**B-STRUCTURES ON G-BUNDLES AND LOCAL TRIVIALITY**

V. G. Drinfeld and Carlos Simpson

1. Let $G$ be a split reductive group scheme over $\mathbb{Z}$ (recall that for any algebraically closed field $k$ there is a bijection $G \mapsto G \otimes k$ between isomorphism classes of such group schemes and isomorphism classes of connected reductive algebraic groups over $k$). Let $B$ be a Borel subgroup of $G$. Let $S$ be a scheme and $X$ a smooth proper scheme over $S$ with connected geometric fibers of pure dimension 1. Our goal is to prove the following theorems.

**Theorem 1.** Any $G$-bundle on $X$ admits a $B$-structure after a suitable surjective etale base change $S' \to S$.

**Theorem 2.** Any $G$-bundle on $X$ becomes Zariski-locally trivial after a suitable etale base change $S' \to S$.

**Theorem 3.** Suppose that $G$ is semisimple. Let $D$ be a subscheme of $X$ such that the projection $D \to S$ is an isomorphism. Set $U := X \setminus D$. Then for any $G$-bundle $F$ on $X$ its restriction to $U$ becomes trivial after a suitable faithfully flat base change $S' \to S$ with $S'$ being locally of finite presentation over $S$. If $S$ is a scheme over $\mathbb{Z}[n^{-1}]$ where $n$ is the order of $\pi_1(G(\mathbb{C}))$ then $S'$ can be chosen to be etale over $S$.

2. Remarks. a) Theorem 2 follows from Theorem 1 because a $B$-bundle on any scheme is Zariski-locally trivial.

b) If $S$ is the spectrum of an algebraically closed field Theorems 1-3 are well known (of course in this case base change is not necessary). In this situation Theorem 3 was proved in [6], while Theorems 1 and 2 follow from the triviality of $G$-bundles over the generic point of $X$. The triviality of the Galois cohomology $H^1(k(X), G)$ was conjectured by J. P. Serre and proved by R. Steinberg [9] and A. Borel and T. A. Springer [3]. Note that Steinberg’s result is for a perfect field of dimension 1—inconvenient here since $k(X)$ is not perfect in characteristic $p$ (whereas of course it is of dimension 1). The restriction to perfect fields was due to the need to have $G$. 

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split, which doesn’t matter here, and the need to apply Rosenlicht’s density theorem. The density theorem was improved by A. Grothendieck [4] and A. Borel and T.A. Springer ([2] Theorem A) to eliminate the perfect hypothesis. The resulting stronger version of Steinberg’s result is announced in [3] section 8.6.

c) Theorem 3 is an answer to a question by A. Beauville and Y. Laszlo. They proved Theorem 3 for $G = SL(n)$ ([1], Lemma 3.5). They showed that in this case it is enough to localize with respect to the Zariski topology of $S$. Theorem 3 is used in [1, 7] to prove the “uniformization” theorem.

d) Since we localize with respect to the etale topology of $S$ the splitness assumption on $G$ is not essential in Theorems 1-3 (in the situation of Theorem 1 one can define the notion of $B$-structure even if $G$ has no Borel subgroups).

e) If $S$ is a scheme over an algebraically closed field $k$ then Theorems 1-3 hold for $G$-bundles where $G$ is a connected affine algebraic group over $k$ which in the case of Theorem 3 must have the property $\text{Hom}(G, G_m) = 0$. This easily follows from Theorems 1-3 for $G$-bundles (the unipotent radical of $G$ does not matter).

3. Theorem 1 for $G$ follows from Theorem 1 for $G/Z^0$ where $Z^0$ is the connected component of the center $Z \subset G$. The same holds for Theorem 2, which also follows from Theorem 1. So we will suppose hereinafter that $G$ is semisimple. We will also assume that all the fibers of $X \to S$ have the same genus $g$.

4. Fix a Cartan subgroup $H \subset B$. Let $\alpha_i : H \to G_m$, $i \in \Delta$, be the simple roots. Denote by $\tilde{\alpha}_i$ the corresponding morphisms $B \to G_m$. For a $B$-bundle $E$ on a smooth projective curve over a field denote by $\deg_i(E)$ the degree of the $G_m$-bundle associated to $E$ and $\tilde{\alpha}_i$.

Let $F$ be a principal $G$-bundle on $X$ ($G$ acts on the scheme $F$ from the right). $B$-structures on $F$ can be identified with sections of $F/B \to X$. Consider the functor $\Phi$ that associates to a scheme $T$ over $S$ the set of $B$-structures on $F \times_S T$ considered as a $G$-bundle on $X \times_S T$. Identifying a section with its graph, the theory of Hilbert schemes shows that $\Phi$ is representable by a scheme $M_F$ locally of finite presentation over $S$. To a point $y \in M_F$ there corresponds a $B$-bundle $E_y$ on the fiber of $X$ over the image of $y$ in $S$. Set $d_i(y) := \deg_i(E_y)$. The functions $d_i$ are locally constant. Set $M_F^+ := \{y \in M_F : d_i(y) < \min(1, 2 - 2g), i \in \Delta\}$.

**Proposition 1.** The morphism $M_F^+ \to S$ is smooth.

**Proposition 2.** The morphism $M_F^+ \to S$ is surjective.
These propositions imply that for some surjective etale morphism \( S' \rightarrow S \) there exists an \( S \)-morphism \( S' \rightarrow M_F \). This is precisely Theorem 1.

**Proof of Proposition 1.** Let \( k \) be a field, \( y \in M_F(k) \). Let \( s \in S(k) \) be the image of \( y \) and \( X_s \) the fiber of \( X \) over \( s \). The point \( y \) corresponds to a section \( \sigma : X_s \rightarrow F/B \) of the morphism \( F/B \rightarrow X \). Standard deformation theory shows that the morphism \( M_F \rightarrow S' \) is smooth in a neighbourhood of \( y \) provided \( H^1(X_s, \sigma^*\Theta) = 0 \) where \( \Theta \) denotes the relative tangent bundle of \( F/B \) over \( X \). \( \sigma^*\Theta \) is the vector bundle associated to the \( B \)-bundle \( E_y \) on \( X_s \) corresponding to \( y \) and the \( B \)-module \( \text{Lie}(G)/\text{Lie}(B) \). So it is easy to see that if \( \deg_i(E_y) < \min(1, 2 - 2g) \) for all \( i \in \Delta \) then \( H^1(X_s, \sigma^*\Theta) = 0 \), Q.E.D.

5. The following statement is stronger than Proposition 2.

**Proposition 3.** Let \( Y \) be a smooth projective curve over an algebraically closed field \( k \) and \( F \) a \( G \)-bundle on \( Y \). Then for any number \( N \) there is a \( B \)-structure on \( F \) such that for the corresponding \( B \)-bundle \( E \) one has \( \deg_i(E) < -N \) for all \( i \in \Delta \).

Proof. 1) It does not matter which \( F \) to consider. Indeed, according to [9] [3] a \( G \)-bundle on \( Y \) is trivial over the generic point of \( Y \). So if \( F \) and \( F' \) are \( G \)-bundles on \( Y \) there is an isomorphism \( h \) between the restrictions of \( F \) and \( F' \) to \( Y \setminus R \) for some finite \( R \subseteq Y \). If \( h \) is fixed then a \( B \)-structure on \( F \) induces a \( B \)-structure on \( F' \) and for the corresponding \( B \)-bundles \( E \) and \( E' \) the inequalities \( -c < \deg_i(E) - \deg_i(E') < c \) hold for some \( c \) depending only on the singularities of \( h \).

2) So we can assume that \( F \) is trivial.

3) Since \( F \) is trivial we can assume that \( G \) is simply connected (otherwise the inverse image \( B' \) of \( B \) in the universal covering \( G' \) is a Borel; let \( F' \) be the trivial \( G' \)-bundle, then a \( B' \)-bundle which induces the \( G' \)-bundle \( F' \) must induce a \( B \)-bundle which induces \( F \)) and \( Y = P^1 \) (choose a finite morphism \( f : Y \rightarrow P^1 \) and consider those \( B \)-structures on the trivial \( G \)-bundle on \( Y \) which are pullbacks via \( f \) of \( B \)-structures on the trivial \( G \)-bundle on \( P^1 \)).

4) Denote by \( \text{Bun}_B \) the stack of \( B \)-bundles on \( P^1 \). Let \( \text{Bun}_B^n \) be the open subset of \( \text{Bun}_B \) corresponding to \( B \)-bundles \( E \) such that \( \deg_i(E) = n \) for all \( i \in \Delta \). There is an integer \( m > 0 \) such that \( \text{Bun}_B^n \neq \emptyset \) if \( m | n \). Choose \( n \) so that \( \text{Bun}_B^n \neq \emptyset \), \( n < -N \), \( n < 0 \). According to Proposition 1 the natural morphism \( \text{Bun}_B^n \rightarrow \text{Bun}_G \) is smooth. So its image \( U \) is open and non-empty. Since \( Y = P^1 \) trivial \( G \)-bundles form an open substack \( V \) of \( \text{Bun}_G \). Since \( G \) is simply connected \( \text{Bun}_G \) is irreducible (see Appendix). So \( U \cap V \neq \emptyset \), Q.E.D.
Remark. If char $k = 0$ one can replace steps 3 and 4 of the above proof by the following argument. There is a homomorphism $r : \text{SL}(2) \to G \otimes Q$ such that 1) $r$ maps diagonal matrices to $H \otimes Q$ and upper-triangular matrices to $B \otimes Q$, 2) $\alpha_i(r(\text{diag}(t, t^{-1})) = t^{m_i}$, $i \in \Delta$, where $m_i$ are positive (e.g., the “principal” homomorphism $\text{SL}(2) \to G \otimes Q$ has these properties). Using $r$ one reduces the problem for the trivial $G$-bundle to that for the trivial $\text{SL}(2)$-bundle. The latter problem is trivial: if $G = \text{SL}(2)$ then a $B$-structure on the trivial $G$-bundle is just a morphism $f : Y \to (G/B) \otimes k = P^1_k$ and the only condition is that deg $f$ should be big enough. We do not know whether this argument works if char $k \neq 0$.

6. Proof of Theorem 3. We can assume that $S$ is affine. According to Theorem 1 one can also suppose that $F$ comes from a $B$-bundle $E'$. Denote by $E$ the $H$-bundle obtained from $E'$ via the homomorphism $B \to H$ and by $F_1$ the $G$-bundle coming from $E$. Since $U$ is affine $E'$ and the $B$-bundle obtained from $E$ via the embedding $H \to B$ are isomorphic over $U$. So the restrictions of $F$ and $F_1$ to $U$ are isomorphic. Therefore one can assume that $F$ comes from an $H$-bundle $E$.

Let us reduce the problem to the case where $G$ is simply connected. Set $A = \text{Hom}(H, G_m)$. $E$ defines a section $u : S \to \text{Hom}(A, \text{Pic} X)$ where $\text{Pic} X$ denotes the relative Picard scheme of $X/S$ and $\text{Hom}$ denotes the scheme of homomorphisms. One can assume that $u(S) \subset \text{Hom}(A, \text{Pic}^0 X)$ (otherwise change $E$ without changing its restriction to $U$). Let $G'$ be the universal cover of $G$, $H'$ the preimage of $H$ in $G'$, $A' := \text{Hom}(H', G_m)$. $A$ is a subgroup of $A'$ and its index is equal to the order $n$ of $\text{Ker}(G' \to G)$. The morphism $\text{Hom}(A', \text{Pic}^0 X) \to \text{Hom}(A, \text{Pic}^0 X)$ is flat, finite, and of finite presentation; if $S$ is a scheme over $\mathbb{Z}[n^{-1}]$ it is etale. Denote by $S'$ the fibered product of $S$ and $\text{Hom}(A', \text{Pic}^0 X)$ over $\text{Hom}(A, \text{Pic}^0 X)$. The pullback of $E$ to $X \times_S S'$ can be lifted to an $H'$-bundle locally with respect to the Zariski topology of $S'$.

So one can assume that $G$ is simply connected and $F$ comes from an $H$-bundle $E$. Since $G$ is simply connected $\text{Hom}(G_m, H)$ is freely generated by simple coroots. So it suffices to show that if $H$-bundles $E_1$ and $E_2$ differ by the image of some $G_m$-bundle via a coroot $\bar{\alpha} : G_m \to H$ then the $G$-bundles corresponding to $E_1$ and $E_2$ are isomorphic locally over $S$. In fact we will show that it is true for $G$ replaced by the subgroup $L \subset G$ generated by $H$ and $r(\text{SL}(2))$ where $r : \text{SL}(2) \to G$ corresponds to $\bar{\alpha}$. It is easy to show that either $L = \text{SL}(2) \times T$ or $L = \text{GL}(2) \times T'$ where $T$ and $T'$ are tori. In the first case it suffices to show that the restriction to $U$ of an $\text{SL}(2)$-bundle on $X$ is trivial locally over $S$. In the second case it is enough to show that that the restriction to $U$ of two $\text{GL}(2)$-bundles on $X$ with the same determinant are isomorphic locally over $S$. The first
statement is proved by Beauville-Laszlo ([1], Lemma 3.5). The proof of the second statement is quite similar (in both cases Zariski localization is enough).

7. In the proof of Theorems 2 and 3 we used Theorem 1. Actually we could use the following weaker version of Theorem 1, which can be proved without using Propositions 2 and 3.

**Proposition 4.** In the situation of Theorem 3 for any G-bundle F on X there is a surjective etale base change S′ → S and a G-bundle F′ on X ×₅ S′ such that the inverse images of F and F′ on U ×₅ S′ are isomorphic and F′ has a B-structure.

Proof (inspired by [5] p. 364, lines 26-34). We can assume that S is strictly henselian, i.e., S is the spectrum of a strictly henselian ring (in this case base change is not necessary). Let s be the closed point of S, Xₙ the fiber of X over s, and Fₙ the restriction of F to Xₙ. According to [9] [3] the restriction of Fₙ to the generic point of Xₙ is trivial. So Fₙ has a B-structure, i.e., it comes from a B-bundle E on Xₙ. Since the restriction of E to U_s := U ∩ Xₙ comes from an H-bundle it is easy to construct a B-bundle E′ on Xₙ such that the restrictions of E and E′ to U_s are isomorphic and deg_i(E′) < min(1, 2 − 2g) for all i ∈ Δ (see Section 4 for the definition of deg_i). Denote by F′ₙ the G-bundle on Xₙ corresponding to E′. The restrictions of F′ₙ and Fₙ to U_s are isomorphic. We will construct a G-bundle F′ on X such that the restrictions of F and F′ to U are isomorphic and the restriction of F′ to Xₙ is equal to Fₙ. According to Proposition 1 such F′ automatically has a B-structure.

For every scheme X etale over X denote by M(X) the set of isomorphism classes of pairs consisting of a G-bundle F′ on X and an isomorphism f between the inverse images of F and F′ on U ×ₓ X. Then M(X) = H⁰(X, jₓj∗G/G) where G is the (etale) sheaf of automorphisms of F and j : U ↪ X. (To see this note first that pairs (F′, f) have no nontrivial automorphisms, consequently that M is the etale sheaf associated to the presheaf M₀ where M₀(X) ⊂ M(X) is the subset of pairs (F′, f) with F′ isomorphic to the inverse image of F, and finally that M₀(X) = H⁰(X, jₓj∗G/G) = H⁰(D, i∗(jₓj∗G/G)) where i : D ↪ X. Since D ≃ S is strictly henselian M(X) = (jₓj∗G)ₓ/Gₓ where w is the point of D over s and Gₓ is the stalk of G at w. Fix a trivialization of F over an etale neighbourhood of w and a local equation t = 0 defining D ⊂ X. These data define isomorphisms Gₓ = G(A{t}), (jₓj∗G)ₓ = G(A{t}[t⁻¹]) where A = H⁰(S, O_S) and A{t} is the henselization of A[t] at the maximal ideal containing t. Fₙ defines an element of G(k{t}[t⁻¹])/G(k{t})
where $k$ is the residue field of $A$, and we have to lift it to an element of $G(A[t][t^{-1}])/G(A[t])$. This can be done, e.g., using the Iwasawa decomposition $G(k[t][t^{-1}]) = B(k[t][t^{-1}])G(k[t])$.

**Appendix**

Let $Y$ be a smooth connected projective curve over an algebraically closed field $k$. Denote by $\text{Bun}_G$ the stack of $G$-bundles on $Y$. The following statement is well known but we could not find a reference with a proof valid for all $G$ without the assumption $\text{char } k = 0$.

**Proposition 5.** If $G$ is simply connected then $\text{Bun}_G$ is irreducible.

**Proof.** A well known deformation-theoretic argument shows that $\text{Bun}_G$ is smooth, so it suffices to show that $\text{Bun}_G$ is connected. The morphism $\text{Bun}_B \to \text{Bun}_G$ is surjective, so the mapping $\pi_0(\text{Bun}_B) \to \pi_0(\text{Bun}_G)$ is also surjective. The mapping $\pi_0(\text{Bun}_H) \to \pi_0(\text{Bun}_B)$ is bijective (use the homomorphism $B \to H$ to prove injectivity; surjectivity follows from the existence of a family of endomorphisms $f_t : B \to B$, $t \in A^1$, such that $f_1 = \text{id}$ and $f_0(B) = H$: just set $f_t(b) := \lambda(t)b\lambda(t)^{-1}$ where $\lambda \in \text{Hom}(G_m, H)$ is a strictly dominant coweight). Since $G$ is simply connected $\pi_0(\text{Bun}_H) = \text{Hom}(G_m, H)$ can be identified with the coroot lattice. So a standard argument (see, e.g., the end of Section 6) shows that it suffices to prove the connectedness of the stack of rank 2 vector bundles with a fixed determinant $M$. Given two such vector bundles $L_1$ and $L_2$ there is a line bundle $A$ such that $L_1 \otimes A$ and $L_2 \otimes A$ are generated by their global sections. Then according to a lemma due to Serre ([8], p.27, Lemma 24) there exist exact sequences

$$0 \to A^{\otimes(-1)} \to L_i \to M \otimes A \to 0, \ i = 1, 2.$$

So $L_1$ and $L_2$ belong to a family of bundles parametrized by the space $\text{Ext}(M \otimes A, A^{\otimes(-1)})$, which is connected, Q.E.D.

**Remark.** In the proof of Theorem 3 we used Proposition 5 in the particular case $Y = \mathbb{P}^1$. In this case there is another proof of Proposition 5: use the well known theorem that all $G$-bundles on $\mathbb{P}^1$ come from $H$-bundles, then compute the dimension of the automorphism group of each $G$-bundle and use the fact that all components of $\text{Bun}_G$ have the same dimension $d = -\dim G$. 
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Physico-Technical Institute of Low Temperatures, Lenin Avenue 47, Kharkov-164, 310164 Ukraine

Laboratoire Emile Picard, UFR-MIG, Université Paul Sabatier, 31062 Toulouse CEDEX, France
E-mail address: simpson@cict.fr