Mathematical Research Letters **3**, 109–122 (1996)

REINHARDT DOMAINS WITH NON-COMPACT AUTOMORPHISM GROUPS

Siqi Fu, A. V. Isaev, and Steven G. Krantz

Abstract. We give an explicit description of smoothly bounded Reinhardt domains with noncompact automorphism groups. In particular, this description confirms a special case of a conjecture of Greene/Krantz.

0. Introduction

Let D be a bounded domain in $\mathbb{C}^n$, $n \geq 2$. Denote by $\mathrm{Aut}(D)$ the group of holomorphic automorphisms of D. The group $\mathrm{Aut}(D)$ with the topology of uniform convergence on compact subsets of D is in fact a Lie group (see [Ko]).

This paper is motivated by known results characterizing a domain by its automorphism group (see e.g. [R], [W], [BP]). More precisely, we assume that $\mathrm{Aut}(D)$ is not compact, i.e. there exist $p \in D$, $q \in \partial D$ and a sequence $\{F_i\}$ in $\mathrm{Aut}(D)$ such that $F_i(p) \to q$ as $i \to \infty$. A point $q \in \partial D$ with the above property is called a *boundary accumulation point for* $\mathrm{Aut}(D)$. An important issue for describing a domain D in terms of $\mathrm{Aut}(D)$ is the geometry of ∂D near a boundary accumulation point q (see e.g. [BP], [GK2]). In particular, we will be interested in the *type* of ∂D at q in the sense of D'Angelo [D'A1], which measures the order of contact that complex varieties passing through q may have with ∂D.

We note in passing that it is known that the Levi form of ∂D must be non-negative at a boundary accumulation point [GK1]. It is desirable to have additional geometric information about boundary accumulation points. We will be discussing the following conjecture that can be found in [GK2]: if D is a bounded domain in $\mathbb{C}^n$ with C^∞-smooth boundary and if q is a boundary accumulation point for $\mathrm{Aut}(D)$, then q is a point of finite type.

Received October 4, 1995.

Mathematics Subject Classification : 32A07, 32H05, 32M05.

The third author was supported by a grant from the National Science Foundation.

For convex domains the conjecture was studied in [Ki]. Here we assume that D is a Reinhardt domain (not necessarily pseudoconvex), i.e. a domain which the standard action of the n-dimensional torus $\mathbb{T}^n$ on $\mathbb{C}^n$,

$$z_j \mapsto e^{i\phi_j} z_j, \qquad \phi_j \in \mathbb{R}, \quad j = 1, \dots, n, \tag{0.1}$$

leaves invariant. The automorphism groups of bounded (and even hyperbolic) Reinhardt domains have been determined in [Su], [Sh2], [Kr]. We will use this description to prove the following classification result.

Theorem. *If D is a bounded Reinhardt domain in $\mathbb{C}^n$ with C^∞-smooth boundary, and if* $\mathrm{Aut}(D)$ *is not compact, then, up to dilations and permutations of coordinates, D is a domain of the form*

$$\left\{|z^1|^2 + P(|z^2|, \dots, |z^p|) < 1\right\}, \tag{0.2}$$

where P is a non-negative polynomial:

$$P(|z^2|, \dots, |z^p|) = \sum_{j=2}^{p} |z^j|^{2m_j} + \sum_{l_2, \dots, l_p} a_{l_2, \dots, l_p} |z^2|^{2l_2} \dots |z^p|^{2l_p},$$

$a_{l_2,\dots,l_p}$ are real parameters, $m_j \in \mathbb{N}$, with the sum taken over all $(p-1)$-tuples $(l_2, \dots, l_p)$, $l_j \in \mathbb{Z}$, $l_j \geq 0$, where at least two entries are nonzero, such that $\sum_{j=2}^{p} \frac{l_j}{m_j} = 1$, and the complex variables $z_1, \dots, z_n$ are divided into p non-empty groups $z^1, \dots, z^p$.

Remark. Domains (0.2) are a special case of an example in [BP]. The above theorem also confirms a conjecture of Catlin and Pinchuk (see [Kra]).

A byproduct of (the proof of) the results presented here is some information about the *size* of the set of boundary accumulation points. For example, one might ask whether boundary accumulation points can be isolated, or whether they can form a relatively open set in the boundary. Our calculations show that, for a bounded C^1-Reinhardt domain, the set S of boundary accumulation points form a manifold of odd dimension between 1 and $2n-1$ inclusive. The case of $\dim S = 2n-1$ (or top dimension) only occurs when the domain under consideration is the ball (up to dilations and permutations of the coordinates).

Note that the domains described in the theorem are not necessarily pseudoconvex or complete Reinhardt. Indeed, consider the following example:

Example 1. Let D be the following bounded domain in $\mathbb{C}^3$

$$D = \left\{(z_1, z_2, z_3) \in \mathbb{C}^3 \colon |z_1|^2 + |z_2|^4 + |z_3|^4 - \frac{3}{2}|z_2|^2|z_3|^2 < 1\right\}.$$

To show that D has noncompact automorphism group consider the following sequence of automorphisms $\{F_i\}$

$$
\begin{aligned}
z_1 &\mapsto \frac{z_1 - a_i}{1 - \overline{a_i} z_1},\\
z_2 &\mapsto \frac{(1-|a_i|^2)^{\frac{1}{4}} z_2}{\sqrt{1 - \overline{a_i} z_1}},\\
z_3 &\mapsto \frac{(1-|a_i|^2)^{\frac{1}{4}} z_3}{\sqrt{1 - \overline{a_i} z_1}},
\end{aligned}
$$

where $|a_i| < 1$, $a_i \to -1$ as $i \to \infty$, and the point $p = (0,0,0) \in D$. Then $F_i(p) = (-a_i, 0, 0) \to (1,0,0) \in \partial D$. Consider next the boundary point $\tilde{q} = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt[4]{2}})$. The complex tangent space at $\tilde{q}$ is

$$\{(z_1, z_2) : z_1 + 2^{\frac{3}{4}} z_3 = 0\},$$

and the Levi form at $\tilde{q}$ is

$$\frac{1}{2^{\frac{3}{2}}}(-3|z_2|^2 + 16|z_3|^2),$$

which is clearly not non-negative. Note that $\tilde{q}$ is *not* a boundary accumulation point.

To show that D is not a complete Reinhardt domain, consider the point $q' = \left(0, \frac{1+\sqrt[4]{2}}{2}, \frac{1+\sqrt[4]{2}}{2}\right)$. It can be verified that $q' \in D$. However, the point $(0,1,0)$ does not lie in D, and therefore D is not complete Reinhardt. □

Since domains (0.2) have real-analytic boundaries, [DF], [L], [D'A2] imply the following result.

Corollary. *If D is a smoothly bounded Reinhardt domain and Aut(D) is non-compact, then D is of finite type. In particular, the Greene/Krantz conjecture holds for D.*

Despite the fact that the description of the automorphism groups of hyperbolic Reinhardt domains essentially coincides with that of bounded Reinhardt domains, the Greene/Krantz conjecture fails for the hyperbolic case:

Example 2. Define $D \subset \mathbb{C}^2$ as follows

$$D = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + (1 - |z_1|^2)^2 |z_2|^2 < 1\}.$$

The domain D is smooth and hyperbolic (see e.g. [PS]). Further, $\mathrm{Aut}(D)$ is non-compact. Indeed, consider the sequence of automorphisms $\{F_i\}$

$$
\begin{aligned}
z_1 &\mapsto \frac{z_1 - a_i}{1 - \overline{a_i} z_1}, \\
z_2 &\mapsto \frac{(1 - \overline{a_i} z_1) z_2}{\sqrt{1 - |a_i|^2}},
\end{aligned}
$$

where, as above, $|a_i| < 1$, $a_i \to -1$ as $i \to \infty$, and the point $p = (0,0) \in D$. Then we have $F_i(p) = (-a_i, 0) \to (1,0) \in \partial D$. Here $q = (1,0)$ is a boundary accumulation point of infinite type since it belongs to the complex affine subspace $\{z_1 = 1\}$ that entirely lies in ∂D.

Note that in this example ∂D has non-negative Levi form at the boundary accumulation point q, but is not globally pseudoconvex. $\square$

1. Proof of Theorem

Following [Sh2] we denote by $\mathrm{Aut}_{\mathrm{alg}}((\mathbb{C}^*)^n)$ the group of algebraic automorphisms of $(\mathbb{C}^*)^n$, i.e. the group of mappings of the form

$$
z_i \mapsto \lambda_i z_1^{a_{i1}} \dots z_n^{a_{in}}, \quad i = 1, \dots n, \tag{1.1}
$$

where $\lambda_i \in \mathbb{C}^*$, $a_{ij} \in \mathbb{Z}$, and $\det(a_{ij}) = \pm 1$. Here $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

For a bounded Reinhardt domain $D \subset \mathbb{C}^n$, denote by $\mathrm{Aut}_{\mathrm{alg}}(D)$ the subgroup of $\mathrm{Aut}(D)$ that consists of algebraic automorphisms of D, i.e. automorphisms induced by mappings from $\mathrm{Aut}_{\mathrm{alg}}((\mathbb{C}^*)^n)$. It is shown in [Sh2], [Kr] that $\mathrm{Aut}(D) = \mathrm{Aut}_0(D) \cdot \mathrm{Aut}_{\mathrm{alg}}(D)$, where $\mathrm{Aut}_0(D)$ is the identity component of $\mathrm{Aut}(D)$ and the dot denotes the composition operation in $\mathrm{Aut}(D)$. We will prove the main theorem using the explicit description of $\mathrm{Aut}_0(D)$ given in [Su], [Sh2], [Kr] together with the following proposition.

Proposition 1.1. *For a smoothly bounded Reinhardt domain $D \subset \mathbb{C}^n$, $\mathrm{Aut}_{\mathrm{alg}}(D)$ is finite up to the action of $\mathbb{T}^n$.*

The proposition will be proved in Section 2.

Corollary 1.2. *For a smoothly bounded Reinhardt domain $D \subset \mathbb{C}^n$, $\mathrm{Aut}(D)$ is non-compact iff $\mathrm{Aut}_0(D)$ is non-compact.*

We will now present the description of $\mathrm{Aut}_0(D)$ from [Su], [Sh2], [Kr]. Any bounded Reinhardt domain in $\mathbb{C}^n$ can—by a biholomorphic mapping of the form (1.1)—be put into a normalized form G written as follows. There exist integers $0 \le s \le t \le p \le n$ and $n_i \ge 1$, $i = 1, \dots, p$, with $\sum_{i=1}^p n_i = n$, and non-negative real numbers α_i^j, $i = 1, \dots, s$, $j = t+1, \dots, p$, such that if we set $z^i = \left(z_{n_1+\cdots+n_{i-1}+1}, \dots, z_{n_1+\cdots+n_i}\right)$, $i = 1, \dots, p$, then $\tilde{G} := G \bigcap \left\{z^i = 0,\, i = 1, \dots, t\right\}$ is a bounded Reinhardt domain in $\mathbb{C}^{n_{t+1}} \times \cdots \times \mathbb{C}^{n_p}$, and G can be written in the form

$$G = \Bigg\{ \left|z^1\right| < 1, \dots, |z^s| < 1,$$

$$\left(\frac{z^{t+1}}{\prod_{i=1}^s \left(1 - |z^i|^2\right)^{\alpha_i^{t+1}} \prod_{j=s+1}^t \exp\left(-\beta_j^{t+1}\,|z^j|^2\right)}, \ \dots\ , \tag{1.2}$$

$$\frac{z^p}{\prod_{i=1}^s \left(1 - |z^i|^2\right)^{\alpha_i^p} \prod_{j=s+1}^t \exp\left(-\beta_j^p\,|z^j|^2\right)} \right) \in \tilde{G} \Bigg\},$$

for some non-negative β_j^k, $j = s+1, \dots, t$, $k = t+1, \dots, p$. A normalized form can be chosen so that $\mathrm{Aut}_0(G)$ is given by the following formulas:

(1.3)
$$\begin{aligned}
&z^i \mapsto \frac{A^i z^i + b^i}{c^i z^i + d^i}, \quad i = 1, \dots, s,\\
&z^j \mapsto B^j z^j + e^j, \quad j = s+1, \dots, t,\\
&z^k \mapsto C^k \frac{\prod_{j=s+1}^t \exp\left(-\beta_j^k \left(2\overline{e^j}^T B^j z^j + |e^j|^2\right)\right) z^k}{\prod_{i=1}^s (c^i z^i + d^i)^{2\alpha_i^k}}, \quad k = t+1, \dots, p,
\end{aligned}$$

where
$$\begin{aligned}
&\begin{bmatrix} A^i & b^i \\ c^i & d^i \end{bmatrix} \in SU(n_i, 1), \quad i = 1, \dots, s,\\
&B^j \in U(n_j), \quad e^j \in \mathbb{C}^{n_j}, \quad j = s+1, \dots, t,\\
&C^k \in U(n_k), \quad k = t+1, \dots, p.
\end{aligned}$$

The above classification implies that $\mathrm{Aut}_0(G)$ is non-compact only if $t > 0$.

We are now going to select only those normalized forms (1.2) with $t > 0$ that can be the images of bounded domains with C^∞-boundaries under mappings of the form (1.1). We will need the following sequence of lemmas.

Lemma 1.3. *Let D be a smoothly bounded Reinhardt domain and $H_{k_1,\dots,k_r}$ a coordinate subspace*

$$H_{k_1,\dots,k_r} = \bigcap_{j=1}^r \{z_{k_j} = 0\}, \quad r < n,$$

such that $\partial D \cap H_{k_1,\dots,k_r} \neq \emptyset$. Then $D \cap H_{k_1,\dots,k_r}$ is a nonempty smoothly bounded set in $H_{k_1,\dots,k_r}$.

Proof. First we prove the lemma for one coordinate hyperplane $H_k = \{z_k = 0\}$. We will show that H_k may only intersect ∂D transversally. Indeed, assume that for some point $q = (q_1, \dots, q_{k-1}, 0, q_{k+1}, \dots, q_n) \in \partial D$, H_k coincides with the complex tangent space to ∂D at q. Consider the affine complex line $S = \{z_j = q_j | j \neq k\}$ that intersects ∂D at q transversally. Then $D \cap S$ is a smooth domain in S near q. On the other hand, there exists $r > 0$ such that, for every $0 < \rho < r$, $D \cap S$ contains a point $(q_1, \dots, q_{k-1}, z_k, q_{k+1}, \dots, q_n)$ with $|z_k| = \rho$. Since D is invariant under rotations in z_k it follows that, near q, $D \cap S$ coincides with the punctured disk $\{0 < |z_k| < r\}$, and therefore is not smooth.

Hence, H_k intersects ∂D transversally everywhere, and $D \cap H_k$ is a nonempty smoothly bounded set which is a finite collection of Reinhardt domains in H_k. An inductive argument now completes the proof. □

Lemma 1.4. *If G is a normalized form of a smoothly bounded Reinhardt domain D, then $s = t$.*

Proof. If $t > s$, then the normalized form G is unbounded in the $z^{s+1}, \dots,$ z^t-directions (see (1.3)). Since D is bounded and G is obtained from D by a mapping of the form (1.1) it follows that, for some i_0,

$$D \cap \{z_{i_0} = 0\} = \emptyset, \qquad \text{and} \qquad \overline{D} \cap \{z_{i_0} = 0\} \neq \emptyset,$$

which is impossible by Lemma 1.3. □

Lemma 1.5. *Let G be a normalized form of a smoothly bounded Reinhardt domain D.*

Then if $p > s$, the following holds:
(i) *For every $s + 1 \le j \le p$ there exists $1 \le i \le s$ such that $\alpha_i^j > 0$;*
(ii) *$s = 1$;*
(iii) *G contains the origin.*
If $p = s$, then G is the unit ball.

Proof. Let $p > s$. Suppose first that $\alpha_i^j = 0$ for all $i = 1, \dots, s$ and $j = s + 1, \dots, p$. Then G is the direct product

$$G = \{|z^1| < 1\} \times \cdots \times \{|z^s| < 1\} \times \tilde{G},$$

and therefore cannot be biholomorphically equivalent to a smoothly bounded domain [HO].

Renumbering the coordinates if necessary, we assume now that there exists $s < k \le p$ such that, for every $s+1 \le j \le k$, there is $1 \le i(j) \le s$ with

$\alpha^j_{i(j)} > 0$, and $\alpha^j_i = 0$ for $j = k+1, \ldots, p$, $i = 1, \ldots, s$. Choose a sequence of points $(z^1_l, \ldots, z^s_l, z^{s+1}_l, \ldots, z^p_l)$ in G such that for all indices $1 \le i \le s$, $|z^i_l| \to 1$ as $l \to \infty$. Since the domain $\tilde{G}$ is bounded, this implies that $z^j_l \to 0$ for $j = s+1, \ldots, k$. Therefore, ∂G intersects the coordinate subspace $H_{n_s+1,\ldots,M(k)}$, where $M(k) = \sum_{j=s+1}^k n_j$. Let $q \in \partial G \cap H_{n_s+1,\ldots,M(k)}$ and let $F = (F_1, \ldots, F_n)$ denote the normalizing mapping for D. It now follows that there exists a sequence $\{q_l\}$ in D such that $F(q_l) \to q$, and therefore $|F^j(q_l)| \to 1$ for $j = 1, \ldots, s$, $F_j(q_l) \to 0$ for $j = n_s+1, \ldots, M(k)$. Since D is smoothly bounded, Lemma 1.3 implies that there exists a coordinate subspace $H_{k_1,\ldots,k_r}$, $r < n$, such that $D \cap H_{k_1,\ldots,k_r} \neq \emptyset$, and $F_j \equiv 0$ for $j = n_s+1, \ldots, M(k)$ on $D \cap H_{k_1,\ldots,k_r}$. Therefore $G_k = G \cap H_{n_s+1,\ldots,M(k)} \neq \emptyset$. At the same time G_k is the direct product

$$G_k = \{|z^1| < 1\} \times \cdots \times \{|z^s| < 1\} \times \ \{\tilde{G} \cap H_{n_s+1,\ldots,M(k)}\}. \tag{1.4}$$

This domain is algebraically equivalent to $F^{-1}(G_k)$, which is a smoothly bounded set by Lemma 1.3. The direct product in (1.4) is nontrivial if $k < p$ or $s > 1$. Therefore, in these cases (as above, for $k = s$) we get a contradiction. Further, since $G_p \neq \emptyset$, G contains the origin.

For $p = s$, G is the direct product

$$G = \{|z^1| < 1\} \times \cdots \times \{|z^s| < 1\},$$

which by the same argument must be trivial, i.e. $s = 1$. Hence, G is the unit ball.

The lemma is proved. □

By Lemma 1.5, G contains the origin; therefore the normalizing mapping for D is of the form

$$z_i \mapsto \lambda_i z_{\sigma(i)},$$

where $\lambda_i \in \mathbb{C}^*$ and σ is a permutation of $\{1, \ldots, n\}$ [Su], [Sh1]. Therefore, to prove the theorem it is sufficient to consider domains of the form

$$G = \left\{ |z^1| < 1, \left(\frac{z^2}{(1 - |z^1|^2)^{\alpha^2}}, \ldots, \frac{z^p}{(1 - |z^1|^2)^{\alpha^p}} \right) \in \tilde{G} \right\}, \tag{1.5}$$

where $\tilde{G}$ is a bounded Reinhardt domain in $\mathbb{C}^{n_2} \times \cdots \times \mathbb{C}^{n_p}$ containing the origin, $\alpha^j > 0$, $j = 2, \ldots, p$.

Lemma 1.6. *If a domain of the form* (1.5) *is smoothly bounded and if* $p \ge 2$*, then for* $j = 2, \ldots, p$*,* $\alpha^j = \frac{1}{2m_j}$*,* $m_j \in \mathbb{N}$*, and* $G \bigcap \left(\cap_{i=2, i \neq j}^p \{z^i = 0\} \right)$ *has the form*

$$\left\{ |z^1|^2 + r^j |z^j|^{2m_j} < 1 \right\}, \quad r^j > 0.$$

Proof. First we observe that, by Lemma 1.3, $\tilde{G}$ is a smoothly bounded domain. Next, fix $2 \leq j \leq p$. Analogously, since $\tilde{G}$ is smooth, then $\tilde{G} \bigcap (\cap_{i=2, i\neq j}^{p} \{z^i = 0\})$ is smooth. Further, $\tilde{G} \bigcap \left(\cap_{i=2, i\neq j}^{p} \{z^i = 0\}\right)$ is invariant under unitary transformations in z^j (see (1.3)). This implies that

$$\tilde{G} \bigcap \left(\cap_{i=2, i\neq j}^{p} \{z^i = 0\}\right) = \{|z^j| < A\} \bigcup \left(\cup_{l=1}^{k} \{\rho_l < |z^j| < \nu_l\}\right) \tag{1.6}$$

for some A, k, $\rho_1, \dots, \rho_k$, $\nu_1, \dots, \nu_k$. Therefore

$$\begin{aligned} G \bigcap \left(\cap_{i=2, i\neq j}^{p} \{z^i = 0\}\right) = &\left\{ |z^1|^2 + \frac{1}{A^{\frac{1}{\alpha^j}}} |z^j|^{\frac{1}{\alpha^j}} < 1 \right\} \bigcup \\ &\bigcup_{l=1}^{k} \left\{ \rho_l (1 - |z_1|^2)^{\alpha^j} < |z^j| < \nu_l (1 - |z_1|^2)^{\alpha^j} \right\}. \end{aligned} \tag{1.7}$$

Since G is smooth, $G \bigcap \left(\cap_{i=2, i\neq j}^{p} \{z^i = 0\}\right)$ is smooth. Together with (1.7) this shows that $\alpha^j = \frac{1}{2m_j}$, $m_j \in \mathbb{N}$, and that in (1.6) one in fact has

$$\tilde{G} \bigcap \left(\cap_{i=2, i\neq j}^{p} \{z^i = 0\}\right) = \{|z^j| < A\}.$$

This proves the lemma. $\square$

The main step in the proof of the theorem is the following proposition.

Proposition 1.7. *Let G be a smoothly bounded domain of the form* (1. 5). *Then G is given by*

$$G = \left\{ |z^1|^2 + P(|z^2|, \dots, |z^p|) < 1 \right\}.$$

Here P is a non-negative polynomial of the form

$$P(|z^2|, \dots, |z^p|) = \sum_{j=2}^{p} r^j |z^j|^{2m_j} + \sum_{l_2, \dots, l_p} a_{l_2, \dots, l_p} |z^2|^{2l_2} \dots |z^p|^{2l_p}, \tag{1.8}$$

where $r^j > 0$, $a_{l_2,\dots,l_p} \in \mathbb{R}$ and the sum is taken over all $(p-1)$-tuples $(l_2, \dots, l_p)$, $l_j \in \mathbb{Z}$, $l_j \geq 0$, where at leat two entries are non-zero, such that $\sum_{j=2}^{p} \frac{l_j}{m_j} = 1$.

Proof. If $p = 1$, then by Lemma 1.5 we see that G is the unit ball. Assume now that $p \geq 2$. We write G, near $q = (1, 0, \dots, 0) \in \partial G$, in the form

$$|z_1|^2 + \phi(z_2, \dots, z_n) < 1, \tag{1.9}$$

where ϕ is a smooth function in a neighbourhood of the origin in $\mathbb{C}^{n-1}$, $\phi(0) = 0$, $\operatorname{grad} \phi(0) = 0$. Since G is invariant under unitary transformations in each of $z^1, \dots z^p$, (see (1.3)), inequality (1.9) is equivalent to

$$|z^1|^2 + \psi(|z^2|, \dots, |z^p|) < 1, \tag{1.10}$$

where

$$\psi(|z^2|, \dots, |z^p|) = \phi(\overbrace{0, \dots, 0}^{n_1 - 1 \text{ times}}; |z^2|, \overbrace{0, \dots, 0}^{n_2 - 1 \text{ times}}; \dots; |z^p|, \overbrace{0, \dots, 0}^{n_p - 1 \text{ times}}).$$

Consider the following family of automorphisms of G:

$$\begin{aligned} z_1 &\mapsto \frac{z_1 - a}{1 - a z_1}, \\ z_i &\mapsto \frac{\sqrt{1 - a^2}\, z_i}{1 - a z_1}, \quad i = 2, \dots, n_1, \\ z^j &\mapsto \frac{(\sqrt{1 - a^2})^{\frac{1}{m_j}} z^j}{(1 - a z_1)^{\frac{1}{m_j}}}, \quad j = 2, \dots, p, \end{aligned}$$

where a is a non-negative parameter close to zero. These automorphisms are holomorphic in a neighbourhood of $\overline{G}$ and map ∂G near q into itself. Therefore (1.10) gives that, on ∂G,

$$\begin{aligned} &\frac{|z_1 - a|^2}{|1 - a z_1|^2} + \sum_{i=2}^{n_1} \frac{(1 - a^2)|z_i|^2}{|1 - a z_1|^2} \\ &+ \psi\left(\frac{(\sqrt{1 - a^2})^{\frac{1}{m_2}} |z^2|}{|1 - a z_1|^{\frac{1}{m_2}}}, \dots, \frac{(\sqrt{1 - a^2})^{\frac{1}{m_p}} |z^p|}{|1 - a z_1|^{\frac{1}{m_p}}} \right) = 1. \end{aligned}$$

It then follows that, on ∂G,

$$\psi\left(\frac{(\sqrt{1 - a^2})^{\frac{1}{m_2}} |z^2|}{|1 - a z_1|^{\frac{1}{m_2}}}, \dots, \frac{(\sqrt{1 - a^2})^{\frac{1}{m_p}} |z^2|}{|1 - a z_1|^{\frac{1}{m_p}}} \right) = \frac{1 - a^2}{|1 - a z_1|^2} \psi(|z^2|, \dots, |z^p|).$$

This implies that

$$\psi(t^{\frac{1}{2m_2}} |z^2|, \dots, t^{\frac{1}{2m_p}} |z^p|) = t \psi(|z^2|, \dots, |z^p|), \tag{1.11}$$

for $(z^2, \dots, z^p)$ in a neighbourhood of the origin and $1 \le t \le 1 + \epsilon$ for some small $\epsilon > 0$.

We will now prove that the homogeneity property (1.11) implies that $\psi(|z^2|, \dots, |z^p|)$ has the form (1.8).

Lemma 1.8. *Let $f(x_1,\dots,x_r)$ be a C^∞-function in a neighbourhood of the origin in $\mathbb{R}^r$. Suppose that there exist $k_j\in\mathbb{N}$, $j=1,\dots,r$, such that*

$$f(t^{\frac{1}{k_1}}x_1,\dots,t^{\frac{1}{k_r}}x_r)=tf(x_1,\dots,x_r), \tag{1.12}$$

for $1\le t\le 1+\epsilon$. Then f has the form

$$f(x_1,\dots,x_r)=\sum_{l_1,\dots,l_r} b_{l_1,\dots,l_r}x_1^{l_1}\dots x_r^{l_r}, \tag{1.13}$$

where $b_{l_1,\dots,l_r}\in\mathbb{R}$, and the sum is taken over all r-tuples $(l_1,\dots,l_r)$, $l_j\in\mathbb{Z}$, $l_j\ge 0$, such that $\sum_{j=1}^r \frac{l_j}{k_j}=1$.

Proof. Differentiating (1.12) at the origin with respect to $x_1,\dots,x_r$, we get

$$\left[t^{\left(\sum_{j=1}^{r}\frac{q_j}{k_j}\right)}\right]\frac{\partial^{q_1+\dots+q_r}f}{\partial x_1^{q_1}\dots\partial x_r^{q_r}}(0)=t\frac{\partial^{q_1+\dots+q_r}f}{\partial x_1^{q_1}\dots\partial x_r^{q_r}}(0),$$

which implies that $\frac{\partial^{q_1+\dots+q_r}f}{\partial x_1^{q_1}\dots\partial x_r^{q_r}}(0)$ may be nonzero only if $\sum_{j=1}^r\frac{q_j}{k_j}=1$. Therefore the Taylor formula for f in a neighbourhood of the origin gives that $f=P+\alpha$, where P is a polynomial as in (1.13), and α is a C^∞-function in a neighbourhood of the origin satisfying (1.12) and such that

$$\alpha(x)=o\left(|x|^N\right) \tag{1.14}$$

for all N, as $x\to 0$.

To show that $\alpha\equiv 0$, we restrict α to the curve

$$x_j(u)=c_ju^{k_1\dots k_{j-1}k_{j+1}\dots k_r},\quad j=1,\dots,r, \tag{1.15}$$

where u is a real parameter close to zero, $c=(c_1,\dots,c_r)\in\mathbb{R}^r$, $|c|=1$. We denote this restriction by $g_c(u)$. Then (1.12) gives

$$g_c(t^{\frac{1}{k_1\dots k_r}}u)=tg_c(u).$$

Differentiating the last equality with respect to t and setting $t=1$ we get

$$\frac{u}{k_1\dots k_r}g_c'(u)=g_c(u).$$

Solving this equation we obtain

$$g_c(u)=A(c)u^{k_1\dots k_r},$$

where $A(c) \in \mathbb{R}$. Further, (1.14) immediately implies that $A(c) = 0$, and since curves of the form (1.15) for all c cover a neighbourhood of the origin, it follows that $\alpha(x) \equiv 0$.

The lemma is proved. $\square$

Property (1.11), Lemma 1.8 and Lemma 1.6 immediately give that $\psi(|z^2|, \dots, |z^p|)$ has the form (1.8). We will now show that equation (1.10) in fact defines G *globally*, not just in a neighbourhood of q. Indeed, fix $1 - |z^1|^2 = \delta$, where δ is small. Then, (1.5) and (1.10) imply that $\tilde{G}$ is given by

$$\tilde{G} = \left\{(z^2, \dots, z^p) \colon \psi(\delta^{\frac{1}{2m_2}}|z^2|, \dots, \delta^{\frac{1}{2m_p}}|z^p|) < \delta\right\}.$$

It now follows from the homogeneity property (1.11) that

$$\tilde{G} = \left\{(z^2, \dots, z^p) \colon \psi\big(|z^2|, \dots, |z^p|\big) < 1\right\}.$$

This completes the proof of the proposition and the theorem. $\square$

2. Proof of Proposition 1.1

Assume that

$$\begin{aligned} &D \cap \{z_i = 0\} \neq \emptyset, \quad i = 1, \dots, k, \\ &D \cap \{z_i = 0\} = \emptyset, \quad i = k+1, \dots, n, \end{aligned}$$

where $0 \leq k \leq n$. For $k = n$ it is shown in [Sh2] that $\mathrm{Aut}_{\mathrm{alg}}(D)$ is finite up to the action of $\mathbb{T}^n$.

Let $k < n$. Since D is smooth, by Lemma 1.3 we have

$$\operatorname{dist}(D, \{z_i = 0\}) > 0, \quad i = k+1, \dots n. \tag{2.1}$$

By [Sh2], every algebraic automorphism of D has the form

$$\begin{aligned} &z_i \mapsto \lambda_i z_{\sigma(i)} z_{k+1}^{a_{i\,k+1}} \dots z_n^{a_{i\,n}}, \quad i = 1, \dots, k, \\ &z_i \mapsto \lambda_i z_{k+1}^{b_{i\,k+1}} \dots z_n^{b_{i\,n}}, \quad i = k+1, \dots, n, \end{aligned} \tag{2.2}$$

where $\lambda_i \in \mathbb{C}^*$, $a_{ij} \in \mathbb{Z}$, $b_{ij} \in \mathbb{Z}$, $\det(b_{ij}) = \pm 1$, and σ is a permutation of $\{1, \dots, k\}$.

Consider the logarithmic image of D, i.e. the domain in $\mathbb{R}^n(x_1, \dots, x_n)$ defined as

$$D_{\log} = \{(\log|z_1|, \dots, \log|z_n|) \in \mathbb{R}^n : (z_1, \dots, z_n) \in D,\ z_1 \dots z_n \neq 0\}.$$

Since D is bounded, we can assume that it lies in the polydisk $\{|z_1| < 1, \dots, |z_n| < 1\}$. Then $D_{\log} \subset \mathbb{R}^n_-$, where $\mathbb{R}^n_- = \{x_1 < 0, \dots, x_n < 0\}$. The mappings (2.2) on $D_{\log}$ now become affine mappings of the form

(2.3)
$$\begin{bmatrix} x_1 \\ \vdots \\ x_k \\ x_{k+1} \\ \vdots \\ x_n \end{bmatrix} \mapsto \begin{bmatrix} x_{\sigma(1)} \\ \vdots \\ x_{\sigma(k)} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} \overbrace{0\dots0}^{k \text{ times}} & a_{1\,k+1} & \dots & a_{1\,n} \\ \vdots \dots \vdots & \vdots & \dots & \vdots \\ 0\dots0 & a_{k\,k+1} & \dots & a_{k\,n} \\ 0\dots0 & b_{k+1\,k+1} & \dots & b_{k+1\,n} \\ \vdots \dots \vdots & \vdots & \dots & \vdots \\ 0\dots0 & b_{n\,k+1} & \dots & b_{n\,n} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ x_{k+1} \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_k \\ \mu_{k+1} \\ \vdots \\ \mu_n \end{bmatrix},$$

where $\mu_j = \log|\lambda_j|$, $j = 1, \dots, n$. Let $D'_{\log}$ denote the projection of $D_{\log}$ to the subspace of the last $n-k$ coordinates $\mathbb{R}^{n-k} = \{x_1 = \dots = x_k = 0\}$. Property (2.1) implies that $D'_{\log}$ is a bounded subset of $\mathbb{R}^{n-k}_-$. Further, for any affine automorphism of $D_{\log}$ of the form (2.3), the mapping

$$x' \mapsto (b_{ij})\, x' + \mu', \tag{2.4}$$

where $x' = (x_{k+1}, \dots, x_n)$, $\mu' = (\mu_{k+1}, \dots, \mu_n)$, is an automorphism of $D'_{\log}$. Since $D'_{\log}$ is bounded, the group $\mathrm{Aff}(D'_{\log})$ of all affine automorphisms of $D'_{\log}$ is clearly compact. Since the b_{ij} are integers, the group of affine transformations of $D'_{\log}$ of the form (2.4) is closed in $\mathrm{Aff}(D'_{\log})$, and therefore is compact. This, in fact, implies that there are only finitely many transformations of $D'_{\log}$ of the form (2.4). Indeed, let

$$\psi_m(x') = B_m x' + \mu'_m$$

be a sequence of distinct transformations of the form (2.4). Then, by choosing a convergent subsequence $\{\psi_{m_l}\}$ and taking into account that the B_{m_l} are integer matrices, we conclude that $B_{m_{l_1}} = B_{m_{l_2}}$ for large m_{l_1}, m_{l_2}. This implies that $\mu'_{m_{l_1}} = \mu'_{m_{l_2}}$ since otherwise $D'_{\log}$ would be invariant under the translation

$$x' \mapsto x' + \mu'_{m_{l_1}} - \mu'_{m_{l_2}},$$

which is impossible because $D'_{\log}$ is bounded. Therefore, the ψ_{m_l} become equal to each other for large m_l, which contradicts our choice of $\{\psi_m\}$.

Now we assume that $k \geq 1$ and will show that if for two affine automorphisms F_1, F_2 of $D_{\log}$ of the form (2.3) the induced automorphisms (2.4) of

$D'_{\log}$ coincide, then F_1 coincides with F_2 up to a mapping from a finite set. Indeed, consider the automorphism $F = F_1^{-1} \circ F_2$. For F the corresponding automorphism of $D'_{\log}$ is the identity. We want to show that such a mapping is determined uniquely by the permutation σ (see (2.3)). Indeed, if f and g are two such mappings with the same σ, then iterating either $f \circ g^{-1}$ or $g \circ f^{-1}$ one can take a point from $D_{\log}$ outside $\mathbb{R}^n_-$ by making one of its first k coordinates positive, unless $f \equiv g$. Hence $\mathrm{Aut}_{\mathrm{alg}}(D)$ is always finite up to the action of $\mathbb{T}^n$.

The proposition is proved. $\square$

Acknowledgements

We wish to thank Nikolai Kruzhilin for a fruitful discussion. We are grateful to Laszlo Lempert for providing useful references. The referee also did an exceptionally careful job, and contributed many helpful remarks. We note that this work was initiated while the third author was the (visiting) Richardson Fellow at the Australian National University.

References

[BP] E. Bedford and S. Pinchuk, *Domains in $\mathbb{C}^{n+1}$ with non-compact automorphism groups*, J. Geom. Anal. **1** (1991), 165–191.

[D'A1] J. D'Angelo, *Real hypersurfaces, orders of contact, and applications*, Ann. Math. **115** (1982), 615–637.

[D'A2] ———, *Several complex variables and the geometry of real hypersurfaces*, Boca Raton, CRS Press, 1993, Studies in advanced mathematics - XIII.

[DF] K. Diederich and J. E. Fornaess, *Pseudoconvex domains with real-analytic boundary*, Ann. Math. **(2)107** (1978), 371–384.

[GK1] R. E. Greene and S. G. Krantz, *Invariants of Bergman geometry and the automorphism groups of domains in $\mathbb{C}^n$*, Proceedings of a Conference on Complex Analysis and Geometry held in Cetraro, 1989, Mediterranean Press, 1992, 107–136.

[GK2] ———, *Techniques for studying automorphisms of weakly pseudoconvex domains*, in J. E. Fornaess (Ed.) Several Complex Variables: Proceedings of the Mittag-Leffler Institute, 1987-1988, Math. Notes, Vol. 38, Princeton University Press, 1993, 389–410.

[HO] A. Huckleberry and E. Ormsby, *Non-existence of proper holomorphic maps between certain complex manifolds*, Manuscr. Math. **26** (1979), 371–379.

[Ki] K.-T. Kim, *On a boundary point repelling automorphism orbits*, J. Math. Anal. Appl. **179** (1993), 463–482.

[Ko] S. Kobayashi, *Intrinsic distances, measures and geometric function theory*, Bull. Amer. Math. Soc. **82** (1976), 357–416.

[Kr] N. G. Kruzhilin, *Holomorphic automorphisms of hyperbolic Reinhardt domains (translated from Russian)*, Math. USSR-Izv. **32** (1989), 15–38.

[Kra] S. G. Krantz, *Convexity in complex analysis*, Several complex variables and complex geometry, Part 1 (Santa Cruz, CA, 1989), Proc. Symp. Pure Math. **52, Part 1** (1991), 119–137.

[L] L. Lempert, *On the boundary behavior of holomorphic mappings*, in A. Howard, P.-M. Wong (Eds.) Contributions to Several Complex Variables in Honour of Wilhelm Stoll, 1986, 193–215.

[PS] E. A. Poletskii and B. V. Shabat, *Invariant metrics (translated from Russian)*, Encycl. Math. Sci., vol. 9 – Several Complex Variables III, Springer-Verlag (1989), 63-111.

[R] J. P. Rosay, *Sur une caractérisation de la boule parmi les domaines de* $\mathbb{C}^n$ *par son groupe d'automorphismes*, Ann. Inst. Fourier (Grenoble) **29** (1979), 91–97.

[Sh1] S. Shimizu, *Automorphisms and equivalence of bounded Reinhardt domains not containing the origin*, Tôhoku Math. J. **40** (1988), 119–152.

[Sh2] ———, *Automorphisms of bounded Reinhardt domains*, Japan J. Math. **15** (1989), 385–414.

[Su] T. Sunada, *Holomorphic equivalence problem for bounded Reinhardt domains*, Math. Ann. **235** (1978), 111–128.

[W] B. Wong, *Characterization of the unit ball in* $\mathbb{C}^n$ *by its automorphism group*, Invent. Math. **41** (1977), 253–257.

Department of Mathematics, University of California, Irvine CA 92717 USA

E-mail address: sfu@math.uci.edu

Centre for Mathematics and Its Applications, The Australian National University, Canberra, ACT 0200 AUSTRALIA

E-mail address: Alexander.Isaev@anu.edu.au

Department of Mathematics, Washington University, St.Louis, MO 63130

E-mail address: sk@math.wustl.edu