Mathematical Research Letters **2**, 537–539 (1995)

ON THE SHAPE OF PROJECTIVE PLANE ALGEBRAIC CURVES

Norbert A'Campo

1. Introduction

In this paper, a *metric curve* is a projective plane algebraic curve with the induced metric from the standard Study-Fubini metric on $\mathbf{P}^2$. In [B] Fedor A. Bogomolov proved that there is no finite upper bound for the diameter of metric curves. His theorem disproves a conjecture of S. Frenkel and proves a conjecture of M. Gromov. The space of curves of fixed degree is compact and hence the least upper bound $\mathrm{diam}(d)$ of the diameters of metric curves of degree d is a real number. In [1] is stated that presumably $\mathrm{diam}(d)$ grows like $\log(d)$. The following theorem extends the result of Bogomolov:

Theorem. *For any $\epsilon > 0$, any compact metric tree admits an ϵ-isometric embedding in a nonsingular metric curve.*

A *metric tree* is a tree equipped with a path metric. A map $f: X \to Y$ between metric spaces (X, dist_X) and (Y, dist_Y) is an *ϵ-isometric embedding* if for any $a, b \in X$ the inequality $|\,\mathrm{dist}_X(a,b) - \mathrm{dist}_Y(f(a), f(b))| < \epsilon$ holds. The proof uses as in [1] a concentration lemma for intersection points.

Question. Does the above theorem hold for compact metric graphs?

2. Concentration of intersection points

Lemma. *Let C be a nonsingular curve given by a homogeneous equation $F(x,y,z) = 0$ of degree $d \geq 3$. For any $\epsilon > 0$ and any $p \in C$ there exist a natural number N and a homogeneous deformation $G(x,y,z)$ of $F(x,y,z)^N$ such that the curve C' given by the equation $G(x,y,z) = 0$ meets the curve*

Received July 28, 1995.

C in less then $g(C) = 1/2(d-1)(d-2)$ points which are all in the ϵ-ball with center p on the metric curve C.

Proof. We consider the Abel-Jacobi map $u\colon \mathrm{Div}(C) \to J(C)$, which sends p to 0, and its restriction $u^{(g)}\colon C^{(g)} \to J(C), g = g(C)$, to the g-th symmetric power of C. The holomorphic map $u^{(g)}$ is surjective [2], hence open. The image U under $u^{(g)}$ of the g-symmetric power of the open ϵ-ball with center p on C is open in $J(C)$ and contains 0. Choose natural numbers R, M such that $rU = J(C)$ for all natural numbers $r \geq R$ and such that $Md^2 \geq R$. Put $N := Mg$. Let X be the vectorfield on $\mathbf{P}^2$ of an infinitesimal projective motion and let s be the projection of its restriction to C in the normal bundle $L := (T\mathbf{P}^2)_{|C}/TC$ of C in $\mathbf{P}^2$. Let S be the divisor of the section s and $u(S) \in J(C)$ its image. In the g-th symmetric power of the open ϵ-ball with center p on C, choose $m := (p_1, \dots, p_g)$ such that $u^{(g)}(m) \in U$ satisfies $Nu(S) = Md^2 u^{(g)}(m)$ in $J(C)$. The Abel-Jacobi Theorem asserts the existence of a meromorphic function f on C with $(f) = -NS + Md^2 \sum_{1 \leq i \leq g} p_i$. The holomorphic section $t := fs^N$ of $L^{\otimes N}$ has its zeros at the points $p_1, \dots, p_g$. To the section t corresponds a deformation $G(x, y, z)$ of $F(x, y, z)^N$ which completes the lemma. $\square$

3. Proof of the Theorem

The Study-Fubini metric of $\mathbf{P}^2$ will be normalized to 1. Any ball of radius strictly smaller then 1 in $\mathbf{P}^2$ is contractible; hence, since any curve C in $\mathbf{P}^2$ carries a nontrivial homology class, its diameter in $\mathbf{P}^2$ exceeds 1 and a fortiori its diameter as a metric curve exceeds 1 too. The sphere L_p with center p of radius 1 in $\mathbf{P}^2$ is a line, so we have an even stronger fact: for any point p on a curve C and for all $q \in C \cap L_p$, $\mathrm{dist}_C(p, q) \geq \mathrm{dist}_{\mathbf{P}^2}(p, q) = 1$.

Subdividing the edges of a metric T tree by introducing new vertices of valency 2 until all edges are of length less then 1/2 does not change T as a metric space, so we need only prove the theorem for metric trees with edges all of length not exceeding 1/2.

Now we will prove the theorem by induction on the number of edges for any compact tree with edges all of length not exceeding 1/2 and any $0 < \epsilon < 1$. To start the induction, observe that trees with 0 or 1 edges admit isometric embeddings in any nonsingular curve of degree $d \geq 3$. Let T be a tree with $n + 1$ edges, $n \geq 1$, all of length not exceeding 1/2, and let $1 > \epsilon > 0$. The tree T admits a decomposition $T = T' \cup E$, where E is a terminal edge of T. By the induction hypothesis, there exists an $\epsilon/4$-isometric embedding $f'\colon T' \to C'$ of T' in a metric nonsingular curve C' of degree $d \geq 3$. Let $p' \in C'$ be the image of the attaching vertex v of the edge E. Using the lemma we construct a curve C'' which intersects the curve C' in at most $g(C')$ points all in a $\epsilon/4$-ball with center p' on C'. Choose $q \in C''$ and $p \in C' \cap C''$ with $\mathrm{dist}_{C''}(q, C' \cap C'') = \mathrm{dist}_{C''}(q, p) \geq 1 - \epsilon/2 \geq 1/2$; choose a distance realizing path γ in C'' from p to q; let $f_E\colon E \to \gamma \subset C''$ be the isometric embedding with $f_E(v) = p$. Moreover, deform the map f' to an $\epsilon/2$-isometric embedding $f'\colon T' \to C'$ with $f'(v) = p$. The maps f' and f_E glue to an $\epsilon/2$-isometric embedding $f' \cup f_E\colon T \to C' \cup C''$. Finally, we deform $C' \cup C''$ by a small deformation to a nonsingular curve C and lift the $\epsilon/2$-isometric embedding $f' \cup f_E\colon T \to C' \cup C''$ to an ϵ-isometric embedding $f : T \to C$.

References

1. F. A. Bogomolov, *On the diameter of plane algebraic curves*, Mathematical Research Letters **1** (1994), 95–98.
2. P. A. Griffiths, translated from the Chinese by Kuniko Weltin, *Introduction to Algebraic Curves*, Translations of Mathematical Monographs, A.M.S., vol. 76, 1989.

Mathematisches Institut der Universität, Rheinsprung 21, CH-4051, Basel, Switzerland

E-mail address: nacampo@math.unibas.ch