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A CONVERSE THEOREM FOR GL_4

J. W. Cogdell and I. I. Piatetski-Shapiro

Let k be a global field. Let k_ν be the completion of k at a place ν and let $\mathbb{A}$ denote the adeles over k. Let $\psi = \otimes\psi_\nu$ be a non-trivial additive character of $\mathbb{A}$ which is trivial on k.

If $\pi = \otimes\pi_\nu$ is an arbitrary irreducible admissible generic representation of $GL_4(\mathbb{A})$ with a central character ω_π trivial on the principal ideles $k^\times$ and $\tau = \otimes\tau_\nu$ is an automorphic cuspidal representation of $GL_m(\mathbb{A})$, $m = 1, 2$, we define $L(\pi \times \tau, s)$ and $\varepsilon(\pi \times \tau, s)$ as in [3]. More specifically, for each place ν of k there is a well defined local L-function $L(\pi_\nu \times \tau_\nu, s)$ and ε-factor $\varepsilon(\pi_\nu \times \tau_\nu, s, \psi_\nu)$ as defined in [5] for non-archimedean ν and [7] for archimedean ν. We then set

$$L(\pi \times \tau, s) = \prod_\nu L(\pi_\nu \times \tau_\nu, s) \quad \text{and} \quad \varepsilon(\pi \times \tau, s) = \prod_\nu \varepsilon(\pi_\nu \times \tau_\nu, s, \psi_\nu).$$

As in [3], we assume that the product for $L(\pi, s)$ converges absolutely in some half-plane. From this it follows that the products for $L(\pi \times \tau, s)$ and $L(\tilde{\pi} \times \tilde{\tau}, s)$ also converge in a half plane [3], where $\tilde{\pi}$ and $\tilde{\tau}$ are the corresponding contragredient representations. We say that $L(\pi \times \tau, s)$ is *nice* if $L(\pi \times \tau, s)$ and $L(\tilde{\pi} \times \tilde{\tau}, s)$ have analytic continuation to entire functions of s, bounded in any vertical strip, and satisfy the standard functional equation

$$L(\pi \times \tau, s) = \varepsilon(\pi \times \tau, s) L(\tilde{\pi} \times \tilde{\tau}, 1 - s).$$

The purpose of this note is to outline the proof of the following Converse Theorems for $GL_4(\mathbb{A})$.

Theorem 1. *Let π be an irreducible admissible generic representation of $GL_4(\mathbb{A})$ whose central character ω_π is trivial on $k^\times$ and whose L-function $L(\pi, s)$ is convergent in some half plane. Assume that $L(\pi \times \tau, s)$ is nice for every cuspidal automorphic representation τ of $GL_2(\mathbb{A})$ and $GL_1(\mathbb{A})$. Then π is a cuspidal automorphic representation of $GL_4(\mathbb{A})$.*

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Theorem 2. *Let π be an irreducible admissible generic representation of $GL_4(\mathbb{A})$ whose central character ω_π is trivial on $k^\times$ and whose L-function $L(\pi, s)$ is convergent in some half plane. Let S be a finite set of finite places. Assume that $L(\pi \times \tau, s)$ is nice for every cuspidal representation τ of $GL_2(\mathbb{A})$ and $GL_1(\mathbb{A})$ which is unramified at the places in S. Then π is quasi-automorphic in the sense that there is an automorphic representation π' of $GL_4(\mathbb{A})$ such that $\pi'_\nu \simeq \pi_\nu$ for all $\nu \notin S$.*

Our proof of these results follows essentially the same method used with Jacquet and Shalika for the case of GL_3 [4,9]. The methods outlined here will also allow us to prove similar results for $GL_n(\mathbb{A})$ requiring control of twists by cuspidal automorphic representations of $GL_m(\mathbb{A})$ with $1 \leq m \leq n-2$. We will present the complete proofs in this general case in a future paper.

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1. Preliminary considerations

Let

$$P = \left\{ \begin{bmatrix} & & & * \\ & GL_3 & & * \\ & & & * \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\}, \qquad \tilde{P} = \left\{ \begin{bmatrix} & & & * \\ & GL_3 & & * \\ & & & * \\ 0 & 0 & 0 & * \end{bmatrix} \right\}$$

$$Q = \left\{ \begin{bmatrix} 1 & * & * & * \\ 0 & & & \\ 0 & & GL_3 & \\ 0 & & & \end{bmatrix} \right\}, \qquad \tilde{Q} = \left\{ \begin{bmatrix} * & * & * & * \\ 0 & & & \\ 0 & & GL_3 & \\ 0 & & & \end{bmatrix} \right\}$$

We refer to P and Q as mirabolic subgroups of GL_4. Let $R = P \cap Q$, $\tilde{R} = \tilde{P} \cap \tilde{Q}$, and $\tilde{R}_1 = P \cap \tilde{Q}$. Then

$$R = \left\{ \begin{bmatrix} 1 & * & * \\ 0 & GL_2 & * \\ 0 & 0 & 1 \end{bmatrix} \right\}, \quad \tilde{R} = \left\{ \begin{bmatrix} * & * & * \\ 0 & GL_2 & * \\ 0 & 0 & * \end{bmatrix} \right\},$$

$$\tilde{R}_1 = \left\{ \begin{bmatrix} * & * & * \\ 0 & GL_2 & * \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$

If π is an irreducible admissible generic representation of $GL_4(\mathbb{A})$ on the space V_π, then π has a Whittaker model $\mathcal{W}(\pi, \psi)$ [3,4,6] and we denote by

$W_\xi(g) \in \mathcal{W}(\pi, \psi)$ the associated Whittaker function of $\xi \in V_\pi$. For each $\xi \in V_\pi$ we define two functions on $GL_4(\mathbb{A})$ by

$$U_\xi(g) = \sum_{p \in N(k)\backslash P(k)} W_\xi(pg)$$

where N is the upper triangular maximal unipotent subgroup of GL_4 as well as of P and

$$V_\xi(g) = \sum_{q \in N(k)\backslash Q(k)} W_\xi(qg).$$

Note that N is also the maximal unipotent subgroup of Q.

For the convergence of these series, we refer the reader to [4] or [3]. Let $F_\xi(g) = U_\xi(g) - V_\xi(g)$. Then $\xi \mapsto F_\xi$ will be an intertwining map from V_π to the space of functions on $\tilde{R}(k)\backslash GL_4(\mathbb{A})$.

Proposition. *$F_\xi(g) \equiv 0$ for every $\xi \in V_\pi$ if and only if π is cuspidal automorphic.*

Proof. Assume that π is cuspidal automorphic. Then we may take V_π to be the associated space of cusp forms on $GL_4(\mathbb{A})$. It is well known [8] that any $\varphi \in V_\pi$ has a Fourier expansion of the form

$$\varphi(g) = \sum_{p \in N(k)\backslash P(k)} W_\varphi(pg).$$

Similarly, since Q is another mirabolic subgroup of GL_4, then repeating the usual proof of the Fourier expansion relative to P as in [8] we have

$$\varphi(g) = \sum_{q \in N(k)\backslash Q(k)} W_\varphi(qg),$$

which implies that if $\varphi \in V_\pi$ then $F_\varphi(g) = U_\varphi(g) - V_\varphi(g) \equiv 0$.

Conversely, if $F_\xi(g) = U_\xi(g) - V_\xi(g) \equiv 0$, then $U_\xi(g) = V_\xi(g)$ is left invariant under the group generated by $P(k)$, $Q(k)$, and the center $Z(k) = Z_4(k)$, that is, $GL_4(k)$. The map $\xi \to U_\xi(g)$ then defines an embedding of V_π into the space of automorphic forms. We can compute directly the integral of $U_\xi(g)$ with respect to any unipotent radical of any standard parabolic and show that it is 0 as in [3]. Similarly, in [3] we have shown that U_ξ is not identically zero. Hence π is cuspidal. □

With this proposition in hand, the idea is to (try to) prove that $F_\xi(g) \equiv 0$ for every $\xi \in V_\pi$. Since $F_\xi(g) = F_{\pi(g)\xi}(I)$, where here and throughout I

is the identity matrix in GL_4, it would suffice to show that $F_\xi(I) = 0$ for every $\xi \in V_\pi$.

Heuristically, the reason to expect to be able to obtain this from twists by GL_2 is that the mirabolics P and Q, with respect to which we have summed W_ξ in forming F_ξ, have intersection R which is essentially GL_2. In fact, R is a semi-direct product of GL_2 and a Heisenberg unipotent group. The presence of the unipotent group will keep us from obtaining directly that $F_\xi \equiv 0$ for all ξ. Note that for the converse theorems for GL_n [3] we worked with a pair of mirabolic subgroups whose intersection was precisely GL_{n-1}.

2. The Basic Lemma

Now let $H = \left\{ \begin{bmatrix} 1 & & \\ & GL_2 & \\ & & 1 \end{bmatrix} \right\}$ and $H^1 = H \cap SL_4$. Let $X = \left\{ x = \begin{bmatrix} 1 & & & x_1 \\ & 1 & & x_2 \\ & & 1 & x_3 \\ & & & 1 \end{bmatrix} \right\}$. Let $\psi_1 : X(\mathbb{A}) \to \mathbb{C}$ be defined by $\psi_1(x) = \psi(x_1)$. Here ψ is the fixed additive character of $k\backslash\mathbb{A}$ with respect to which the Whittaker model is defined.

At the heart of the converse theorems for GL_4 is the following lemma.

Basic Lemma. *Assume that $L(\pi \times \tau, s)$ is nice, where τ is a given generic automorphic representation occurring as a subspace of the space of automorphic forms on $GL_2(\mathbb{A})$. Then*

$$\int_{H^1(k)\backslash H^1(\mathbb{A})} \int_{X(k)\backslash X(\mathbb{A})} F_\xi(xh)\psi_1^{-1}(x)\varphi(h)dx\ dh = 0$$

for all $\varphi \in V_\tau$ and $\xi \in V_\pi$.

Proof. The proof basically involves formal computations similar to those in [4] or [9] combined with the techniques used in the proof of Theorem 1 of [3]. □

If we now appeal to Langlands' spectral theory for $SL_2(\mathbb{A})$ as in the proof of Proposition 6.4 of [3] we obtain the following corollary.

Corollary 1. *Assume that $L(\pi \times \tau, s)$ is nice for every cuspidal automorphic representation of $GL_2(\mathbb{A})$ and $GL_1(\mathbb{A})$. Then*

$$\int_{X(k)\backslash X(\mathbb{A})} F_\xi(x)\psi_1^{-1}(x)\ dx = 0$$

for all $\xi \in V_\pi$.

We would like to show that $F_\xi(I) = 0$ for all $\xi \in V_\pi$. Consider $F_\xi(x)$ with $x \in X(\mathbb{A})$. Since X is abelian and F_ξ is left invariant under $X(k)$, we have a Fourier expansion

$$F_\xi(I) = \sum_{\lambda \in k^3} \int_{X(k)\backslash X(\mathbb{A})} F_\xi(x)\psi^{-1}(\textstyle\sum \lambda_i x_i)\ dx.$$

From Corollary 1 to the Basic Lemma it follows that $\int F_\xi(x)\psi_1^{-1}(x)dx = 0$ for all $\xi \in V_\pi$. The group $\tilde{R}_1$ normalizes X and will hence act on the characters of X. If we consider the element $r = \begin{bmatrix} \lambda_1 & \lambda' \\ 0 & I_3 \end{bmatrix} \in \tilde{R}_1(k)$, where $\lambda' = (\lambda_2, \lambda_3, \lambda_4)$ then $F_\xi(rg) = F_\xi(g)$ and hence $F_{\pi(r)\xi}(x) = F_\xi(r^{-1}xr)$. Hence we have

$$\int_{X(k)\backslash X(\mathbb{A})} F_\xi(x)\psi_1^{-1}(rxr^{-1})\ dx = 0.$$

Now $\psi_1(rxr^{-1}) = \psi(\lambda_1x_1+\lambda_2x_2+\lambda_3x_3)$, for all $\lambda_1 \neq 0$ and λ_2, λ_3 arbitrary.

Inserting this into the Fourier expansion, we obtain

$$F_\xi(I) = F_\xi \begin{bmatrix} 1 & & x_1 \\ & I_2 & \\ & & 1 \end{bmatrix}$$

for all x_1.

This as far as we are able to get using our Basic Lemma. We record this in the following corollary.

Corollary 2. *Assume that $L(\pi \times \tau, s)$ is nice for every cuspidal automorphic representation of $GL_2(\mathbb{A})$ and $GL_1(\mathbb{A})$. Then, for every $\xi \in V_\pi$, $F_\xi \begin{bmatrix} 1 & & x_1 \\ & I_2 & \\ & & 1 \end{bmatrix}$ is a constant depending only on ξ. This constant is in fact $F_\xi(I)$.*

3. Some congruence type subgroups

To proceed, we will need to impose certain local conditions on our vector ξ to insure that $F_\xi(I) = 0$. We will then have to work around these local conditions. In order to accomplish this, we need the following lemmas.

If ν is a finite place of k, let $\mathcal{O}_\nu$ denote the ring of integers of k_ν, and let $\mathcal{P}_\nu$ denote the prime ideal of $\mathcal{O}_\nu$. For our application, we may assume that the local additive character ψ_ν is normalized, i.e., that ψ_ν is trivial on $\mathcal{O}_\nu$ and non-trivial on $\mathcal{P}_\nu^{-1}$. This will be the case at almost all finite places and can be arranged to hold at any given finite set of places S .

Given an integer $n_\nu \geq 0$ we consider the open compact congruence group

$$\begin{aligned} K_{0,\nu}(\mathcal{P}_\nu^{n_\nu}) = \{g = (g_{i,j}) \in GL_4(\mathcal{O}_\nu)\ |&(i)\ g_{i,j} \in \mathcal{P}_\nu^{n_\nu} \text{ for } i = 2,3 \text{ and } j = 1;\\ &(ii)\ g_{i,j} \in \mathcal{P}_\nu^{n_\nu} \text{ for } i = 4 \text{ and } j = 2,3;\\ &(iii)\ g_{4,1} \in \mathcal{P}_\nu^{2n_\nu}\}. \end{aligned}$$

(As usual, $g_{i,j}$ represents the entry of g in the i-th row and j-th column.)

Lemma 1. *Let ν be a finite place of k as above and let (π_ν, V_{π_ν}) be an irreducible admissible generic representation of $GL_4(k_\nu)$ with central character ω_{π_ν} . Then there is a vector $\xi_\nu^o \in V_{\pi_\nu}$ and a non-negative integer n_ν such that*

(1) *for any $g \in K_{0,\nu}(\mathcal{P}_\nu^{n_\nu})$ we have $\pi_\nu(g)\xi_\nu^o = \omega_{\pi_\nu}(g_{4,4})\xi_\nu^o$*

(2) $\displaystyle\int_{\mathcal{P}_\nu^{-1}} \pi_\nu \begin{bmatrix} 1 & & x_1 \\ & I_2 & \\ & & 1 \end{bmatrix} \xi_\nu^o \, dx_1 = 0.$

Proof. We will work in the Whittaker model $\mathcal{W}(\pi_\nu, \psi_\nu)$ of π_ν. The proof depends on the following two facts about the Whittaker model of generic representations [1,2,5].

(i) The map $\xi_\nu \mapsto W_{\xi_\nu} \begin{bmatrix} h & \\ & 1 \end{bmatrix}$, with $h \in GL_3(k_\nu)$, is injective on V_{π_ν}.

(ii) If f is any smooth function on $GL_3(k_\nu)$ such that $f(nh) = \psi_\nu(n)f(h)$ for $n \in N_3(k_\nu)$, the upper triangular maximal unipotent subgroup of $GL_3(k_\nu)$, and f is compactly supported modulo $N_3(k_\nu)$ then there exists $\xi_\nu \in V_{\pi_\nu}$ such that $W_{\xi_\nu} \begin{bmatrix} h & \\ & 1 \end{bmatrix} = f(h)$.

Let $T_\nu \subset GL_3(\mathcal{O}_\nu)$ be the open set defined by the extra condition $t = (t_{i,j}) \in T_\nu$ iff $|t_{3,1}|_\nu = 1$. Let $\xi_\nu^o \in V_{\pi_\nu}$ be determined by

$$W_{\xi_\nu^o} \begin{bmatrix} h & \\ & 1 \end{bmatrix} = \begin{cases} \psi_\nu(n) & h = nt \in N_3(k_\nu)T_\nu \\ 0 & h \notin N_3(k_\nu)T_\nu \end{cases}.$$

This completely determines ξ_ν^o by (i). Then ξ_ν^o satisfies the conditions of the lemma for n_ν sufficiently large since ξ_ν^o is fixed by some compact open subgroup of the form $K_\nu(\mathcal{P}^{m_\nu}) = \{g \in GL_4(\mathcal{O}_\nu) \mid g \equiv I_4 \mod \mathcal{P}^{m_\nu}\}$. □

Let S be a finite set of finite places and for each $\nu \in S$ let n_ν be a non-negative integer. Let $\mathcal{N} = \prod_{\nu \in S} \mathcal{P}_\nu^{n_\nu}$ and let $K_{0,S}(\mathcal{N}) = \prod_{\nu \in S} K_{0,\nu}(\mathcal{P}_\nu^{n_\nu})$. Let $G^S = \prod_{\nu \notin S} GL_4(k_\nu)$. Let $\Gamma = GL_4(k) \cap K_{0,S}(\mathcal{N})G^S$, $\Gamma_1 = \Gamma \cap \tilde{P}(k)$, and $\Gamma_2 = \Gamma \cap \tilde{Q}(k)$.

Lemma 2. *Γ is generated by Γ_1 and Γ_2.*

Proof. This is an elementary matrix calculation. $\square$

4. Proof of Theorem 1

Let ν_0 be a non-archimedean place of k as in Section 3 and take $\xi^o_{\nu_o} \in V_{\pi_{\nu_0}}$ satisfying the conditions of Lemma 1. Let $\xi = \xi^o_{\nu_0} \otimes \xi'$ where ξ' is an arbitrary vector in $\pi^{\nu_0} = \otimes_{\nu \neq \nu_0} \pi_\nu$. Then combining Corollary 2 of the Basic Lemma with our local conditions on ξ we have

$$F_\xi(I) = \mathrm{Vol}(\mathcal{P}_{\nu_0}^{-1})^{-1} \int_{\mathcal{P}_{\nu_0}^{-1}} F_\xi \begin{bmatrix} 1 & & x_1 \\ & I_2 & \\ & & 1 \end{bmatrix} dx_1 = 0$$

for this restricted set of ξ.

Now let $G' = K_{0,\nu_0}(\mathcal{P}_{\nu_0}^{n_{\nu_0}}) G^{\nu_0}$. Then for every ξ of the form $\xi = \xi^o_{\nu_0} \otimes \xi'$ we have $U_\xi(I) = V_\xi(I)$. Hence for $g \in G'$ we have $U_\xi(g) = U_{\pi(g)\xi}(I) = V_{\pi(g)\xi}(I) = V_\xi(g)$. Moreover, for $g \in G'$ we have $U_\xi(g)$ is left invariant under $G' \cap \tilde{P}(k) = \Gamma_1$ and $V_\xi(g)$ is left invariant under $G' \cap \tilde{Q}(k) = \Gamma_2$. By Lemma 2, Γ_1 and Γ_2 generate $\Gamma = G' \cap GL_4(k)$. So the map $\xi \mapsto U_\xi$ embeds the representation π^{ν_0} in the space $\mathcal{A}(\Gamma \backslash G')$ of automorphic functions on $\Gamma \backslash G'$ increasing slowly at infinity. Since $GL_4(\mathbb{A}) = GL_4(k)G'$ by weak approximation [10] and $GL_4(k) \cap G' = \Gamma$, we can extend π^{ν_0} to an automorphic representation of $GL_4(\mathbb{A})$. If we let π_0 denote an irreducible component of the extended representation, we see that π_0 coincides with π at all places except possibly at ν_0.

Let $\nu_1 \neq \nu_0$ be a second finite place. Then by the procedure above we get a corresponding π_1 which is automorphic and agrees with π except possibly at the place ν_1. But then π_1 and π_0 are two automorphic representations of $GL_4(\mathbb{A})$ which agree at all but at most two places. By the strong multiplicity one theorem for GL_4, as presented in [6], we must have that π_1 and π_2 are both constituents of the same automorphic induced representation $\Xi = \mathrm{Ind}(\sigma_1 \otimes \cdots \otimes \sigma_r)$ with each σ_i a cuspidal representation of some $GL_{m_i}(\mathbb{A})$, $m_i \leq 4$.

Suppose $r > 1$. Each σ_i is of the form $\sigma_i' |\det|^{t_i}$ where σ_i' is unitary cuspidal and $t_i \in \mathbb{R}$. We may assume that the σ_i are ordered so that $t_1 \geq \cdots \geq t_r$, since this does not effect the constituents. Since each proper parabolic subgroup of GL_4 corresponds to a partition $4 = m_1 + \cdots + m_r$, then one of the extremal representations σ_1 or σ_r must be a cuspidal representation of $GL_2(\mathbb{A})$ or $GL_1(\mathbb{A})$. Let us first assume it is σ_r. Let $\tau = \tilde{\sigma}_r$. Let S be a finite set of places, containing all archimedean places and the places ν_0 and ν_1, such that π, π_0, π_1, and all σ_i are unramified outside S. Then, if we consider the partial L-functions, we have by [5] and [7]

$$\begin{aligned} L^S(\pi \times \tau, s) &= L^S(\pi_0 \times \tau, s) = L^S(\pi_1 \times \tau, s) \\ &= \prod L^S(\sigma_i \times \tau, s) \\ &= \prod L^S(\sigma_i' \times \tilde{\sigma}_r', s + t_i - t_r). \end{aligned}$$

The factor $L^S(\sigma_r' \times \tilde{\sigma}_r', s)$ will have a pole at $s = 1$. The terms $L^S(\sigma_i' \times \tilde{\sigma}_r', s + t_i - t_r)$ can have no zeros in $\mathrm{Re}(s + t_i - t_r) \geq 1$, and hence no zeros in $\mathrm{Re}(s) \geq 1$ since $t_i \geq t_r$. Thus $L^S(\pi \times \tau, s)$ and hence $L(\pi \times \tau, s)$ will also have a pole at $s = 1$ as the local factors at S are never zero. But this contradicts our assumption that $L(\pi \times \tau, s)$ is nice. If we assume that σ_1 is a cuspidal representation of $GL_2(\mathbb{A})$ or $GL_1(\mathbb{A})$, then applying the same argument to $L(\tilde{\pi} \times \sigma_1, s)$ will again contradict the assumption that this L-function has no poles. Hence $r = 1$.

In this case, $\pi_0 = \pi_1 = \sigma_1$ is cuspidal. Comparing the local factors at ν_0 and ν_1, we see that $\pi = \pi_0 = \pi_1$ and hence π is cuspidal. $\square$

5. Proof of Theorem 2

Since $L(\pi \times \tau, s)$ is nice for all cuspidal τ which are unramified at S, we may apply the Basic Lemma for such τ and we find that

$$\int_{H^1(k)\backslash H^1(\mathbb{A})} \int_{X(k)\backslash X(\mathbb{A})} F_\xi(xh)\psi_1^{-1}(x)\varphi(h)dx\ dh = 0$$

for all $\varphi \in V_\tau$ and $\xi \in V_\pi$.

For each place $\nu \in S$ choose a vector ξ_ν^o as in Lemma 1 relative to some congruence subgroup $K_{0,\nu}(\mathcal{P}_\nu^{n_\nu})$ and fix $\xi_S^o = \otimes_{\nu \in S} \xi_\nu^o$. Define G' and Γ as in Lemma 2. So $\mathcal{N} = \prod_{\nu \in S} \mathcal{P}_\nu^{n_\nu}$, $K_{0,S}(\mathcal{N}) = \prod_{\nu \in S} K_{0,\nu}(\mathcal{P}_\nu^{n_\nu})$, and $G^S = \prod_{\nu \notin S} GL_4(k_\nu)$. Then $G' = K_{0,S}(\mathcal{N})G^S$ and $\Gamma = GL_4(k) \cap G'$.

Let $\pi^S = \otimes_{\nu \notin S} \pi_\nu$. Consider vectors of the form $\xi = \xi_S^o \otimes \xi^S$ with $\xi^S \in V_{\pi^S}$. By construction, ξ_S^o is invariant under the maximal compact subgroup $GL_2(\mathcal{O}_S)$ of $GL_2(k_S)$ embedded naturally in $H(k_S)$. Hence for ξ of the form $\xi = \xi_S^o \otimes \xi^S$ and representations τ of $GL_2(\mathbb{A})$ which are ramified at the places in S we also have

$$\int_{H^1(k)\backslash H^1(\mathbb{A})} \int_{X(k)\backslash X(\mathbb{A})} F_\xi(xh)\psi_1^{-1}(x)\varphi(h)dx\ dh = 0$$

for $\varphi \in V_\tau$.

Now, applying the spectral theory for $SL_2(\mathbb{A})$ as in Corollary 1 of the Basic Lemma, we again conclude that

$$\int_{X(k)\backslash X(\mathbb{A})} F_\xi(x)\psi_1^{-1}(x)\ dx = 0$$

but only for $\xi \in V_\pi$ of the form $\xi = \xi_S^o \otimes \xi^S$.

The group G' acts on the vectors of the form $\xi = \xi_S^o \otimes \xi^S$. Hence if we take $r \in \tilde{R}_1(k) \cap G'$ of the form $r = \begin{bmatrix} \lambda_1 & \lambda' \\ 0 & I_3 \end{bmatrix}$ and repeat the arguments leading to Corollary 2 of the Basic Lemma, we get

$$\int_{X(k)\backslash X(\mathbb{A})} F_\xi(x)\psi_1^{-1}(rxr^{-1})dx = 0$$

for all $\xi = \xi_S^o \otimes \xi^S$ and for all $r \in \tilde{R}_1(k) \cap G'$. Now $\psi(rxr^{-1}) = \psi(\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3)$, so that

$$\text{(A)} \qquad \int_{X(k)\backslash X(\mathbb{A})} F_\xi(x)\psi^{-1}(\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3)dx = 0$$

for all $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in k^3$ with $|\lambda_1|_\nu = 1$ and $|\lambda_i|_\nu \leq 1$ for $i = 2, 3$ and for all $\nu \in S$. Since

$$\int_{\mathcal{P}_\nu^{-1}} \pi_\nu \begin{bmatrix} 1 & & x_1 \\ & I_2 & \\ & & 1 \end{bmatrix} \xi_\nu^o\ dx_1 = 0$$

for all $\nu \in S$ by our choice of ξ, (A) is also valid for all λ with $|\lambda_1|_\nu < 1$ with $|\lambda_i|_\nu$ arbitrary for $i = 2, 3$ for all $\nu \in S$. If we have some $|\lambda_i|_\nu > 1$ for some $\nu \in S$, then (A) follows because ξ_ν^o is invariant under $K_{0,\nu}(\mathcal{P}_\nu^{n_\nu}) \cap \tilde{R}_1(k_\nu) = \tilde{R}_1(\mathcal{O}_\nu)$ for $\nu \in S$. Thus (A) holds for all λ with $\lambda_1 \neq 0$. If we now expand $F_\xi(x)$ in a Fourier expansion as in the proof of Corollary 2 of the Basic Lemma, we now have $F_\xi(I) = F_\xi \begin{bmatrix} 1 & & x_1 \\ & I_2 & \\ & & 1 \end{bmatrix}$ is a constant depending only on ξ, again for those ξ of the form $\xi = \xi_S^o \otimes \xi^S$.

We now proceed exactly as in the proof of Theorem 1 to embed π^S in the space $\mathcal{A}(\Gamma\backslash G')$ of automorphic forms increasing slowly at infinity. We can extend this uniquely to an automorphic representation of $GL_4(\mathbb{A})$ by weak approximation. If we take π' to be an irreducible component of the extended representation, we see that π' coincides with π at all places except possibly the places in S. $\square$

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Department of Mathematics, Oklahoma State University, Stillwater, OK 74078
E-mail address: cogdell@littlewood.math.okstate.edu

Department of Mathematics, Yale University, New Haven, CT 06520
E-mail address: ilya@math.yale.edu