A NOTE ON THE TOTAL CURVATURE
OF A KÄHLER MANIFOLD

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Given a complete manifold with non-negative Ricci curvature, it is a very interesting geometric problem of how curvature decays at infinity. While it is not true that the curvature decays in a strong sense, it is possible that the average of the scalar curvature decays at least linearly. Such a statement is certainly consistent with the Cohn-Vossen inequality which holds for surfaces. The significance of such an inequality is also clear because of its relevance with the work of the first author [1] on the attempt to prove the conjecture of the second author that a complete noncompact Kähler manifold with positive bisectional curvature is biholomorphic to the complex euclidean space.

The purpose of this note is to prove a weaker version of the conjecture for Kähler manifolds.

Theorem 1. Suppose $M$ is a complex $n$-dimensional ($n \geq 3$) complete noncompact Kähler manifold with bounded and nonnegative holomorphic bisectional curvature such that

$$ R_{\alpha \beta} \geq \epsilon R \quad \text{on } M, $$

where $0 < \epsilon < +\infty$ is a constant and $R$ is the scalar curvature. Then we have

$$ \int_{B(x_0, \gamma)} R(x) dx \leq \frac{C(n, \epsilon)}{\gamma^2} \text{Vol } B(x_0, \gamma) $$

for any $x_0 \in M$ and $0 < \gamma < +\infty$, where $B(x_0, \gamma)$ is the geodesic ball of radius $\gamma$ centered at $x_0$ and the constant $C(n, \epsilon)$ depends only on $n$ and $\epsilon$.

Since both assumptions and conclusion are scaling invariant, we only need to prove (2) for $\gamma = 1$. That is, we only need to prove

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**Theorem 2.** Suppose $M$ is a complex $n$-dimensional ($n \geq 3$) complete noncompact Kähler manifold such that

\begin{equation}
\epsilon R \leq R_{\alpha \beta \beta}^\alpha \leq K_0 \quad \text{on } M,
\end{equation}

where $0 < \epsilon, K_0 < +\infty$ are constants. Then

\begin{equation}
\int_{B(x_0,1)} R(x)dx \leq c(n, \epsilon) \text{ Vol } B(x_0,1)
\end{equation}

for any $x_0 \in M$, where $c(n, \epsilon)$ depends only on $n$ and $\epsilon$, and is independent of $K_0$.

Fix a point $x_0 \in M$. Suppose $\varphi(x, t)$ is the solution of the heat equation

\begin{equation}
\begin{cases}
\frac{\partial \varphi}{\partial t} = \Delta \varphi & \text{on } M \times [0, \infty), \\
\varphi(x, 0) = \left[\frac{1}{1 + \gamma(x, x_0)}\right]^{2n} x \in M,
\end{cases}
\end{equation}

where $\gamma(x, x_0)$ denotes the distance between $x$ and $x_0$. We have $\varphi(x, t) \in C^\infty(M \times (0, \infty))$,

\begin{equation}
\begin{cases}
\left[\frac{C_1}{1 + \gamma(x, x_0)}\right]^{2n} \leq \varphi(x, t) \leq \left[\frac{C_2}{1 + \gamma(x, x_0)}\right]^{2n} & \text{on } M \times [0, 1] \\
|\nabla_i \varphi(x, t)| \leq \frac{C_3}{[1 + \gamma(x, x_0)]^{2n+1}} & \text{on } M \times [0, 1],
\end{cases}
\end{equation}

where $0 < C_1, C_2, C_3 < +\infty$ depend only on $n$.

**Lemma 3.**

\begin{equation}
|\nabla_\alpha \nabla_\beta \varphi(x, t)| \leq \frac{C_4}{[1 + \gamma(x, x_0)]^{2n+1}} \left(\frac{1}{t}\right) \quad \text{on } M \times [0, 1],
\end{equation}

where $0 < C_4 < +\infty$ depends only on $n$.

**Proof.** From (5) we have

\begin{equation}
\frac{\partial}{\partial t} |\nabla_\alpha \nabla_\beta \varphi|^2 = \Delta |\nabla_\alpha \nabla_\beta \varphi|^2 - |\nabla_\gamma \nabla_\alpha \nabla_\beta \varphi|^2 - |\nabla_\gamma \nabla_\alpha \nabla_\beta \varphi|^2 \\
+ 2R_{\alpha \beta \gamma \delta} \nabla_\alpha \nabla_\beta \varphi \cdot \nabla_\delta \nabla_\gamma \varphi - 2R_{\alpha \beta} \nabla_\beta \nabla_\gamma \varphi \cdot \nabla_\gamma \nabla_\alpha \varphi.
\end{equation}

Choose a coordinate system such that at one point

\[
\nabla_\alpha \nabla_\beta \varphi = \begin{cases}
0 & \alpha \neq \beta \\
\ell \ell_\alpha & \alpha = \beta.
\end{cases}
\]
Then
\[
2R_{\alpha \beta \gamma \delta} \nabla_\alpha \nabla_\beta \varphi \cdot \nabla_\delta \nabla_\gamma \varphi - 2R_{\alpha \beta} \nabla_\gamma \varphi \cdot \nabla_\gamma \nabla_\alpha \varphi \\
= 2R_{\alpha \beta} \ell_\alpha \ell_\beta \ell_\gamma - 2R_{\alpha \beta} \ell_\alpha \ell_\beta ^2 \\
= - \sum_{\alpha, \beta} R_{\alpha \beta} \ell_\alpha \ell_\beta \ell_\gamma - 2R_{\alpha \beta} \ell_\alpha \ell_\beta ^2 \\
= - \sum_{\alpha, \beta} R_{\alpha \beta} (\ell_\alpha - \ell_\beta )^2 \leq 0.
\]

(9)

Thus
\[
\frac{\partial}{\partial t} |\nabla_\alpha \nabla_\beta \varphi|^2 \leq \Delta |\nabla_\alpha \nabla_\beta \varphi|^2 - |\nabla_\gamma \nabla_\alpha \nabla_\beta \varphi|^2 - |\nabla_\gamma \nabla_\alpha \nabla_\beta \varphi|^2.
\]

Combining (6) and (10) we can establish (7). For the details, one can see Shi-Yau [2]. \(\square\)

Remark. If we do not assume the nonnegativity of the holomorphic bisectional curvature, estimate (7) is still true with the constant \(C_4\) depends not only on \(n\) but also on \(K_0\).

Let
\[
\psi(x) = \varphi(x, 1) \quad x \in M.
\]

Then
\[
\psi(x) \in C^\infty(M),
\]

(12)

\[
\frac{C_1}{(1 + \gamma)^{2n}} \leq \psi(x) \leq \frac{C_2}{(1 + \gamma)^{2n}},
\]

(13)

\[
|\nabla_\alpha \psi(x)| \leq \frac{C_3}{(1 + \gamma)^{2n+1}},
\]

(14)

\[
|\nabla_\alpha \nabla_\beta \psi(x)| \leq \frac{C_4}{(1 + \gamma)^{2n+1}},
\]

(15)

For the remaining part of this note, we always denote
\[
\gamma = \gamma(x) = \gamma(x, x_0), \quad x \in M.
\]

(16)
Suppose \( U(x) \) is the function defined by

\[
(17) \quad U(x) = -\int_M G(x,y)\psi(y)dy, \quad x \in M
\]

where \( G(x,y) > 0 \) is the Green function on \( M \). Then

\[
(18) \quad \Delta U(x) = \psi(x) \quad x \in M.
\]

If \( G(x,y) \) does not exist on \( M \), we can use elliptic equation theory to solve (18). Thus (13), (14), and (15) give

\[
(19) \quad \begin{cases}
\frac{C_1}{(1+\gamma)^{2n}} \leq \Delta U \leq \frac{C_2}{(1+\gamma)^{2n}} \\
|\nabla_\alpha(\Delta U)| \leq \frac{C_3}{(1+\gamma)^{2n+1}} \\
|\nabla_\alpha \nabla_\beta(\Delta U)| \leq \frac{C_4}{(1+\gamma)^{2n+1}}.
\end{cases} \quad \text{on } M
\]

Since \( R_{ij} \geq 0 \) on \( M \), we have

\[
(20) \quad \frac{C(n)\gamma(x,y)^2}{\text{Vol } B(x,\gamma(x,y))} \leq G(x,y) \leq \frac{\widetilde{C(n)}\gamma(x,y)^2}{\text{Vol } B(x,\gamma(x,y))}, \quad \forall x, y \in M
\]

where \( 0 < C(n), \widetilde{C(n)} < +\infty \) depend only on \( n \).

Combining (13), (17), (18), (19) and (20) we can show that

\[
(21) \quad -\frac{C_7(\gamma + 1)^2\text{Vol } B(x_0,1)}{\text{Vol } B(x_0,\gamma + 1)} \leq U(x) \leq -\frac{C_8(\gamma + 1)^2\text{Vol } B(x_0,1)}{\text{Vol } B(x_0,\gamma + 1)} \quad \forall x \in M
\]

\[
(22) \quad |\nabla_\alpha U(x)| \leq \frac{C_9(\gamma + 1)^2\text{Vol } B(x_0,1)}{\text{Vol } B(x_0,\gamma + 1)} \quad x \in M
\]

\[
(23) \quad |\nabla_\alpha \nabla_\beta U(x)| \leq \frac{C_9(\gamma + 1)^2\text{Vol } B(x_0,1)}{\text{Vol } B(x_0,\gamma + 1)} \quad x \in M,
\]

where \( 0 < C_7, C_8, C_9 < +\infty \) depend only on \( n \). The proof of (23) is similar to the proof of (7). The only difference is to replace \( \frac{\partial \varphi}{\partial t} \) in (5) by \( \psi(x) \) in (18).
Using the interchange formula for covariant derivatives, we have (convention: $\Delta = \frac{1}{2} \nabla_\alpha \nabla_\alpha + \frac{1}{2} \nabla_\pi \nabla_\pi$)

$$\Delta(\nabla_\beta \nabla_\gamma U) = \nabla_\beta \nabla_\gamma (\Delta U) + \frac{1}{2} R_{\theta \beta} \nabla_\theta U$$

$$+ \frac{1}{2} R_{\gamma \theta} \nabla_\theta U - R_{\alpha \beta \gamma \theta} \nabla_\pi \nabla_\theta U$$

$$\Delta |\nabla_\beta \nabla_\gamma U| = 2 \text{Re}\{\nabla_\beta \nabla_\gamma U \cdot \nabla_\beta \nabla_\gamma (\Delta U)$$

$$+ |\nabla_\alpha \nabla_\beta \nabla_\gamma U|^2 + |\nabla_\pi \nabla_\beta \nabla_\gamma U|^2$$

$$+ 2 R_{\theta \beta} \nabla_\theta \nabla_\gamma U \cdot \nabla_\beta \nabla_\gamma U - 2 R_{\alpha \beta \gamma \theta} \nabla_\pi \nabla_\theta U \cdot \nabla_\beta \nabla_\gamma U\}.$$  

Thus

$$2 \int_M \text{Re}\{\nabla_\beta \nabla_\gamma U \cdot \nabla_\beta \nabla_\gamma (\Delta U)\} dx$$

$$+ \int_M [||\nabla_\alpha \nabla_\beta \nabla_\gamma U||^2 + ||\nabla_\pi \nabla_\beta \nabla_\gamma U||^2] dx$$

$$+ 2 \int_M [R_{\theta \beta} \nabla_\theta \nabla_\gamma U \cdot \nabla_\beta \nabla_\gamma U - R_{\alpha \beta \gamma \theta} \nabla_\pi \nabla_\theta U \cdot \nabla_\beta \nabla_\gamma U] dx$$

$$= 0.$$  

Choose a coordinate system such that at one point

$$\nabla_\alpha \nabla_\beta U = \begin{cases} 0 & \alpha \neq \beta \\ \lambda_\alpha & \alpha = \beta. \end{cases}$$

Then

$$R_{\theta \beta} \nabla_\theta \nabla_\gamma U \cdot \nabla_\beta \nabla_\gamma U - R_{\alpha \beta \gamma \theta} \nabla_\pi \nabla_\theta U \cdot \nabla_\beta \nabla_\gamma U$$

$$= \sum_{\alpha,\beta} R_{\alpha \pi \beta \theta} (\lambda_\alpha - \lambda_\beta)^2 \geq 0.$$  

(24) can be written as

$$2 \int_M \text{Re}\{\nabla_\beta \nabla_\gamma U \cdot \nabla_\beta \nabla_\gamma (\Delta U)\} dx$$

$$+ \int_M [||\nabla_\alpha \nabla_\beta \nabla_\gamma U||^2 + ||\nabla_\pi \nabla_\beta \nabla_\gamma U||^2] dx$$

$$+ 2 \int_M \sum_{\alpha,\beta} R_{\alpha \pi \beta \theta} (\lambda_\alpha - \lambda_\beta)^2 dx = 0.$$
But from (19) and (23) we have

\[
2 | \int_M \text{Re} \{ \nabla_\beta \nabla_\gamma U \cdot \nabla_\beta \nabla_\gamma (\Delta U) \} dx |
\]

(28)

\[
\leq 2 \int_M | \nabla_\beta \nabla_\gamma U | \cdot | \nabla_\beta \nabla_\gamma (\Delta U) | dx
\]

\[
\leq \int_M \frac{2C_3 C_5 \text{Vol}B(x_0,1)}{(1 + \gamma)^{2n+1} \text{Vol}B(x_0,\gamma + 1)} dx \leq C_{10} \text{Vol} B(x_0,1),
\]

where \(0 < C_{10} < +\infty\) depends only on \(n\).

Combining (27) and (28) we get

(29)

\[
\int_M \{ | \nabla_\alpha \nabla_\beta \nabla_\gamma U |^2 + | \nabla_\alpha \nabla_\beta \nabla_\gamma U |^2 \} dx \leq C_{10} \text{Vol} B(x_0,1)
\]

(30)

\[
2 \int_M \sum_{\alpha, \beta} R_{\alpha \beta} (\lambda_\alpha - \lambda_\beta)^2 dx \leq C_{10} \text{Vol} B(x_0,1).
\]

On the other hand, we have

\[
\int_M R(\Delta U)^2 dx = \int_M R\Delta U \cdot \Delta U dx
\]

\[
= \int_M R\Delta U \cdot \nabla_\alpha \nabla_\beta U dx
\]

\[
= - \int_M R\nabla_\alpha (\Delta U) \cdot \nabla_\beta U dx - \int_M \nabla_\alpha R \cdot \Delta U \cdot \nabla_\beta U dx
\]

\[
= - \int_M R\nabla_\alpha \nabla_\beta \nabla_\gamma U \cdot \nabla_\beta U dx - \int_M \nabla_\gamma R \nabla_\alpha \Delta U \cdot \nabla_\alpha U dx
\]

(31)

\[
= \int_M \nabla_\alpha \nabla_\beta U \cdot \nabla_\beta R \nabla_\gamma \Delta U dx + \int_M \nabla_\gamma R \nabla_\alpha \Delta U \cdot \nabla_\alpha U dx
\]

\[
= \int_M R\nabla_\alpha \nabla_\beta U \cdot \nabla_\beta \nabla_\alpha U dx + \int_M \nabla_\beta R \nabla_\alpha \nabla_\alpha U \cdot \nabla_\beta U dx
\]

\[
+ \int_M R\nabla_\alpha \nabla_\beta \nabla_\gamma U \cdot \Delta U dx + \int_M R\nabla_\alpha \nabla_\gamma (\Delta U) \cdot \nabla_\alpha U dx.
\]
\[
\int_M \nabla_\beta R \cdot \nabla_\alpha \nabla_\beta U \cdot \nabla_\pi U \, dx
\]

\[
= \int_M \nabla_\gamma R_{\beta \gamma} \cdot \nabla_\alpha \nabla_\beta U \cdot \nabla_\pi U \, dx
\]

\[
= - \int_M R_{\beta \gamma} \nabla_\gamma \nabla_\alpha \nabla_\beta U \cdot \nabla_\pi U \, dx - \int_M R_{\beta \gamma} \nabla_\alpha \nabla_\beta U \cdot \nabla_\gamma \nabla_\pi U \, dx
\]

\[
= - \int_M R_{\beta \gamma} \nabla_\alpha \nabla_\gamma \nabla_\beta U \cdot \nabla_\pi U \, dx - \int_M R_{\beta \gamma} \nabla_\alpha \nabla_\beta U \cdot \nabla_\gamma \nabla_\pi U \, dx
\]

\[
= \int_M \nabla_\gamma \nabla_\beta U \cdot \nabla_\alpha [R_{\beta \gamma} \nabla_\pi U] \, dx - \int_M R_{\beta \gamma} \nabla_\alpha \nabla_\beta U \cdot \nabla_\gamma \nabla_\pi U \, dx
\]

\[
(32) \quad = \int_M \nabla_\alpha R_{\beta \gamma} \cdot \nabla_\gamma \nabla_\beta U \cdot \nabla_\pi U \, dx + \int_M R_{\beta \gamma} \nabla_\alpha \nabla_\beta U \cdot \Delta U \, dx
\]

\[
- \int_M R_{\beta \gamma} \nabla_\alpha \nabla_\beta U \cdot \nabla_\gamma \nabla_\pi U \, dx
\]

\[
= \int_M \nabla_\beta R_{\alpha \gamma} \cdot \nabla_\gamma \nabla_\beta U \cdot \nabla_\pi U \, dx + \int_M R_{\beta \gamma} \nabla_\alpha \nabla_\beta U \cdot \Delta U \, dx
\]

\[
- \int_M R_{\beta \gamma} \nabla_\alpha \nabla_\beta U \cdot \nabla_\gamma \nabla_\pi U \, dx
\]

\[
= - \int_M R_{\alpha \gamma} \nabla_\beta (\Delta U) \cdot \nabla_\pi U \, dx - \int_M R_{\alpha \gamma} \nabla_\gamma \nabla_\beta U \cdot \nabla_\beta \nabla_\pi U \, dx
\]

\[
+ \int_M R_{\beta \gamma} \nabla_\gamma \nabla_\beta U \cdot \Delta U \, dx - \int_M R_{\beta \gamma} \nabla_\alpha \nabla_\beta U \cdot \nabla_\gamma \nabla_\pi U \, dx.
\]

Combining (31) and (32) we get

\[
\int_M R(\Delta U)^2 \, dx = \int_M R \nabla_\alpha \nabla_\beta U \cdot \nabla_\pi \nabla_\beta U \, dx
\]

\[
+ 2 \int_M R_{\alpha \gamma} \nabla_\gamma \nabla_\pi U \cdot \Delta U \, dx
\]

\[
- 2 \int_M R_{\alpha \gamma} \nabla_\gamma \nabla_\pi U \cdot \nabla_\beta \nabla_\pi U \, dx.
\]

(33)

If we define a function

\[
F(x) = R(\Delta U)^2 - R \nabla_\alpha \nabla_\gamma \nabla_\beta U \cdot \nabla_\pi \nabla_\beta U
\]

\[
+ 2 R_{\alpha \gamma} \nabla_\gamma \nabla_\beta U \cdot \nabla_\beta \nabla_\pi U
\]

\[
- 2 R_{\alpha \gamma} \nabla_\gamma \nabla_\pi U \cdot \Delta U.
\]

Then

\[
(35) \quad \int_M F(x) \, dx = 0.
\]
With the coordinate (25) we know that

\begin{equation}
|\nabla_\alpha \nabla_\beta U| = \sqrt{\sum_\alpha \lambda_\alpha^2}.
\end{equation}

Define \( \Omega \subseteq M \) such that

\begin{equation}
\Omega = \{ x \in M \big| \frac{|\nabla_\beta \nabla_\gamma U|}{\sqrt{n}} \big| \leq \frac{|\nabla_\beta \nabla_\gamma U|}{(2n + 10)^{10}} \text{ for } \alpha = 1, 2, \ldots, n \}.
\end{equation}

It is easy to see that for any \( x \in M \setminus \Omega \), there exist \( \alpha \) and \( \beta \) (say \( \alpha = 1, \beta = 2 \)) such that

\[
|\lambda_1 - \lambda_2| \geq \frac{|\nabla_\beta \nabla_\gamma U|}{(2n + 10)^{10}}.
\]

Thus

\begin{equation}
\sum_{\alpha, \beta} R_{\alpha \beta \gamma \gamma} (\lambda_\alpha - \lambda_\beta)^2 \geq R_{1T22} \frac{|\nabla_\alpha \nabla_\beta U|^2}{(2n + 10)^{10}} \geq \frac{\epsilon}{(2n + 10)^{10}} R|\nabla_\alpha \nabla_\beta U|^2
\end{equation}

\[ \forall x \in M \setminus \Omega \]

\[
\int_{M \setminus \Omega} R|\nabla_\alpha \nabla_\beta U|^2 dx \leq \frac{(2n + 10)^{10}}{\epsilon} \int_{M \setminus \Omega} \sum_{\alpha, \beta} R_{\alpha \beta \gamma \gamma} (\lambda_\alpha - \lambda_\beta)^2 dx \]

\[
\leq \frac{(2n + 10)^{10}}{\epsilon} \int_M \sum_{\alpha, \beta} R_{\alpha \beta \gamma \gamma} (\lambda_\alpha - \lambda_\beta)^2 dx.
\]

Combining (30) and (39) we know that

\begin{equation}
\int_{M \setminus \Omega} R|\nabla_\alpha \nabla_\beta U|^2 dx \leq \frac{(2n + 10)^{10}}{2\epsilon} C_{10} \text{Vol } B(x_0, 1).
\end{equation}

It is easy to see that

\begin{equation}
|F(x)| \leq C_{11}(n) R|\nabla_\alpha \nabla_\beta U|^2 \quad \forall x \in M.
\end{equation}

Thus

\begin{equation}
|\int_{M \setminus \Omega} F(x)dx| \leq \int_{M \setminus \Omega} |F(x)|dx \leq C_{11} \int_{M \setminus \Omega} R|\nabla_\alpha \nabla_\beta U|^2 dx \leq C_{12}(n, \epsilon) \text{Vol } B(x_0, 1).
\end{equation}
From (35) we get
\[ \int_{\Omega} F(x)dx = -\int_{M \setminus \Omega} F(x)dx. \]
Thus from (42)
\[ |\int_{\Omega} F(x)dx| \leq C_{12}(n, \epsilon) \text{ Vol } B(x_0, 1). \]
Since on \( \Omega \) we have
\[ \lambda_{\alpha} \sim \frac{|\nabla_{\beta} \nabla_{\gamma} U|}{\sqrt{n}} \quad \alpha = 1, 2, \ldots, n. \]
From (34) we have
\[ F(x) \sim R \cdot n|\nabla_{\beta} \nabla_{\gamma} U|^2 - R|\nabla_{\beta} \nabla_{\gamma} U|^2 + 2 \sum_{\alpha} R_{\alpha \pi} \cdot \frac{1}{n} |\nabla_{\beta} \nabla_{\gamma} U|^2 - 2 \sum_{\alpha} R_{\alpha \pi} |\nabla_{\beta} \nabla_{\gamma} U|^2, \]
\[ F(x) \sim \sum_{\alpha} R_{\alpha \pi} (n - 3 + \frac{2}{n}) |\nabla_{\beta} \nabla_{\gamma} U|^2, \quad \forall x \in \Omega. \]
Since \( n \geq 3 \), we have \( n - 3 + \frac{2}{n} \geq \frac{2}{n} \). (44) is not precise. Precisely we have
\[ F(x) \geq \frac{1}{n} R |\nabla_{\beta} \nabla_{\gamma} U|^2 \quad x \in \Omega. \]
(43), (45) \implies
\[ \int_{\Omega} R |\nabla_{\beta} \nabla_{\gamma} U|^2 dx \leq C_{13}(n, \epsilon) \text{ Vol } B(x_0, 1). \]
(40), (46) \implies
\[ \int_{M} R |\nabla_{\beta} \nabla_{\gamma} U|^2 dx \leq C_{14}(n, \epsilon) \text{ Vol } B(x_0, 1). \]
(19) \implies
\[ |\nabla_{\beta} \nabla_{\gamma} U|^2 \geq \frac{C_1^2}{n(1 + \gamma)^{4n}}, \quad x \in M. \]
(47), (48) \implies
\[ \int_{M} \frac{R(x)dx}{(1 + \gamma(x,x_0))^{4n}} \leq C_{15}(n, \epsilon) \text{ Vol } B(x_0, 1). \]
Thus finally we have
\[ \int_{B(x_0,1)} R(x)dx \leq C_{16}(n, \epsilon) \text{ Vol } B(x_0, 1). \]
Theorem 2 is proved.
References


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