*Mathematical Research Letters* **2**, 823–829 (1995)

# $B$-STRUCTURES ON $G$-BUNDLES AND LOCAL TRIVIALITY

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**1.** Let $G$ be a split reductive group scheme over $\mathbb{Z}$ (recall that for any algebraically closed field $k$ there is a bijection $G \mapsto G \otimes k$ between isomorphism classes of such group schemes and isomorphism classes of connected reductive algebraic groups over $k$). Let $B$ be a Borel subgroup of $G$. Let $S$ be a scheme and $X$ a smooth proper scheme over $S$ with connected geometric fibers of pure dimension 1. Our goal is to prove the following theorems.

**Theorem 1.** *Any $G$-bundle on $X$ admits a $B$-structure after a suitable surjective etale base change $S' \to S$.*

**Theorem 2.** *Any $G$-bundle on $X$ becomes Zariski-locally trivial after a suitable etale base change $S' \to S$.*

**Theorem 3.** *Suppose that $G$ is semisimple. Let $D$ be a subscheme of $X$ such that the projection $D \to S$ is an isomorphism. Set $U := X \setminus D$. Then for any $G$-bundle $F$ on $X$ its restriction to $U$ becomes trivial after a suitable faithfully flat base change $S' \to S$ with $S'$ being locally of finite presentation over $S$. If $S$ is a scheme over $\mathbb{Z}[n^{-1}]$ where $n$ is the order of $\pi_1(G(\mathbb{C}))$ then $S'$ can be chosen to be etale over $S$.*

**2. Remarks.** a) Theorem 2 follows from Theorem 1 because a $B$-bundle on any scheme is Zariski-locally trivial.

b) If $S$ is the spectrum of an algebraically closed field Theorems 1-3 are well known (of course in this case base change is not necessary). In this situation Theorem 3 was proved in [6], while Theorems 1 and 2 follow from the triviality of $G$-bundles over the generic point of $X$. The triviality of the Galois cohomology $H^1(k(X), G)$ was conjectured by J. P. Serre and proved by R. Steinberg [9] and A. Borel and T. A. Springer [3]. Note that Steinberg's result is for a perfect field of dimension 1—inconvenient here since $k(X)$ is not perfect in characteristic $p$ (whereas of course it is of dimension 1). The restriction to perfect fields was due to the need to have $G$

Received October 9, 1995.

The first author was partially supported by INTAS grant No.94-4720 and DKNT grant No. 11.3/21.

split, which doesn't matter here, and the need to apply Rosenlicht's density theorem. The density theorem was improved by A. Grothendieck [4] and A. Borel and T.A. Springer ([2] Theorem A) to eliminate the perfect hypothesis. The resulting stronger version of Steinberg's result is announced in [3] section 8.6.

c) Theorem 3 is an answer to a question by A. Beauville and Y. Laszlo. They proved Theorem 3 for $G = \mathrm{SL}(n)$ ([1], Lemma 3.5). They showed that in this case it is enough to localize with respect to the Zariski topology of $S$. Theorem 3 is used in [1, 7] to prove the "uniformization" theorem.

d) Since we localize with respect to the etale topology of $S$ the splitness assumption on $G$ is not essential in Theorems 1-3 (in the situation of Theorem 1 one can define the notion of $B$-structure even if $G$ has no Borel subgroups).

e) If $S$ is a scheme over an algebraically closed field $k$ then Theorems 1-3 hold for $\mathcal{G}$-bundles where $\mathcal{G}$ is a connected affine algebraic group over $k$ which in the case of Theorem 3 must have the property $\mathrm{Hom}(\mathcal{G}, G_m) = 0$. This easily follows from Theorems 1-3 for $G$-bundles (the unipotent radical of $\mathcal{G}$ does not matter).

**3.** Theorem 1 for $G$ follows from Theorem 1 for $G/Z^0$ where $Z^0$ is the connected component of the center $Z \subset G$. The same holds for Theorem 2, which also follows from Theorem 1. So we will suppose hereinafter that $G$ is semisimple. We will also assume that all the fibers of $X \to S$ have the same genus $g$.

**4.** Fix a Cartan subgroup $H \subset B$. Let $\alpha_i : H \to G_m$, $i \in \Delta$, be the simple roots. Denote by $\bar{\alpha}_i$ the corresponding morphisms $B \to G_m$. For a $B$-bundle $E$ on a smooth projective curve over a field denote by $\deg_i(E)$ the degree of the $G_m$-bundle associated to $E$ and $\bar{\alpha}_i$.

Let $F$ be a principal $G$-bundle on $X$ ($G$ acts on the scheme $F$ from the right). $B$-structures on $F$ can be identified with sections of $F/B \to X$. Consider the functor $\Phi$ that associates to a scheme $T$ over $S$ the set of $B$-structures on $F \times_S T$ considered as a $G$-bundle on $X \times_S T$. Identifying a section with its graph, the theory of Hilbert schemes shows that $\Phi$ is representable by a scheme $M_F$ locally of finite presentation over $S$. To a point $y \in M_F$ there corresponds a $B$-bundle $E_y$ on the fiber of $X$ over the image of $y$ in $S$. Set $d_i(y) := \deg_i(E_y)$. The functions $d_i$ are locally constant. Set $M_F^+ := \{y \in M_F : d_i(y) < \min(1, 2-2g), i \in \Delta\}$.

**Proposition 1.** *The morphism $M_F^+ \to S$ is smooth.*

**Proposition 2.** *The morphism $M_F^+ \to S$ is surjective.*

These propositions imply that for some surjective etale morphism $S' \to S$ there exists an $S$-morphism $S' \to M_F$. This is precisely Theorem 1.

**Proof of Proposition 1.** Let $k$ be a field, $y \in M_F(k)$. Let $s \in S(k)$ be the image of $y$ and $X_s$ the fiber of $X$ over $s$. The point $y$ corresponds to a section $\sigma : X_s \to F/B$ of the morphism $F/B \to X$. Standard deformation theory shows that the morphism $M_F \to S$ is smooth in a neighbourhood of $y$ provided $H^1(X_s, \sigma^*\Theta) = 0$ where $\Theta$ denotes the relative tangent bundle of $F/B$ over $X$. $\sigma^*\Theta$ is the vector bundle associated to the $B$-bundle $E_y$ on $X_s$ corresponding to $y$ and the $B$-module $\mathrm{Lie}(G)/\mathrm{Lie}(B)$. So it is easy to see that if $\deg_i(E_y) < \min(1, 2-2g)$ for all $i \in \Delta$ then $H^1(X_s, \sigma^*\Theta) = 0$, Q.E.D.

**5.** The following statement is stronger than Proposition 2.

**Proposition 3.** *Let $Y$ be a smooth projective curve over an algebraically closed field $k$ and $F$ a $G$-bundle on $Y$. Then for any number $N$ there is a $B$-structure on $F$ such that for the corresponding $B$-bundle $E$ one has $\deg_i(E) < -N$ for all $i \in \Delta$.*

Proof. 1) It does not matter which $F$ to consider. Indeed, according to [9] [3] a $G$-bundle on $Y$ is trivial over the generic point of $Y$. So if $F$ and $F'$ are $G$-bundles on $Y$ there is an isomorphism $h$ between the restrictions of $F$ and $F'$ to $Y \setminus R$ for some finite $R \subset Y$. If $h$ is fixed then a $B$-structure on $F$ induces a $B$-structure on $F'$ and for the corresponding $B$-bundles $E$ and $E'$ the inequalities $-c < \deg_i(E) - \deg_i(E') < c$ hold for some $c$ depending only on the singularities of $h$.

2) So we can assume that $F$ is trivial.

3) Since $F$ is trivial we can assume that $G$ is simply connected (otherwise the inverse image $B'$ of $B$ in the universal covering $G'$ is a Borel; let $F'$ be the trivial $G'$-bundle, then a $B'$-bundle which induces the $G'$-bundle $F'$ must induce a $B$-bundle which induces $F$) and $Y = P^1$ (choose a finite morphism $f : Y \to P^1$ and consider those $B$-structures on the trivial $G$-bundle on $Y$ which are pullbacks via $f$ of $B$-structures on the trivial $G$-bundle on $P^1$).

4) Denote by $\mathrm{Bun}_B$ the stack of $B$-bundles on $P^1$. Let $\mathrm{Bun}_B^n$ be the open subset of $\mathrm{Bun}_B$ corresponding to $B$-bundles $E$ such that $\deg_i(E) = n$ for all $i \in \Delta$. There is an integer $m > 0$ such that $\mathrm{Bun}_B^n \neq \emptyset$ if $m \mid n$. Choose $n$ so that $\mathrm{Bun}_B^n \neq \emptyset$, $n < -N$, $n < 0$. According to Proposition 1 the natural morphism $\mathrm{Bun}_B^n \to \mathrm{Bun}_G$ is smooth. So its image $U$ is open and non-empty. Since $Y = P^1$ trivial $G$-bundles form an open substack $V$ of $\mathrm{Bun}_G$. Since $G$ is simply connected $\mathrm{Bun}_G$ is irreducible (see Appendix). So $U \cap V \neq \emptyset$, Q.E.D.

**Remark.** If $\operatorname{char} k = 0$ one can replace steps 3 and 4 of the above proof by the following argument. There is a homomorphism $r : \mathrm{SL}(2) \to G \otimes Q$ such that 1) $r$ maps diagonal matrices to $H \otimes Q$ and upper-triangular matrices to $B \otimes Q$, 2) $\alpha_i(r(\operatorname{diag}(t, t^{-1})) = t^{m_i}$, $i \in \Delta$, where $m_i$ are *positive* (e.g., the "principal" homomorphism $\mathrm{SL}(2) \to G \otimes Q$ has these properties). Using $r$ one reduces the problem for the trivial $G$-bundle to that for the trivial $\mathrm{SL}(2)$-bundle. The latter problem is trivial: if $G = \mathrm{SL}(2)$ then a $B$-structure on the trivial $G$-bundle is just a morphism $f : Y \to (G/B) \otimes k = P^1_k$ and the only condition is that $\deg f$ should be big enough. We do not know whether this argument works if $\operatorname{char} k \neq 0$.

**6. Proof of Theorem 3**. We can assume that $S$ is affine. According to Theorem 1 one can also suppose that $F$ comes from a $B$-bundle $E'$. Denote by $E$ the $H$-bundle obtained from $E'$ via the homomorphism $B \to H$ and by $F_1$ the $G$-bundle coming from $E$. Since $U$ is affine $E'$ and the $B$-bundle obtained from $E$ via the embedding $H \to B$ are isomorphic over $U$. So the restrictions of $F$ and $F_1$ to $U$ are isomorphic. Therefore one can assume that $F$ comes from an $H$-bundle $E$.

Let us reduce the problem to the case where $G$ is simply connected. Set $A = \mathrm{Hom}(H, G_m)$. $E$ defines a section $u : S \to \mathrm{Hom}(A, \operatorname{Pic} X)$ where $\operatorname{Pic} X$ denotes the relative Picard scheme of $X/S$ and Hom denotes the *scheme* of homomorphisms. One can assume that $u(S) \subset \mathrm{Hom}(A, \operatorname{Pic}^0 X)$ (otherwise change $E$ without changing its restriction to $U$). Let $G'$ be the universal cover of $G$, $H'$ the preimage of $H$ in $G'$, $A' := \mathrm{Hom}(H', G_m)$. $A$ is a subgroup of $A'$ and its index is equal to the order $n$ of $\operatorname{Ker}(G' \to G)$. The morphism $\mathrm{Hom}(A', \operatorname{Pic}^0 X) \to \mathrm{Hom}(A, \operatorname{Pic}^0 X)$ is flat, finite, and of finite presentation; if $S$ is a scheme over $\mathbb{Z}[n^{-1}]$ it is etale. Denote by $S'$ the fibered product of $S$ and $\mathrm{Hom}(A', \operatorname{Pic}^0 X)$ over $\mathrm{Hom}(A, \operatorname{Pic}^0 X)$. The pullback of $E$ to $X \times_S S'$ can be lifted to an $H'$-bundle locally with respect to the Zariski topology of $S'$.

So one can assume that $G$ is simply connected and $F$ comes from an $H$-bundle $E$. Since $G$ is simply connected $\mathrm{Hom}(G_m, H)$ is freely generated by simple coroots. So it suffices to show that if $H$- bundles $E_1$ and $E_2$ differ by the image of some $G_m$-bundle via a coroot $\check{\alpha} : G_m \to H$ then the $G$-bundles corresponding to $E_1$ and $E_2$ are isomorphic locally over $S$. In fact we will show that it is true for $G$ replaced by the subgroup $L \subset G$ generated by $H$ and $r(\mathrm{SL}(2))$ where $r : \mathrm{SL}(2) \to G$ corresponds to $\check{\alpha}$. It is easy to show that either $L = \mathrm{SL}(2) \times T$ or $L = \mathrm{GL}(2) \times T'$ where $T$ and $T'$ are tori. In the first case it suffices to show that the restriction to $U$ of an $\mathrm{SL}(2)$-bundle on $X$ is trivial locally over $S$. In the second case it is enough to show that that the restriction to $U$ of two $\mathrm{GL}(2)$-bundles on $X$ with the same determinant are isomorphic locally over $S$. The first statement is proved by Beauville-Laszlo ([1] , Lemma 3.5). The proof of the second statement is quite similar (in both cases Zariski localization is enough).

**7.** In the proof of Theorems 2 and 3 we used Theorem 1. Actually we could use the following weaker version of Theorem 1, which can be proved without using Propositions 2 and 3.

**Proposition 4.** *In the situation of Theorem 3 for any $G$-bundle $F$ on $X$ there is a surjective etale base change $S' \to S$ and a $G$-bundle $F'$ on $X \times_S S'$ such that the inverse images of $F$ and $F'$ on $U \times_S S'$ are isomorphic and $F'$ has a $B$-structure.*

Proof (inspired by [5] p. 364, lines 26-34). We can assume that $S$ is strictly henselian, i.e., $S$ is the spectrum of a strictly henselian ring (in this case base change is not necessary). Let $s$ be the closed point of $S$, $X_s$ the fiber of $X$ over $s$, and $F_s$ the restriction of $F$ to $X_s$. According to [9] [3] the restriction of $F_s$ to the generic point of $X_s$ is trivial. So $F_s$ has a $B$-structure, i.e., it comes from a $B$-bundle $E$ on $X_s$. Since the restriction of $E$ to $U_s := U \cap X_s$ comes from an $H$-bundle it is easy to construct a $B$-bundle $E'$ on $X_s$ such that the restrictions of $E$ and $E'$ to $U_s$ are isomorphic and $\deg_i(E') < \min(1, 2-2g)$ for all $i \in \Delta$ (see Section 4 for the definition of $\deg_i$). Denote by $F'_s$ the $G$-bundle on $X_s$ corresponding to $E'$. The restrictions of $F'_s$ and $F_s$ to $U_s$ are isomorphic. We will construct a $G$-bundle $F'$ on $X$ such that the restrictions of $F$ and $F'$ to $U$ are isomorphic and the restriction of $F'$ to $X_s$ is equal to $F_s$. According to Proposition 1 such $F'$ automatically has a $B$-structure.

For every scheme $\tilde{X}$ etale over $X$ denote by $M(\tilde{X})$ the set of isomorphism classes of pairs consisting of a $G$-bundle $F'$ on $\tilde{X}$ and an isomorphism $f$ between the inverse images of $F$ and $F'$ on $U \times_X \tilde{X}$. Then $M(\tilde{X}) = H^0(\tilde{X}, j_* j^* \mathcal{G}/\mathcal{G})$ where $\mathcal{G}$ is the (etale) sheaf of automorphisms of $F$ and $j : U \hookrightarrow X$. (To see this note first that pairs $(F', f)$ have no nontrivial automorphisms, consequently that $M$ is the etale sheaf associated to the presheaf $M_0$ where $M_0(\tilde{X}) \subset M(\tilde{X})$ is the subset of pairs $(F', f)$ with $F'$ isomorphic to the inverse image of $F$, and finally that $M_0(\tilde{X}) = H^0(\tilde{X}, j_* j^* \mathcal{G})/H^0(\tilde{X}, \mathcal{G})$.) In particular we have $M(X) = H^0(X, j_* j^* \mathcal{G}/\mathcal{G}) = H^0(D, i^*(j_* j^* \mathcal{G}/\mathcal{G}))$ where $i : D \hookrightarrow X$. Since $D \simeq S$ is strictly henselian $M(X) = (j_* j^* \mathcal{G})_w / \mathcal{G}_w$ where $w$ is the point of $D$ over $s$ and $\mathcal{G}_w$ is the stalk of $\mathcal{G}$ at $w$. Fix a trivialization of $F$ over an etale neighbourhood of $w$ and a local equation $t = 0$ defining $D \subset X$. These data define isomorphisms $\mathcal{G}_w = G(A\{t\})$, $(j_* j^* \mathcal{G})_w = G(A\{t\}[t^{-1}])$ where $A = H^0(S, O_S)$ and $A\{t\}$ is the henselization of $A[t]$ at the maximal ideal containing $t$. $F_s$ defines an element of $G(k\{t\}[t^{-1}])/G(k\{t\})$

where $k$ is the residue field of $A$, and we have to lift it to an element of $G(A\{t\}[t^{-1}])/G(A\{t\})$. This can be done, e.g., using the Iwasawa decomposition $G(k\{t\}[t^{-1}]) = B(k\{t\}[t^{-1}])G(k\{t\})$.

## Appendix

Let $Y$ be a smooth connected projective curve over an algebraically closed field $k$. Denote by $\mathrm{Bun}_G$ the stack of $G$-bundles on $Y$. The following statement is well known but we could not find a reference with a proof valid for all $G$ without the assumption $\operatorname{char} k = 0$.

**Proposition 5.** *If $G$ is simply connected then* $\mathrm{Bun}_G$ *is irreducible.*

Proof. A well known deformation-theoretic argument shows that $\mathrm{Bun}_G$ is smooth, so it suffices to show that $\mathrm{Bun}_G$ is connected. The morphism $\mathrm{Bun}_B \to \mathrm{Bun}_G$ is surjective, so the mapping $\pi_0(\mathrm{Bun}_B) \to \pi_0(\mathrm{Bun}_G)$ is also surjective. The mapping $\pi_0(\mathrm{Bun}_H) \to \pi_0(\mathrm{Bun}_B)$ is bijective (use the homomorphism $B \to H$ to prove injectivity; surjectivity follows from the existence of a family of endomorphisms $f_t : B \to B$, $t \in A^1$, such that $f_1 = \mathrm{id}$ and $f_0(B) = H$: just set $f_t(b) := \lambda(t)b\lambda(t)^{-1}$ where $\lambda \in \mathrm{Hom}(G_m, H)$ is a strictly dominant coweight). Since $G$ is simply connected $\pi_0(\mathrm{Bun}_H) = \mathrm{Hom}(G_m, H)$ can be identified with the coroot lattice. So a standard argument (see, e.g., the end of Section 6) shows that it suffices to prove the connectedness of the stack of rank 2 vector bundles with a fixed determinant $\mathcal{M}$. Given two such vector bundles $L_1$ and $L_2$ there is a line bundle $\mathcal{A}$ such that $L_1 \otimes \mathcal{A}$ and $L_2 \otimes \mathcal{A}$ are generated by their global sections. Then according to a lemma due to Serre ([8], p.27, Lemma 24) there exist exact sequences

$$0 \to \mathcal{A}^{\otimes(-1)} \to L_i \to \mathcal{M} \otimes \mathcal{A} \to 0, \ i = 1, 2.$$

So $L_1$ and $L_2$ belong to a family of bundles parametrized by the space $\mathrm{Ext}(\mathcal{M} \otimes \mathcal{A}, \mathcal{A}^{\otimes(-1)})$, which is connected, Q.E.D.

**Remark.** In the proof of Theorem 3 we used Proposition 5 in the particular case $Y = P^1$. In this case there is another proof of Proposition 5: use the well known theorem that all $G$-bundles on $P^1$ come from $H$-bundles, then compute the dimension of the automorphism group of each $G$-bundle and use the fact that all components of $\mathrm{Bun}_G$ have the same dimension $d = -\dim G$.

## Acknowledgements

The authors are grateful to A. Beauville and Y. Laszlo for a stimulating question and to A. A. Beilinson and V. A. Ginzburg for useful discussions.

## References

1. A. Beauville and Y. Laszlo, Conformal blocks and generalized theta functions, Commun. Math. Physics **164** (1994), 385-419
2. A. Borel and T.A. Springer, Rationality properties of linear algebraic groups, Proc. Symp. Pure Math. A.M.S. **9** (1966), 26-32.
3. A. Borel and T.A. Springer, Rationality properties of linear algebraic groups II, Tôhoku Math. J. **20** (1968), 443-497.
4. M. Demazure and A. Grothendieck, Schémas en groupes, S.G.A. 3.
5. G. Faltings, A proof for the Verlinde formula, Journal of Algebraic Geometry **3** (1994), 347-374.
6. G. Harder, Halbeinfache Gruppenschemata über Dedekindringen, Invent. Math. **4** (1967), 165-191
7. Y. Laszlo and C. Sorger, The line bundles on the moduli of parabolic $G$-bundles over curves and their sections, to appear
8. C. S. Seshadri, Fibrés vectoriels sur les courbes algébriques, Astérisque **96** (1982)
9. R. Steinberg, Regular elements of semisimple algebraic groups, Publ. Math. IHES **25** (1965), 281-312

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