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L^p-$L^{p'}$-ESTIMATES FOR HYPERBOLIC EQUATIONS

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Introduction

This expository article is based on the author's recent three papers ([4], [5], [6]) which were, or will be, published elsewhere.

We shall consider the following Cauchy problem:

$$\text{(C.P.)} \qquad \begin{cases} Pu = 0 \\ D_t^k u_{|t=0} = g_k \quad (k = 0, 1, \cdots, m-1). \end{cases}$$

The operator $P = P(D_t, D_x)$ is a homogeneous, constant-coefficient partial differential operator of degree m in $D_t, D_{x_1}, \cdots, D_{x_n}$ which is strictly hyperbolic; the symbol $p(\tau, \xi)$ can be factorized as

$$\begin{cases} p(\tau, \xi) = (\tau - \varphi_1(\xi)) \cdots (\tau - \varphi_m(\xi)) \\ \varphi_1(\xi) > \cdots > \varphi_m(\xi) \quad (\xi \neq 0). \end{cases}$$

For the sake of simplicity, we shall assume that each characteristic root $\varphi_l(\xi)$ $(l = 1, 2, \cdots, m)$ is either identically positive for $\xi \neq 0$ or identically negative. The wave equation $(p(\tau, \xi) = \tau^2 - |\xi|^2)$ is a typical example of the kind of operators we have in mind.

We shall express the solution to (C.P.) as

$$u(t) = \sum_{k=0}^{m-1} E_k(t) g_k,$$

and study the behavior of the operator norm $\|E_k(t)\|_{L^p \to L^{p'}}$ with respect to the time variable t, where $1 \leq p \leq 2$, $1/p + 1/p' = 1$.

In the case of the wave equation, the behavior is well known (Strichartz [3]) and can be applied to many semi-linear problems. Our main goal is to extend them to more general hyperbolic equations.

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In this problem, all we need is the L^p-$L^{p'}$-boundedness of $E_k(1)$ because of the following:

$$[E_k(t)g_k](x) = t^k [E_k(1)(g_k(t\cdot))](t^{-1}x). \tag{1}$$

Furthermore, each $E_k(1)$ is a linear combination of Fourier multipliers of the following type (modulo a regularizing operator):

$$M_k = F^{-1} e^{i\varphi(\xi)} a_k(\xi) F, \tag{2}$$

where

$$\begin{cases} \varphi(\xi) \in C^\omega(\mathbf{R}^n \setminus 0) \text{ is homogeneous of order } 1, \\ a_k(\xi) \in C^\infty(\mathbf{R}^n) \text{ is homogeneous of order } -k \text{ for large } |\xi|. \end{cases}$$

We shall assume $\varphi(\xi) > 0$ ($\xi \neq 0$), since it is one of the characteristic roots $\{\varphi_l(\xi)\}_{l=1}^m$. (Estimates for the case $\varphi(\xi) < 0$ can easily be derived from those for the case $\varphi(\xi) > 0$). Furthermore, we shall assume $n \geq 2$ since the case $n = 1$ is trivial.

The following result is our starting point.

Theorem 1 (Brenner [1]**).** *If the Gaussian curvature of the hypersurface*

$$\Sigma = \{\xi \in \mathbf{R}^n \setminus 0; \varphi(\xi) = 1\}, \tag{3}$$

never vanishes, then M_k is L^p-$L^{p'}$-bounded if $k > (n+1)(\frac{1}{p} - \frac{1}{2})$. This inequality can be replaced by an equality if $p \neq 1$.

In the case of equations of order 2 (e.g. the wave equation), the hypersurfaces Σ defined by their characteristic roots satisfy the assumption of Theorem 1. The following corollary is easily obtained if we notice equation (1).

Corollary 1 (Strichartz [3]**).** *Suppose $m = 2$. Then, for $t > 0$,*

$$\|E_k(t)g\|_{L^{p'}} \leq C t^{k-2n(\frac{1}{p}-\frac{1}{2})} \|g\|_{L^p} \tag{4}$$

if $k \geq (n+1)(\frac{1}{p} - \frac{1}{2})$. Here the constant $C > 0$ is independent of t and g.

Is the boundedness in Theorem 1 also true if the Gaussian curvature of Σ might vanish? The answer is *no*. That is what we shall discuss in this article. Although the L^p-boundedness of M_k is independent of any geometrical properties of Σ (Seeger-Sogge-Stein [2]), Σ has an essential effect on the L^p-$L^{p'}$-boundedness.

1. Convex case

In this section, we shall see that the L^p-$L^{p'}$-boundedness of operator M_k defined by (2) is really dependent on a particular geometrical property of hypersurface Σ defined by (3). Here we assume that Σ is convex. If the Gaussian curvature of Σ never vanishes, Σ is automatically convex. But the converse is not true.

In order to describe the dependence, we shall introduce an index for hypersurfaces in $\mathbf{R}^n$ (that is, submanifolds of codimension 1). We define $\gamma(\Sigma)$ to be the maximal order of contact of the hypersurface Σ to its tangent hyperplanes. More precisely

Definition 1. Let Σ be a hypersurface in $\mathbf{R}^n$. Then, for a point $p \in \Sigma$ and for a plane H (of dimension 2) which contains the normal line of Σ at p, we define the index $\gamma(\Sigma; p, H)$ to be the order of contact of the curve $\Sigma \cap H$ to the line $T \cap H$ at p. Here T denotes the tangent hyperplane of Σ at p. Furthermore, we define the index $\gamma(\Sigma)$ by

$$\gamma(\Sigma) = \sup_p \sup_H \gamma(\Sigma; p, H).$$

Example 1. Let $\Sigma = \{\xi \in \mathbf{R}^n \setminus 0; \varphi(\xi) = 1\}$, where

$$\varphi(\xi) = (\xi_1^{2N} + \xi_2^{2N} + \cdots + \xi_n^{2N})^{1/(2N)} \tag{5}$$

($\xi = (\xi_1, \xi_2, \ldots, \xi_n)$, $N \in \mathbf{N}$). Then $\gamma(\Sigma) = 2N$.

Example 2. Let $q(\xi)$ be a polynomial of order d. Suppose $\Sigma = \{\xi \in \mathbf{R}^n; q(\xi) = 0\}$ is compact and $\nabla q(\xi) \neq 0$ on Σ. Then $\gamma(\Sigma) \leq d$.

With this index, we can state our result. The following is a generalization of Theorem 1.

Theorem 2 ([4]). *Suppose Σ defined by (3) is convex. Then M_k is L^p-$L^{p'}$-bounded if $k > (2n - \frac{2(n-1)}{\gamma(\Sigma)})(\frac{1}{p} - \frac{1}{2})$. This inequality can be replaced by an equality if $p \neq 1$.*

Theorem 2 in the case $\gamma(\Sigma) = 2$ is equivalent to Theorem 1. The following theorem says that the L^p-$L^{p'}$-boundedness of M_k is really dependent on the index $\gamma(\Sigma)$.

Theorem 3 ([4]). *Let $\varphi(\xi)$ be defined by (5) and $a_k(\xi) = |\xi|^{-k}\chi(\xi)$, where $\chi(\xi) \in C^\infty(\mathbf{R}^n)$ equals 1 for large $|\xi|$ and vanishes near the origin. Then M_k is not L^p-$L^{p'}$-bounded if $k < (2n - \frac{(n-1)}{N})(\frac{1}{p} - \frac{1}{2})$.*

Now we shall apply Theorem 2 to our problem (C.P.). By virtue of Example 2, we can apply Theorem 2 with $\gamma(\Sigma) \leq m$. In fact, $\Sigma_l = \{\xi \in$

$\mathbf{R}^n; \varphi_l(\xi) = \pm 1\}$ is a component of $\{\xi \in \mathbf{R}^n; q(\xi) = 0\}$, where $q(\xi) = p(\pm 1, \xi)$. Hence we can have the following corollary which is equivalent to Corollary 1 in the case $m = 2$.

Corollary 2 ([4]). *Suppose P in (C.P.) satisfies the convexity condition (that is, all components of $\{\xi \in \mathbf{R}^n; p(\pm 1, \xi) = 0\}$ are convex). Then the estimate (4) holds if $k \geq (2n - \frac{2(n-1)}{m})(\frac{1}{p} - \frac{1}{2})$ and $p \neq 1$.*

2. Non-convex case

In Section 1, L^p-$L^{p'}$-estimates are given in the case that P in (C.P.) satisfies the convexity condition. But there exist hyperbolic operators which do not satisfy the convexity condition. The operator

$$P = (D_t^2 - 4D_{x_1}^2 - D_{x_2}^2)(D_t^2 - D_{x_1}^2 - 4D_{x_2}^2) - \varepsilon(D_{x_1}^2 + D_{x_2}^2)^2$$

($\varepsilon > 0$ small enough) is one example.

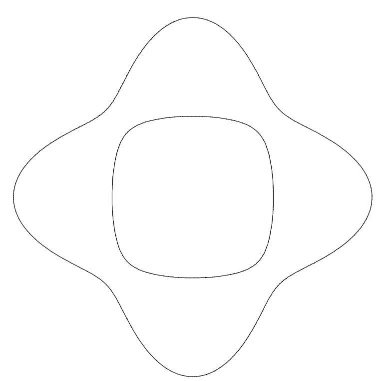

Figure 1. $\{\xi \in \mathbf{R}^2; p(\pm 1, \xi) = 0\}$

Now, we run up against a question. Can we remove the convexity assumption in Theorem 2? The objective of this section is to answer this question. We shall show that the answer is *yes* for $n = 2$ but is *no* for general n. In order to describe the answer, we shall introduce another index.

Definition 2. Let Σ be a hypersurface in $\mathbf{R}^n$, and let $\gamma(\Sigma; p, H)$ be the same index as in Definition 1. Then we define index $\gamma_0(\Sigma)$ by

$$\gamma_0(\Sigma) = \sup_p \inf_H \gamma(\Sigma; p, H).$$

We easily obtain $\gamma_0(\Sigma) \le \gamma(\Sigma)$ from Definitions 1 and 2. Equality holds when $n = 2$. But, in the case $n \ge 3$, $\gamma_0(\Sigma)$ and $\gamma(\Sigma)$ do not necessarily coincide with each other. Here is an example.

Example 3. Suppose $n \ge 3$. Let $\Sigma = \{\xi \in \mathbf{R}^n \setminus 0; \varphi(\xi) = 1\}$, where

$$\varphi(\xi) = \left\{(\xi_1^2 + \cdots + \xi_{n-1}^2 - \xi_n^2)^{2N} + \xi_n^{4N}\right\}^{1/(4N)} \tag{6}$$

($\xi = (\xi_1, \xi_2, \ldots, \xi_n)$, $N \in \mathbf{N}$). Then $\gamma(\Sigma) = 4N$, $\gamma_0(\Sigma) = 2N$.

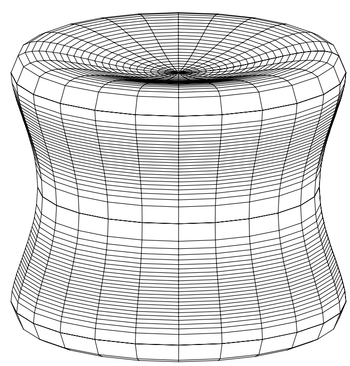

Figure 2. Σ $(n = 3,\ N = 1)$

We shall state our result. Let Σ be defined by (3) again.

Theorem 4 ([5])**.** *M_k is L^p-$L^{p'}$-bounded if $k > (2n - \frac{2}{\gamma_0(\Sigma)})(\frac{1}{p} - \frac{1}{2})$. This inequality can be replaced by an equality if $p \ne 1$.*

This result says that the convexity assumption in Theorem 2 can be removed in the case $n = 2$. Although it appears to be a bad result for larger n, it is optimal in the following sense.

Theorem 5 ([5])**.** *Suppose $n \ge 3$. Let $\varphi(\xi)$ be defined by (6) and $a_k(\xi) = |\xi|^{-k}\chi(\xi)$, where $\chi(\xi) \in C^\infty(\mathbf{R}^n)$ equals 1 for large $|\xi|$ and vanishes near the origin. Then M_k is not L^1-L^∞-bounded if $k < n - \frac{1}{2N}$.*

We can easily see, by Plancherel's theorem, that Theorem 4 is optimal for $p = 2$. Theorem 5 states that it is also optimal for another endpoint, $p = 1$. It also states that we cannot remove the convexity assumption in Theorem A in the case $n \ge 4$.

By the same reason as we obtain Corollary 2 from Theorem 2, we can obtain the following corollary from Theorem 4 with $\gamma_0(\Sigma) \le m$, which is a generalization of Corollaries 1 and 2 in the case $n = 2$.

Corollary 3. *The estimate (4) holds if* $k \geq (2n - \frac{2}{m})(\frac{1}{p} - \frac{1}{2})$ *and* $p \neq 1$.

3. Special case ($n = 3$)

In previous sections, we have exactly determined the relation between the L^p-$L^{p'}$-boundedness of M_k defined by (2) and a geometrical property of Σ defined by (3) in the case $n = 2$. But, in the case $n \geq 3$, there still remains the problem "What about the optimality of Theorem 4 for $1 < p < 2$?".

To our surprise, it is not optimal in spite of the optimality for $p = 1, 2$, and the boundedness is dependent more intricately on a geometrical property of Σ in the non-convex case. In the following, we shall exhibit such a strange phenomenon in the simplest case $n = 3$, $\gamma_0(\Sigma) = 2$ which we shall assume hereafter. We can improve Theorem 4 in this special case.

From now on, we shall microlocalize the problem. That is, we shall assume that $a_k(\xi)$ in (2) is supported in a sufficiently small conic neighborhood of a particular point $\omega \in S^2$, say $\omega = (0, 0, 1)$. Then Σ defined by (3) can be expressed as

$$\{(y, h(y)); y = (y_1, y_2) \in U\}$$

in the neighborhood, where $h(y) \in C^\omega(U)$ and $U \subset \mathbf{R}^2$ is an open neighborhood of the origin. For the sake of simplicity, we shall assume $\nabla h(0, 0) = 0$. We remark that

$$\operatorname{rank} h''(0, 0) \neq 0,$$

which is derived from the assumption that $\gamma_0(\Sigma) = 2$. In particular,

$$\det h''(0, 0) \neq 0$$

if the Gaussian curvature of Σ never vanishes.

In order to show how the boundedness is dependent on Σ, we shall classify functions $h(y)$ which are real analytic at the origin and satisfy $\nabla h(0, 0) = 0$, $\operatorname{rank} h''(0, 0) \neq 0$. We may assume that $h''_{11}(0, 0) \neq 0$, otherwise rotate the variable y appropriately. Then we define the functions $b_0(y_2)$ and $b_1(y_2)$, which are real analytic at the origin, by the equations

$$\text{(7)} \qquad \begin{cases} h'_1(b_1(y_2), y_2) = 0, b_1(0) = 0 \\ b_0(y_2) = h(b_1(y_2), y_2). \end{cases}$$

They are uniquely determined near the origin by the implicit function theorem. The curve $\{(b_1(t), t, b_0(t))\}$ is the *ridge* of the *mountain* Σ when we see it parallel to the y_2-axis.

Definition 3. Let b_j be defined by (7) and δ_j be the smallest integer $m \geq 2$ such that $b_j^{(m)}(0) \neq 0$ $(j = 0, 1)$. Then we say that $h(y)$ is of type I if $\delta_0 < \infty$, type II if $\delta_0 = \infty, \delta_1 < \infty$, and type III if $\delta_0 = \delta_1 = \infty$.

We remark that the definition above is well-defined. That is, when $\tilde{h}(y) = h(yT)$ satisfies $\tilde{h}''_{11}(0,0) \neq 0$ for an orthogonal matrix T, $\tilde{h}$ defines the same δ_0, and the same δ_1 if $\delta_0 = \infty$.

Typical examples are the following:

Example 4. The function $h(y)$ is of type I with $\delta_0 = 2$ if and only if $\det h''(0,0) \neq 0$.

Example 5. [I] $h(y) = 1 - (y_1^2 - y_2^N)$ is of type I ($b_0 = y_2^N$, $b_1 = 0$).
[II] $h(y) = 1 - (y_1 - y_2^N)^2$ is of type II ($b_0 = 0$, $b_1 = y_2^N$).
[III] $h(y) = 1 - y_1^2$ is of type III ($b_0 = b_1 = 0$).

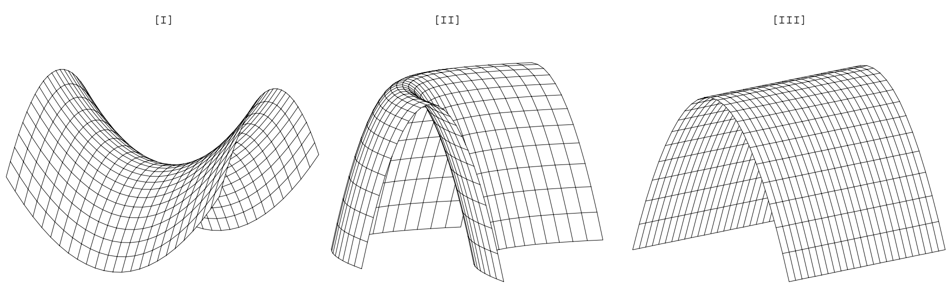

Figure 3. $\{(y, h(y))\}$

Each type has a different type of estimate.

Theorem 6 ([6]). *Suppose $h(y)$ is of type $*$ ($*$=I, II, III). Then M_k is L^p-$L^{p'}$-bounded if $k > k_*(p)$. This inequality can be replaced by an equality if $p \neq 1$ and $* \neq II$. Here*

$$k_*(p) = \begin{cases} (5 - \frac{2}{\delta_0})(\frac{1}{p} - \frac{1}{2}) & \text{if } * = I \\ \max\{6(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2}, (5 - \frac{1}{2\delta_1 - 1})(\frac{1}{p} - \frac{1}{2})\} & \text{if } * = II \\ 5(\frac{1}{p} - \frac{1}{2}) & \text{if } * = III. \end{cases}$$

We easily see that Theorem 6 is an improvement of Theorem 4 in the case $n = 3$ and $\gamma_0(\Sigma) = 2$. In fact, since Σ defined by (3) is a compact analytic hypersurface, it cannot contain any lines. Hence Σ is expressed microlocally by a function of type either I or II.

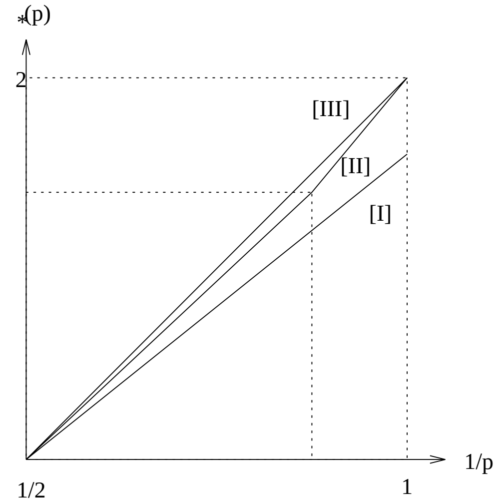

Figure 4. Graph of $k_*(p)$

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