surfaces S on $\mathbb{P}$ (this is immediate from (ii)).

However, $L^2.S = 0$ if $S = X_1$ or X_2.

Note that if instead X is the product of two non-hyperelliptic curves, then the analogous divisor class L satisfies the same inequalities, except that (ii) must be replaced by the weaker condition

(ii)′ $L.C \geq 0$ for all curves C on $\mathbb{P}$.

References

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