Mathematical Research Letters **3**, 211–229 (1996)

K3 SURFACES, LORENTZIAN KAC–MOODY ALGEBRAS AND MIRROR SYMMETRY

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Abstract. We consider the variant of the Mirror Symmetry Conjecture for K3 surfaces which relates "geometry" of curves on a general member of a family of K3 surfaces with "algebraic functions" on the moduli of the mirror family. Lorentzian Kac–Moody algebras are involved in this construction. We give several examples when this conjecture is valid.

0. Introduction

In this paper we want to interpret our results [GN1], [GN2] and [N10] from the viewpoint of mirror symmetry for K3 surfaces.

This paper was written during our stay at SFB 170 "Geometrie und Analysis" of Mathematical Institute of Georg-August-University in September – October 1995. We are grateful to the Mathematical Institute for hospitality.

1. Mirror symmetry for K3 surfaces

Let S be an *even hyperbolic lattice*, i.e., a free $\mathbb{Z}$-module of rank $n+1$ with an integral even symmetric bilinear form of signature $(1,n)$. This lattice S may appear in two ways in connection with algebraic K3 surfaces:

(A) $S = S_X$ is the *Picard lattice* of a K3 surface X. These K3 surfaces form a family

$$\mathcal{M}_S = \{ \text{ K3 surface } X \mid S \subset S_X \}$$

of dimension $20 - \dim S$. See [N1], [N6] for definition which is actually based on local (G.N. Tjurina in [Š]) and global Torelli Theorem [P-Š–Š] and epimorphicity of Torelli map [Ku] for K3 surfaces. A general member X of this family has the Picard lattice $S_X = S$.

Received October 12, 1995.

The first author was supported by SFB 170 "Geometrie und Analysis".

This second author was partially supported by a grant of the Russian Fund of Fundamental Research; Grant of ISF MI6000; Grant of ISF and Russian Government MI6300; Grant of AMS; SFB 170 "Geometrie und Analysis".

(B) $S = ([c]^{\perp}_{T})/[c]$, for the lattice of transcendental cycles $T = T_X$ (the *transcendental lattice*) of a K3 surface X where $c \in T$ is a primitive element of T with $c^2 = 0$. (We consider K3 surfaces over $\mathbb{C}$; then $T_X = (S_X)^{\perp}_{H^2(X,\mathbb{Z})}$.) These K3 surfaces X form a family

$$\mathcal{M}_{T^{\perp}} = \{ \text{ K3 surface } X \mid T_X \subset T\}$$

of dimension $\dim S$. A general member X of this family has $T_X = T$.

These two families $\mathcal{M}_S$ and $\mathcal{M}_{T^{\perp}}$ are called *dual* (or *mirror symmetric*, or *mirror*). This is how mirror symmetry for K3 surfaces (inspired by explanation of the Arnold's strange duality [A] for 14 exceptional unimodal singularities of functions) had first appeared in [P], [DN] and [N2], [D1]. In particular, in [N2] there was developed some lattice theory for the exact calculation of these dual families. The new understanding of mirror symmetry for K3 (see for example [D2]) which is due to the modern physics and mirror symmetry for Calabi–Yau threefolds (see [COGP] and [Mor1], [Mor2]) is related with the fact that one can calculate the lattice S for the situation (B) using Yukawa coupling at the point defined by c at infinity of $\mathcal{M}_{T^{\perp}}$. Moreover, there are no quantum corrections for the model (A).

For the model (A), the lattice S is related with the geometry of curves and is the intersection form of all curves on a general K3 surface $X \in \mathcal{M}_S$. For the model (B), the lattice S is related with the geometry of moduli $\mathcal{M}_{T^{\perp}}$ at an appropriate point at infinity of the mirror (dual) family $\mathcal{M}_{T^{\perp}} \subset \overline{\mathcal{M}}_{T^{\perp}}$. Thus, for any question related with the geometry of curves on a general member X of the family $\mathcal{M}_S$, one can ask about its analog for the dual family $\mathcal{M}_{T^{\perp}}$ from the point of view of the geometry of the moduli $\mathcal{M}_{T^{\perp}}$.

An effect we want to discuss here is the following:

It turns out that in some cases "geometry" of irreducible and effective classes of divisors on general $X \in \mathcal{M}_S$ is related with interesting "algebraic functions" on the dual family $\mathcal{M}_{T^{\perp}}$. This relation involves Lorentzian Kac–Moody Lie algebras (and conjecturally some physics).

Here an element of $S = S_X$ is called *irreducible* (respectively *effective*) if it contains an *irreducible* (respectively *effective*) curve. The moduli $\mathcal{M}_{T^{\perp}}$ is a quotient of a Hermitian symmetric domain of type IV by some arithmetic group G, and "algebraic function" means here an automorphic form with respect to G on this domain.

2. Geometry of irreducible and effective classes of divisors on a K3 surface

In this section, we consider a hyperbolic lattice S from the point of view of the model (A). Thus, now $S = S_X$ is the Picard lattice of a K3 surface

X. Then elements of S reflect some geometry of curves. An element $h \in S$ is called *irreducible* if it contains an irreducible curve on X. An element $h \in S$ is called *effective* if it is a finite sum of irreducible elements. For K3 surfaces, the sets of the effective and irreducible classes may be described purely arithmetically using only the intersection form of the lattice S (up to automorphisms of S). Now we give this description.

It is sufficient to describe the set $\Delta^{\mathrm{ir}} \subset S$ of all irreducible elements. It is well-known (and very easy to see) that $h^2 \geq -2$ if $h \in \Delta^{\mathrm{ir}}$. In particular, $\Delta^{\mathrm{ir}} = \Delta^{\mathrm{ir}}_{-2} \cup \Delta^{\mathrm{ir}}_{\geq 0}$ where $\Delta^{\mathrm{ir}}_{-2} = \{\delta \in \Delta^{\mathrm{ir}} \mid \delta^2 = -2\}$, $\Delta^{\mathrm{ir}}_{\geq 0} = \{h \in \Delta^{\mathrm{ir}} \mid h^2 \geq 0\}$. An element δ belongs to $\Delta^{\mathrm{ir}}_{-2}$ if and only if it contains a non-singular irreducible rational curve (exceptional curve) on X. An element $h \in S$ is called *nef* if $h \cdot C \geq 0$ for any irreducible curve C on X. We denote by $\mathrm{NEF}(S)$ the set of all nef elements of S. It is known that the sets $\mathrm{NEF}(S)$ and $\Delta^{\mathrm{ir}}_{\geq 0}$ almost coincide. Obviously, $x^2 \geq 0$ if $x \in \mathrm{NEF}(S)$, and $\mathbb{N}\Delta^{\mathrm{ir}}_{\geq 0} \subset \mathrm{NEF}(X)$. If $c \in \mathrm{NEF}(S)$ and $c^2 = 0$, then $c \in \Delta^{\mathrm{ir}}_{\geq 0}$ if and only if c is primitive [P-Š–Š]. If $h \in \mathrm{NEF}(S)$ and $h^2 > 0$, then $h \in \Delta^{\mathrm{ir}}_{\geq 0}$ if and only if there does not exist primitive $c \in \mathrm{NEF}(S)$ with $c^2 = 0$ such that $c \cdot h = 1$ (in particular, $2h \in \Delta^{\mathrm{ir}}_{\geq 0}$). See [SD]. Thus, the set $\Delta^{\mathrm{ir}}_{\geq 0}$ is completely determined by the set $\mathrm{NEF}(S)$. In what follows, we use the set $\mathrm{NEF}(S)$ instead of $\Delta^{\mathrm{ir}}_{\geq 0}$ since it is more convenient to work with.

Since the lattice S is hyperbolic, the cone

$$V(S \otimes \mathbb{R}) = \{x \in S \otimes \mathbb{R} \mid x^2 > 0\}$$

is the union of two half cones $\pm V^+(S \otimes \mathbb{R})$ where $V^+(S \otimes \mathbb{R})$ contains the class of a hyperplane section. It is easy to see that

$$\mathrm{NEF}(S) = \{h \in S \mid h \in \overline{V^+(S \otimes \mathbb{R})} - \{0\}, \quad h \cdot \Delta^{\mathrm{ir}}_{-2} \geq 0\}.$$

Thus, $\mathrm{NEF}(S)$ is completely defined by $\Delta^{\mathrm{ir}}_{-2}$. Moreover, there exists a group-theoretical description of both sets. Let

$$\mathbb{R}_{++}\mathcal{M} = \{x \in \overline{V^+(S \otimes \mathbb{R})} - \{0\} \mid x \cdot \Delta^{\mathrm{ir}}_{-2} \geq 0\}$$

be a cone and $\mathcal{M} = \mathbb{R}_{++}\mathcal{M}/\mathbb{R}_{++}$ its set of rays. Then $\mathrm{NEF}(S) = S \cap \mathbb{R}_{++}\mathcal{M}$. Let $W^{(2)}(S) \subset O(S)$ be the group generated by all reflections $s_\delta : x \mapsto x + (x \cdot \delta)\delta$, $x \in S$, of the lattice S in elements $\delta \in S$ with $\delta^2 = -2$. It is easy to see that the group $W^{(2)}(S)$ is discrete in the corresponding hyperbolic space $\mathcal{L}^+(S) = V^+(S)/\mathbb{R}_{++}$ and $\mathcal{M}$ is the fundamental domain of $W^{(2)}(S)$ with the set $\Delta^{\mathrm{ir}}_{-2}$ of vectors orthogonal to $\mathcal{M}$. It means that $\delta \in \Delta^{\mathrm{ir}}_{-2}$ if and only if $\delta \in S$, $\delta^2 = -2$ and the inequality $\delta \cdot x \geq 0$ defines a face of $\mathcal{M}$ (or of $\mathbb{R}_{++}\mathcal{M}$) of codimension one. This gives the description of the both sets NEF(S) and $\Delta^{\rm ir}_{-2}$ of S in terms of the group $W^{(2)}(S)$: the real convex cone $\mathbb{R}_{++}$NEF(S) is a fundamental domain for the group $W^{(2)}(S)$ acting on $V^+(S\otimes\mathbb{R})$ with the set of orthogonal vectors $\Delta^{\rm ir}_{-2}$.

Let EF(S) be the set of all effective elements of S and EF$(S)_{\ge -2}$, EF$(S)_{-2}$ and EF$(S)_{\ge 0}$ are the sets of all elements $x\in$ EF(S) with $x^2\ge -2$, $x^2=-2$ and $x^2\ge 0$ respectively. Using Riemann-Roch Theorem, one can see that

$$\mathrm{EF}(S)_{-2}=\{\delta\in S\mid \delta^2=-2, \delta\cdot \mathrm{NEF}(S)\ge 0\},$$
$$\mathrm{EF}(S)_{\ge 0}=S\cap\overline{V^+(S\otimes\mathbb{R})}-\{0\},\quad \text{and,}$$
$$\mathrm{EF}(S)_{\ge -2}=\mathrm{EF}(S)_{-2}\cup \mathrm{EF}(S)_{\ge 0}.$$

3. Kac–Moody algebras associated to a K3 surface

In this section, we define Kac–Moody algebras associated to a K3 surface X with the Picard lattice $S=S_X$. (See [Ka1], [Ka2], [Ka3], [Bo1] and [GN1] for the theory of Kac–Moody algebras.) Such an algebra will be a generalized Kac–Moody (Lie) superalgebra without odd real simple roots. It is defined by a set ${}_s\Delta$ of simple roots which is divided in a set of simple real (even) roots ${}_s\Delta^{\rm re}$ and a set of simple imaginary roots ${}_s\Delta^{\rm im}$; the last set is divided in a set of even simple imaginary roots ${}_s\Delta^{\rm im}_{\bar 0}$ and a set of simple odd imaginary roots ${}_s\Delta^{\rm im}_{\bar 1}$.

We put ${}_s\Delta^{\rm re}=\Delta^{\rm ir}_{-2}$ and the sets ${}_s\Delta^{\rm im}_{\bar 0}$, ${}_s\Delta^{\rm im}_{\bar 1}$ are some sequences of nef elements of S. Each imaginary root α defines an element of NEF(S) but one can repeat each element of NEF(S) finite number of times in each set ${}_s\Delta^{\rm im}_{\bar 0}$ and ${}_s\Delta^{\rm im}_{\bar 1}$.

The *generalized Kac–Moody superalgebra without odd real simple roots* $\mathfrak{g}=\mathfrak{g}''(S,\,{}_s\Delta^{\rm im})$ is a Lie superalgebra generated by h_r, e_r, f_r where $r\in {}_s\Delta^{\rm im}$. All h_r are even, e_r, f_r are even (respectively odd) if r is even (respectively odd). The algebra $\mathfrak{g}$ has the following defining relations:

(1) The map $r\mapsto h_r$ for $r\in {}_s\Delta$ gives an embedding of $S\otimes\mathbb{R}$ into $\mathfrak{g}''(S,{}_s\Delta^{\rm im})$ as an abelian subalgebra (it is even since all h_r are even). In particular, all elements h_r commute.

(2) $[h_r, e_{r'}]=-(r\cdot r')e_{r'}$, and $[h_r, f_{r'}]=(r\cdot r')f_{r'}$.

(3) $[e_r, f_{r'}]=h_r$ if $r=r'$, and is 0 if $r\ne r'$.

(4) $(\mathrm{ad}\ e_r)^{1+r\cdot r'}e_{r'}=(\mathrm{ad}\ f_r)^{1+r\cdot r'}f_{r'}=0$ if $r\ne r'$ and $r\in {}_s\Delta^{\rm re}$.

(5) If $r\cdot r'=0$, then $[e_r, e_{r'}]=[f_r, f_{r'}]=0$.

The superalgebra $\mathfrak{g}=\mathfrak{g}''(S,{}_s\Delta^{\rm im})$ is graded by S as follows. Let

$$\widetilde{Q}_+=\sum_{\alpha\in {}_s\Delta}\mathbb{Z}_+\alpha\subset S$$

be the integral cone (semi-group) generated by all simple roots. We have

$$\mathfrak{g} = \left(\bigoplus_{\alpha \in Q_+ - \{0\}} \mathfrak{g}_\alpha \right) \bigoplus \mathfrak{g}_0 \bigoplus \left(\bigoplus_{\alpha \in Q_+ - \{0\}} \mathfrak{g}_{-\alpha} \right)$$

where h_r has degree 0, and e_r and f_r have degree $r \in \widetilde{Q}_+$ and $-r \in -\widetilde{Q}_+$ respectively ($r \in {}_s\Delta$); and $\mathfrak{g}_0 = S \otimes \mathbb{R}$. A non-zero $\alpha \in \pm\widetilde{Q}_+$ is called a *root* if $\mathfrak{g}_\alpha$ is non-zero. Let Δ be the set of all roots and $\Delta_\pm = \Delta \cap \pm\widetilde{Q}_+$. For a root $\alpha \in \Delta$ we set $\mathrm{mult}_{\bar{0}}\alpha = \dim \mathfrak{g}_{\alpha,\bar{0}}$, $\mathrm{mult}_{\bar{1}}\alpha = -\dim \mathfrak{g}_{\alpha,\bar{1}}$ and

$$\mathrm{mult}\ \alpha = \mathrm{mult}_{\bar{0}}\alpha + \mathrm{mult}_{\bar{1}}\alpha = \dim \mathfrak{g}_{\alpha,\bar{0}} - \dim \mathfrak{g}_{\alpha,\bar{1}}.$$

The integer mult α is called the multiplicity of α. According to the general theory of Kac–Moody algebras, the set of roots is the union of real and imaginary roots: $\Delta = \Delta^{\mathrm{re}} \cup \Delta^{\mathrm{im}}$. The set of real roots is $\Delta^{\mathrm{re}} = W^{(2)}(S)({}_s\Delta^{\mathrm{re}})$ (in particular, $\alpha^2 = -2$ if $\alpha \in \Delta^{\mathrm{re}}$). The set of imaginary roots is $\Delta^{\mathrm{im}} = \{\alpha \in \Delta \mid \alpha^2 \geq 0\}$. It follows that $\Delta^{\mathrm{re}}_+ := \Delta^{\mathrm{re}} \cap \Delta_+ = \mathrm{EF}(S)_{-2}$ and $\Delta^{\mathrm{im}}_+ := \Delta^{\mathrm{im}} \cap \Delta_+ \subset \mathrm{EF}(S)_{\geq 0}$. If $\alpha \in \Delta^{\mathrm{re}}$, then $\mathrm{mult}_{\bar{0}}\alpha = 1$, $\mathrm{mult}_{\bar{1}}\alpha = 0$ and mult $\alpha = 1$. Thus, we can rewrite the decomposition above using "geometry" of K3 surfaces as follows:

$$\mathfrak{g} = \left(\bigoplus_{\alpha \in \mathrm{EF}(S)_{\geq -2}} \mathfrak{g}_\alpha \right) \bigoplus \mathfrak{g}_0 \bigoplus \left(\bigoplus_{\alpha \in \mathrm{EF}(S)_{\geq -2}} \mathfrak{g}_{-\alpha} \right).$$

Here $\mathfrak{g}_\alpha = 0$ if $\alpha \notin \Delta$.

Since the lattice S is hyperbolic, the algebra $\mathfrak{g} = \mathfrak{g}''(S, {}_s\Delta^{\mathrm{im}})$ is *Lorentzian*. Moreover ${}_s\Delta^{\mathrm{re}}$ is the whole set $\Delta^{\mathrm{ir}}_{-2}$, therefore the algebra $\mathfrak{g}$ has *restricted arithmetic type* (see [N10]). Thus, the algebras associated to a K3 surface are *Lorentzian generalized Kac–Moody superalgebras of restricted arithmetic type without odd real simple roots.*

In what follows, we restrict ourselves to considering S with a lattice Weyl vector for $\Delta^{\mathrm{ir}}_{-2}$.

Definition. An element $\rho \in S \otimes \mathbb{Q}$ is called a *lattice Weyl vector* if $\rho \cdot \delta = 1$ for any $\delta \in {}_s\Delta^{\mathrm{re}} = \Delta^{\mathrm{ir}}_{-2}$.

There are three cases when a lattice Weyl vector does exist:

(i) $\Delta^{\mathrm{ir}}_{-2} = \emptyset$, then we can take any $\rho \in S \otimes \mathbb{Q}$;

(ii) $\dim S = 2$ and $\Delta^{\mathrm{ir}}_{-2} \neq \emptyset$, then the set $\Delta^{\mathrm{ir}}_{-2}$ is linearly independent and does not contain more than 2 elements;

(iii) $\dim S \ge 3$, $\Delta^{\mathrm{ir}}_{-2} \ne \emptyset$ and a lattice Weyl vector ρ exists. It follows from general results [N4], [N5] and [N10], that the set of hyperbolic lattices S with this property is finite up to isomorphism. These lattices S are divided in two classes. Firstly, it is easy to see that ρ is a nef element of S and for large $n \in \mathbb{N}$ the linear system $|h|$ of $h = n\rho \in S$ is free. If $\rho^2 > 0$ (this case is called *elliptic*), the linear system $|h|$ gives an embedding of X into a projective space such that all non-singular rational curves on X have the same degree n. All these cases are known (see [N3], [N7], [N8]). If $\rho^2 = 0$ (this case is called *parabolic*), then $|h|$ gives an elliptic fibration of X over a projective line such that all non-singular rational curves of X have the same degree n over the projective line. The list of such S is not known yet.

We would like to mention that in the case (iii) the fundamental polyhedron $\mathcal{M} = \mathbb{R}_{++}\mathrm{NEF}(S)/\mathbb{R}_{++}$ for the action of $W^{(2)}(S)$ on the hyperbolic space $\mathcal{L}^+(S)$ is a very right and beautiful polyhedron: it is a fundamental polyhedron for a reflection group and it touches a sphere with centre $\mathbb{R}_{++}\rho$.

The case (iii) is especially interesting for us because it is very exceptional: there is only finite number of possibilities. Moreover, we want to get some relations between the sets $\mathrm{NEF}(S)$ and $\mathrm{EF}(S)$ which are different only in the cases (ii) and (iii). The case (iii) is also related with automorphic forms on multi-dimensional ($\dim \ge 3$) Hermitian domains.

From now on we assume that S has a lattice Weyl vector ρ. For $a \in \mathrm{NEF}(S)$, let $m(a)'_{\bar 0}$, $m(a)'_{\bar 1}$ are equal to the numbers of times we repeat a in the sequences ${}_s\Delta^{\mathrm{im}}_{\bar 0}$ and ${}_s\Delta^{\mathrm{im}}_{\bar 1}$ respectively. We set $m(a)' = m(a)'_{\bar 0} - m(a)'_{\bar 1}$. Let a_0 be a primitive element of $\mathrm{NEF}(S)$ with $a_0^2 = 0$. In this case we define "corrected" invariants $m(ta_0)$, $t \in \mathbb{N}$, using the identity of power series:

$$\prod_{n\in\mathbb{N}} (1 - q^n)^{m(na_0)'} = 1 - \sum_{t\in\mathbb{N}} m(ta_0) q^t.$$

For $a \in \mathrm{NEF}(S)$ with $a^2 > 0$ we set $m(a) = m(a)'$.

We have the following Weyl–Kac–Borcherds denominator identity for Kac–Moody superalgebra $\mathfrak{g} = \mathfrak{g}''(S, {}_s\Delta^{\mathrm{im}})$ (see [Ka1], [Bo1], and [GN1]):

(*)

$$\begin{aligned}
&\Phi(z)\\
&= \sum_{w\in W^{(2)}(S)} \det(w) \Big(\exp(2\pi i (w(\rho)\cdot z)) - \sum_{a\in \mathrm{NEF}(S)} m(a) \exp(2\pi i (w(\rho + a)\cdot z)) \Big)\\
&= \exp\Big(2\pi i(\rho\cdot z)\Big) \prod_{\alpha\in \mathrm{EF}(S)_{\ge -2}} \Big(1 - \exp\left(2\pi i(\alpha\cdot z)\right)\Big)^{\mathrm{mult}\ \alpha},
\end{aligned}$$

where z belongs to the complexified cone $\Omega(S) = S \otimes \mathbb{R} + iV^+(S \otimes \mathbb{R})$ of $V^+(S \otimes \mathbb{R})$. The function $\Phi(z)$ is called the *denominator function* of $\mathfrak{g} = \mathfrak{g}''(S, {}_s\Delta^{\text{im}})$.

Considering different sequences ${}_s\Delta^{\text{im}}$ of simple imaginary roots from $\text{NEF}(S)$ we get different denominator identities which one can consider as multi-dimensional identities relating the sets of effective and nef (or irreducible) elements of S. Actually these identities depend only on the integral function $m(a)$, $a \in \text{NEF}(S)$. If this function is given, one can calculate $m(a)'$ and find all possible non-negative integers $m(a)'_{\bar{0}}$, $m(a)'_{\bar{1}}$ with $m(a)' = m(a)'_{\bar{0}} - m(a)'_{\bar{1}}$ which define Kac–Moody superalgebras $\mathfrak{g} = \mathfrak{g}''(S, {}_s\Delta^{\text{im}})$ with the fixed denominator function. One also can consider the function (*) as some kind of integral

$$\Phi(z) = \int_{C \subset X} \xi(C, z)$$

along effective curves on the K3 surface X with $S_X = S$. This integral could be correctly defined because we only use effective classes in the formula (*).

4. A variant of mirror conjecture

Now we consider the hyperbolic lattice S using the model (B). We consider only the simplest case when

$$T = S \oplus U(k), \quad k \in \mathbb{N}, \quad U(k) = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}.$$

Let c_1, c_2 be a basis of $U(k)$ with this intersection matrix. Then $z \mapsto \mathbb{C}(z \oplus (-z^2/2)c_1 \oplus (1/k)c_2)$ defines an embedding corresponding to the cusp defined by c_1 of the complexified cone $\Omega(S)$ to the connected component $\Omega(T)_0$ of the Hermitian symmetric domain of type IV

$$\Omega(T) = \{\mathbb{C}\omega \subset T \otimes \mathbb{C} \mid \omega^2 = 0,\ \omega \cdot \overline{\omega} > 0\}.$$

The choice of $\omega_0 = z \oplus (-z^2/2)c_1 \oplus (1/k)c_2 \in \mathbb{C}\omega_0$ is determined by the normalization $\omega_0 \cdot c_1 = 1$. For this normalization, the local moduli of K3 are identified with $S \otimes \mathbb{C}$ and Yukawa coupling coincides with the intersection pairing of the lattice S. This normalization is prescribed by mirror symmetry for K3. The quotient $(\mathcal{M}_{T^\perp})_0 = O(T)' \setminus \Omega(T)_0$ is a connected component of the dual (mirror) family of K3 surfaces for an appropriate subgroup $O(T)'$ of finite index of $O(T)$.

Mirror Conjecture. *There exists a choice of* $k \in \mathbb{N}$ *and a sequence* ${}_s\Delta^{\rm im} \subset NEF(S)$ *of simple imaginary roots such that the denominator function* $\Phi(z)$ *of* $\mathfrak{g}''(S, {}_s\Delta^{\rm im})$ *is a holomorphic automorphic form with respect to* $O(T)'$ *on the domain* $\Omega(T)_0$*, i.e.* $\Phi(z)$ *is an "algebraic function" on the dual moduli* $\mathcal{M}_{T^\perp}$ *(model (B)). The form* $\Phi(z)$ *has the following sense from the point of view of the model (A):* $\Phi(z)$ *is written in the form (*) using "geometry of curves" (effective and irreducible or nef classes of divisors) of a general member* X *with* $S_X = S$ *of the family* $\mathcal{M}_S$ *and it gives an identity (*) between effective and nef divisor classes on* X*. Moreover, we suppose that for the automorphic form* $\Phi(z)$ *it is possible to give exact formulae for Fourier coefficients* $m(a)$ *of the left side and multiplicities* $\operatorname{mult}\alpha$ *of the right side of (*). Besides, the generalized Kac–Moody superalgebra* $\mathfrak{g}''(S, {}_s\Delta^{\rm im})$ *should also be related with geometry of curves and moduli of K3 (and conjecturally with some physics).*

It is very important that the zero divisor of $\Phi(z)$ in the domain where the product (*) converges has multiplicity one and is contained in the discriminant

$$\mathcal{D} = O(T)' \setminus \Big(\bigcup_{\delta \in T,\ \delta^2 = -2} D_\delta \Big)$$

of moduli $\mathcal{M}_{T^\perp}$ of K3 surfaces where $D_\delta = \{\mathbb{C}\omega \in \Omega(T) \mid \omega \cdot \delta = 0\}$. Therefore, in some sense, $\Phi(z)$ shows how far we are from the discriminant.

In the rest part of the paper we give several examples when this conjecture is valid.

5. Example 1

For the first example, $\dim S = 3$, $S \cong U(4) \oplus \langle -2 \rangle$. (In what follows we denote by $K(t)$ a lattice which one gets by multiplication on $t \in \mathbb{Q}$ of the symmetric bilinear form of the lattice K.) The set $\Delta^{\rm ir}_{-2}(S) = \{\delta_1, \delta_2, \delta_3\}$ generates the lattice S and has the intersection matrix

$$(\delta_i \cdot \delta_j) = \begin{pmatrix} -2 & 2 & 2 \\ 2 & -2 & 2 \\ 2 & 2 & -2 \end{pmatrix}$$

which defines the lattice S. The fundamental polyhedron $\mathcal{M}$ is the right triangle with the vertices at infinity. The lattice Weyl vector ρ is equal to $\rho = (\delta_1 + \delta_2 + \delta_3)/2$. The element $h = 2\rho$ has the square $h^2 = 6$ and the linear system $|h|$ gives an embedding of a K3 surface X with $S_X = S$ as an intersection of a quadric and a cubic in $\mathbb{P}^4$. For this embedding, all non-singular rational curves on X are three conics corresponding to

$\delta_1, \delta_2, \delta_3$. Their sum is a hyperplane section of X. The lattice T is equal to $T = U(4) \oplus S \cong 2U(4) \oplus \langle -2 \rangle$. The orthogonal complement $T^\perp$ is isomorphic to a hyperbolic lattice $S' \cong U(4) \oplus K$ where K is a negative definite lattice of rank 15 with the discriminant quadratic form $q_{U(4)} \oplus q_{\langle 2 \rangle}$. It follows from results of [N2] that the lattice S' is unique and the moduli space of K3 surfaces $\mathcal{M}_{S'}$ is irreducible. (It would be very interesting to determine this family using equations and to give an algebraic description of the automorphic form $F_1(Z)$ which we shall describe. We hope to do this later.)

Let us consider another basis f_2, f_3, f_{-2} of $S \otimes \mathbb{Q}$ where

$$\delta_1 = 2f_2 - f_3,\ \delta_2 = 2f_{-2} - f_3,\quad \delta_3 = f_3.$$

These elements have the intersection matrix

$$(f_i \cdot f_j) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Thus the lattice S is a sublattice of $M_0 = \mathbb{Z}f_2 \oplus \mathbb{Z}f_3 \oplus \mathbb{Z}f_{-2}$ of index 4. We have $M_0 = U \oplus \langle -2 \rangle$ where $U = \mathbb{Z}f_2 \oplus \mathbb{Z}f_{-2}$ and $\mathbb{Z}f_3 = \langle -2 \rangle$. These lattices are related as follows: $S \cong 2(M_0)^*$. We consider coordinates (z_3, z_2, z_1) where $z = z_3 f_2 + z_2 f_3 + z_1 f_{-2} \in M_0 \otimes \mathbb{C} = S \otimes \mathbb{C}$. We introduce the lattice $L = U \oplus M_0$ where $U = \mathbb{Z}f_1 \oplus \mathbb{Z}f_{-1}$ with $f_1^2 = f_{-1}^2 = 0$ and $f_1 \cdot f_{-1} = 1$. We use z as a coordinate for the point $Z = \mathbb{C}((-z^2/2)f_1 + z + f_{-1}) \in \Omega(L)$ of the domain $\Omega(L)$ of the type IV corresponding to L. We also identify Z with the matrix

$$Z = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} \in \mathbb{H}_2$$

where $\mathbb{H}_2$ is the Siegel upper-half plane.

Let us consider the classical function $\Delta_5(Z)$ (see [F]) which is the product of all even theta-constants

$$\Delta_5(Z) = \prod_{(a,b)} \vartheta_{a,b}(Z), \qquad (Z = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} \in \mathbb{H}_2)$$

with

$$\vartheta_{a,b}(Z) = \sum_{l \in \mathbb{Z}^2} \exp\left(\pi i (Z[l + \frac{1}{2}a] + {}^t bl)\right) \qquad\qquad (Z[l] = {}^t lZl).$$

The product is taken over all vectors $a, b \in (\mathbb{Z}/2\mathbb{Z})^2$ such that ${}^t ab \equiv 0 \bmod 2$. (There are exactly ten different (a, b).) This is the automorphic cusp form of weight 5 with a character with respect to $Sp_4(\mathbb{Z})/\{\pm E_4\} \cong O^+(L)/\{\pm E_5\}$ where $O^+(L)$ is the subgroup of $O(L)$ which fixes two connected components of $\Omega(L)$ (see [GN1]). The function $F_1(Z) = \frac{1}{64}\Delta_5(Z)$ has integral Fourier coefficients.

Theorem 1. *The function $F_1(Z)$ gives the solution of Mirror Conjecture of §4 for the lattice S and $U(4) = \mathbb{Z}c_1 \oplus \mathbb{Z}c_2$ where $c_1 = 2f_1, c_2 = 2f_{-1}$. Therefore $F_1(Z)$ is an "algebraic function" on the moduli $\mathcal{M}_{T^\perp}$ where $T = U(4) \oplus S$, and it defines the identity (*) for $S = S_X$ of the general member X of the mirror family $\mathcal{M}_S$. Moreover, $F_1(Z)$ defines the corresponding Kac–Moody superalgebras $\mathfrak{g}''(S, {}_s\Delta^{\mathrm{im}})$.*

Proof. The function $F_1(Z)$ as a function on $\Omega(L)$ is automorphic with respect to $O^+(L)$. We have the equality $T = 2L^*$ because $U(4) = 2U^*$ and $S = 2(M_0)^*$. It follows that $F_1(Z)$ is automorphic with respect to $O^+(T) = O^+(L)$ and defines then an "algebraic function" on the moduli $\mathcal{M}_{T^\perp} = O^+(T) \setminus \Omega(T)$.

It is proved in [GN1] that for the coordinate z which we introduced above, the function $F_1(Z)$ can be written in the form

(5.1)
$$\begin{aligned} F_1(Z) &= \sum_{w\in W^{(2)}(S)} \det(w)\Big(\exp(\pi i(w(\rho)\cdot z)) - \sum_{a\in \mathrm{NEF}(S)} m(a)\exp(\pi i(w(\rho+a)\cdot z))\Big) \\ &= \exp\Big(\pi i(\rho\cdot z)\Big) \prod_{\alpha\in \mathrm{EF}(S)_{\geq -2}} \Big(1-\exp\left(\pi i(\alpha\cdot z)\right)\Big)^{\mathrm{mult}\ \alpha} \end{aligned}$$

with integral coefficients $m(a)$ and mult α. For $c_1 = 2f_1$ and $k = 4$, we should consider the coordinate $z' = z/2$ (mirror symmetry coordinate) instead of the coordinate z. For this coordinate z', from (5.1) we get

(5.2)
$$\begin{aligned} F_1(Z) &= \sum_{w\in W^{(2)}(S)} \det(w)\Big(\exp(2\pi i(w(\rho)\cdot z')) - \sum_{a\in \mathrm{NEF}(S)} m(a)\exp(2\pi i(w(\rho+a)\cdot z'))\Big) \\ &= \exp\Big(2\pi i(\rho\cdot z')\Big) \prod_{\alpha\in \mathrm{EF}(S)_{\geq -2}} \Big(1-\exp\left(2\pi i(\alpha\cdot z')\right)\Big)^{\mathrm{mult}\ \alpha}. \end{aligned}$$

This proves Theorem 1.

To get an exact formula for the coefficients $m(a)$ and mult α in (5.1), we need two types of Jacobi forms. The Jacobi form of the first type is the form of weight 5 and index 1/2

$$\psi_{5,\frac{1}{2}}(z_1, z_2) = \eta(z_1)^9\, \vartheta_{11}(z_1,\, z_2).$$

Here $\eta(z_1) = q^{1/24} \prod_{n\ge 1} (1-q^n)$ is Dedekind eta-function and

$$\vartheta_{11}(z_1,\, z_2) = \sum_{n\in\mathbb{Z}} (-1)^n \exp\big(\frac{\pi i}{4}(2n+1)^2 z_1 + \pi i\,(2n+1)z_2\big)$$
$$= -q^{1/8} r^{-1/2} \prod_{n\ge 1} (1-q^{n-1}r)(1-q^n r^{-1})(1-q^n)$$

is the classical Jacobi theta-function, where we put

$$z_1 \in \mathbb{H}_1 = \{z_1 = x+iy \in \mathbb{C} \,|\, y>0\},\ \ z_2\in\mathbb{C},$$
$$q = \exp(2\pi i z_1),\ \ r = \exp(2\pi i z_2),\ \ p = \exp(2\pi i z_3).$$

The holomorphic function $\psi_{5,\frac{1}{2}}(z_1,z_2)$ is a Jacobi form of index one-half with a multiplier system. It means that the following identities are satisfied

$$\psi_{5,\frac{1}{2}}(\frac{az_1+b}{cz_1+d},\, \frac{z_2}{cz_1+d}) = v_\eta^{12}(g)(cz_1+d)^5 \exp(\pi i\,\frac{cz_2^2}{cz_1+d})\,\psi_{5,\frac{1}{2}}(z_1,\, z_2),$$
$$\psi_{5,\frac{1}{2}}(z_1, z_2+pz_1+q) = (-1)^{p+q}\exp(-\pi i(p^2 z_1 + 2pz_2))\,\psi_{5,\frac{1}{2}}(z_1,z_2),$$

where $p,q\in\mathbb{Z},\ g = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ and

$$\eta(\frac{az_1+b}{cz_1+d}) = v_\eta(g)(cz_1+d)^{1/2}\eta(z_1).$$

Here $v_\eta(g)$ is a 24th root of unity.

Let us consider the Fourier coefficients of $\eta^d(z_1)$

$$q^{d/24}\prod_{n\ge 1}(1-q^n)^d = \sum_m \tau_d(m) q^{m/24}.$$

Then we have the following Fourier expansion of $\psi_{5,\frac{1}{2}}(z_1,z_2)$:

$$\psi_{5,\frac{1}{2}}(z_1,z_2) = \eta(z_1)^9\,\vartheta_{11}(z_1,\, z_2)$$
$$= \sum_{\substack{n,l\equiv 1\, mod\, 2\\ n>0,\, 4n-l^2>0}} (-1)^{\frac{l-1}{2}}\,\tau_9(4n-l^2)\,\exp(\pi i(nz_1+lz_2)).$$

The second type of Jacobi forms which we need are special Jacobi forms of weight zero (weak Jacobi forms in terms of [EZ]). The ring of all weak

Jacobi forms has two generators as an algebra over $SL_2(\mathbb{Z})$-modular forms (see [EZ, §9]). One of these generators is the function

$$\phi_{0,1}(z_1,z_2)=\frac{1}{144\Delta(z_1)}\big(E_4^2(z_1)E_{4,1}(z_1,z_2)-E_6(z_1)E_{6,1}(z_1,z_2)\big)$$

where

$$\Delta(z_1)=q\prod_n(1-q^n)^{24},$$
$$E_4(z_1)=1+240\sum_{n\ge1}\sigma_3(n)q^n,$$
$$E_6(z_1)=1-504\sum_{n\ge1}\sigma_5(n)q^n,$$

are the cusp forms of weight 12 and the Eisenstein series of weights 4 and 6 for $SL_2(\mathbb{Z})$. $E_{k,1}(z_1,z_2)$ is the Jacobi-Eisenstein series of weight k and index one which has the following integral Fourier coefficients (see [EZ, §2])

$$E_{k,1}(z_1,z_2)=\zeta(3-2k)^{-1}\sum_{\substack{n,l\in\mathbb{Z}\\ 4n-l^2\ge0}}H(k-1,4n-l^2)\,q^nr^l,$$

where $H(k,N)$ are H. Cohen's numbers (see [C]). If $D=(-1)^kN$ is a discriminant of a quadratic field, then $H(k,N)=L(1-k,\big(\frac{D}{\cdot}\big))$ are values of the Dirichlet L-function with the character $\chi_D(n)=\big(\frac{D}{n}\big)$. We recall that $\zeta(-5)=-1/252$ and $\zeta(-9)=-1/132$.

The weak Jacobi form $\phi_{0,1}$ has the Fourier expansion with integral coefficients

$$\phi_{0,1}(z_1,z_2)=\sum_{\substack{n,l\in\mathbb{Z},n\ge0\\ 4n-l^2\ge-1}}f(n,l)\,\exp\left(2\pi i\left(nz_1+lz_2\right)\right)$$
$$=(r^{-1}+10+r)+q(10r^{-2}-64r^{-1}+108-64r+10r^2)+\dots$$

which depend only on the "norm" $4n-l^2$ of (n,l)

$$f(n,l)=c_1(4n-l^2)\qquad\text{and}\qquad c_1(m)=0\ \text{ for } m<-1.$$

Moreover, the function

$$C_1(z_1)=\sum_{\substack{m\ge-1\\ m\equiv0,\,3\,mod\,4}}c_1(m)q^m$$

is an automorphic form of weight $-\frac{1}{2}$. It is easy to get a formula for this function using the Cohen's modular forms of half integral weight $k-\frac{1}{2}$

$$\mathcal{H}_{k-1}(z_1)=\sum_{n\ge 0}H(k-1,n)q^n$$

(see [C] and [EZ, §5]). One has

$$\begin{aligned}\Delta(4z_1)C_1(z_1)&=\frac{1}{12}\big(11E_6(4z_1)\mathcal{H}_5(z_1)-21E_8(4z_1)\mathcal{H}_3(z_1)\big)\\&=q^3+10q^4+\cdots .\end{aligned}$$

We would like to note that the function:

$$11E_6(4z_1)\mathcal{H}_5(z_1)-21E_8(4z_1)\mathcal{H}_3(z_1)$$

is a cusp form of weight $11\frac{1}{2}$ for $\Gamma_0(4)$.

Using the functions introduced above and Theorem 4.1 of [GN1], we can write the identity (5.1) in the following form:

$$\begin{aligned}&F_1(Z)\\&=\sum_{\substack{n,l,m\equiv 1 \bmod 2\\ n,m>0}}\ \sum_{d|(n,l,m)}(-1)^{\frac{l+d+2}{2}}\,\tau_9\,(\frac{4nm-l^2}{d^2})\,q^{n/2}\,r^{l/2}\,p^{m/2}\\&=(qrp)^{1/2}\prod_{\substack{n,\,l,\,m\in\mathbb{Z}\\(n,l,m)>0}}\big(1-q^nr^lp^m\big)^{c_1(4nm-l^2)},\end{aligned}$$

where $(n,l,m)>0$ means that $n\ge 0$, $m\ge 0$, and l is an arbitrary integral if $n+m>0$, and $l<0$ if $n=m=0$.

6. Example 2

For this example, $\dim S=3$ and $S\cong U(8)\oplus\langle -2\rangle$. The set $\Delta^{\text{ir}}_{-2}(S)=\{e_1,e_2,e_3,e_4\}$ generates the lattice S and has the intersection matrix

$$(e_i\cdot e_j)=\begin{pmatrix}-2&2&6&2\\2&-2&2&6\\6&2&-2&2\\2&6&2&-2\end{pmatrix}$$

which defines the lattice S. The fundamental polyhedron $\mathcal{M}$ is the right quadrangle with the vertices at infinity. The lattice Weyl vector ρ is given by the equality $\rho = (e_1 + e_3)/4 = (e_2 + e_4)/4$. The element $h = 4\rho$ has the square $h^2 = 8$ and the linear system $|h|$ gives an embedding of a K3 surface X with $S_X = S$ as an intersection of three quadrics in $\mathbb{P}^5$ (this follows easily from general results of [SD]). For this embedding, all four non-singular rational curves on X have degree 4. The curves $e_1 + e_3$ and $e_2 + e_4$ give two hyperplane sections of X. The lattice T is $T = U(8) \oplus S \cong 2U(8) \oplus \langle -2\rangle$. The orthogonal complement $T^\perp$ is isomorphic to a hyperbolic lattice $S' \cong U(8) \oplus K$ where K is a negative definite lattice of the rank 15 with the discriminant quadratic form $q_{U(8)} \oplus q_{\langle 2\rangle}$. It follows from results of [N2], that the lattice S' is unique and the moduli space $\mathcal{M}_{S'}$ is irreducible.

We describe below an automorphic form connected to the lattice S. This form gives the solution of the mirror conjecture in §4 in this case. We consider a hyperbolic lattice M_0 with the basis f_2, f_3, f_{-2} having the intersection matrix

$$(f_i \cdot f_j) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -4 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Let us take the hyperbolic plane U with the standard basis f_1, f_{-1} where $f_1^2 = f_{-1}^2 = 0$, $f_1 \cdot f_{-1} = 1$, the lattice $L = U \oplus M_0$ and the domain $\Omega(L)$. We use the coordinate $z' = z_3' f_2 + z_2' f_3 + z_1' f_{-2}$ for a point $Z' = \mathbb{C}((-(z')^2/2)f_1 + z' + f_{-1})$ in this domain. We define

$$\{\delta_1 = -f_3,\ \delta_2 = 4f_2 + f_3,\ \delta_3 = 4f_2 + 3f_3 + 4f_{-2},\ \delta_4 = f_3 + 4f_{-2}\} \subset M_0.$$

Then $\delta_i \cdot \delta_j = 2e_i \cdot e_j$. Thus, the sublattice $M_{II} \subset M_0$ generated by $\delta_1, \dots, \delta_4$ is isomorphic to $S(2) \cong U(16) \oplus \langle -4\rangle$. Equivalently, $S = M_{II}(1/2)$. We identify this lattices replacing e_i by δ_i.

In [GN1, §5] there was constructed an automorphic cusp form $F_2(Z')$ of weight 2 with a character with respect to $O^+(L)/\{\pm E_5\}$. This function has the following representation with integral coefficients

(6.1)
$$\begin{aligned} &F_2(Z') \\ &= \sum_{w \in W^{(2)}(S)} \det(w) \Bigl(\exp(\frac{\pi i}{2}(w(\rho)\cdot z')) - \sum_{a \in \mathrm{NEF}(S)} m(a) \exp(\frac{\pi i}{2}(w(\rho + a)\cdot z')) \Bigr) \\ &= \exp\left(\frac{\pi i}{2}(\rho \cdot z')\right) \prod_{\alpha \in \mathrm{EF}(S)_{\geq -2}} \left(1 - \exp\left(\frac{\pi i}{2}(\alpha \cdot z')\right)\right)^{\mathrm{mult}\ \alpha}. \end{aligned}$$

Theorem 2. *The function $F_2(Z')$ gives the solution of Mirror Conjecture of §4 for the lattice $S = M_{II}(1/2) \cong 2U(8) \oplus \langle -2 \rangle$ and $U(8) = [\mathbb{Z}c_1 \oplus \mathbb{Z}c_2](1/2)$ where $c_1 = 4f_1, c_2 = 4f_{-1}$. Therefore, $F_2(Z)$ is an "algebraic function" on the moduli $\mathcal{M}_{T^\perp}$ (model (B)) where $T = U(8) \oplus S$, and it defines the identity (*) for $S = S_X$ of the general member X of the mirror family $\mathcal{M}_S$ (model (A)). Moreover, it defines the corresponding Kac–Moody superalgebras $\mathfrak{g}''(S, {}_s\Delta^{\mathrm{im}})$.*

Proof. The function $F_2(Z')$ as a function on $\Omega(L)$ is automorphic with respect to $O^+(L)$. We have $T(2) = 4L^*$ because $U(16) = 4U^*$ and $M_{II} = 4(M_0)^*$. It follows that $F_2(Z')$ is automorphic with respect to $O^+(T) = O^+(T(2)) = O^+(L)$ and defines an "algebraic function" on the moduli $\mathcal{M}_{T^\perp} = O^+(T) \setminus \Omega(T)$.

For $c_1 = 4f_1, c_2 = 4f_{-1}$ and $U(8)$ we should use the mirror symmetry coordinate $z'' = z'/2$. We should also remember that $S = M_{II}(1/2)$. From (6.1), we get the identity

(6.1′)

$$
\begin{aligned}
&F_2(Z') \\
&= \sum_{w \in W^{(2)}(S)} \det(w) \left(\exp(2\pi i (w(\rho)\cdot z'')) - \sum_{a \in \mathrm{NEF}(S)} m(a) \exp(2\pi i (w(\rho + a)\cdot z'')) \right) \\
&= \exp\left(2\pi i (\rho \cdot z'')\right) \prod_{\alpha \in \mathrm{EF}(S)_{\geq -2}} \left(1 - \exp\left(2\pi i (\alpha \cdot z'')\right)\right)^{\mathrm{mult}\ \alpha},
\end{aligned}
$$

where we use the intersection pairing of the lattice S and $z'' \in S \otimes \mathbb{C}$. This proves Theorem 2.

The function $F_2(Z')$ (see [GN, §5]) is connected with the Jacobi functions

$$
\begin{aligned}
\phi_{0,2}(z_1, z_2) &= \frac{1}{288\Delta_{12}(z_1)} \left(E_4(z_1)E_{4,1}^2(z_1, z_2) - E_{6,1}^2(z_1, z_2)\right) \\
&= \sum_{n,l} c_2(8n - l^2) \exp\left(2\pi i \left(n z_1 + l z_2\right)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\psi_{2,\frac{1}{2}}(z_1, z_2) &= -\eta^3(\tau)\vartheta_{11}(z_1, z_2) \\
&= \sum_{n \equiv 1\, mod\, 4 \quad l \equiv 1\, mod\, 2} (-1)^{\frac{l+1}{2}} \tau_3\left(2n - l^2\right) \exp\left(\frac{\pi i}{2} n z_1 + \pi i l z_2\right)).
\end{aligned}
$$

The coefficients $\tau_3(n)$ are given by the Jacobi formula

$$\eta^3(z_1) = \sum_{m\ge 1} \left(\frac{-4}{m}\right) m\, q^{m^2/8},$$

where

$$\left(\frac{-4}{m}\right) = \begin{cases} \ \ 1 & \text{if } m \equiv \ \ 1 \mod 4 \\ -1 & \text{if } m \equiv -1 \mod 4 \\ \ \ 0 & \text{if } m \equiv \ \ 0 \mod 2 \end{cases}$$

is the generalized symbol of the quadratic residue. The numbers $c_2(n)$, which define the Fourier coefficients of the Jacobi form $\psi_{2,\frac{1}{2}}$, are Fourier coefficients of an automorphic form of weight $-1/2$. One can express them in terms of Cohen's numbers $H(3,N)$ and $H(5,N)$. Using these functions we can rewrite the identity (6.1) in the following form (see [GN1, §5])

$$\begin{aligned} &F_2(Z') \\ &= \sum_{N\ge 1} \sum_{\substack{2mn-l^2=N^2 \\ n,m\equiv 1 \, mod\, 4 \\ n>0,\ l\equiv 1\, mod\, 2,}} (-1)^{\frac{l+1}{2}} \left(\frac{-4}{N}\right) N \sum_{d\,|\,(n,l,m)} \left(\frac{-4}{d}\right) q^{n/4} r^{l/2} p^{m/4} \\ &= q^{1/4} r^{-1/2} p^{1/4} \prod_{\substack{n,\,l,\,m\in\mathbb{Z} \\ (n,l,m)>0}} (1-q^n r^l p^m)^{c_2(8nm-l^2)}, \end{aligned}$$

where $(n,l,m)>0$ means that $n\ge 0$, $m\ge 0$, l is an arbitrary integral if $n+m>0$, and $l>0$ if $n=m=0$; $\quad q=\exp{(2\pi i z_1')}$, $r=\exp{(2\pi i z_2')}$, $p=\exp{(2\pi i z_3')}$.

Let us consider the coordinate $Z = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} \in \mathbb{H}_2$ where $z_1 = z_1'$, $z_2 = z_2'$, $z_3 = z_3'/2$. The function $F_2(Z)$ is the cusp form of weight 2 with a character (with values in the group of fourth roots of unity) for the double extension of the paramodular group

$$\Gamma_2 := \left\{ \begin{pmatrix} * & 2* & * & * \\ * & * & * & 2^{-1}* \\ * & 2* & * & * \\ 2* & 2* & 2* & * \end{pmatrix} \in Sp_4(\mathbb{Q}), \quad \text{all } * \in \mathbb{Z} \right\}.$$

This function is a lifting of Jacobi form $\psi_{2,\frac{1}{2}}$ (see [G2], [G3], [G4]).

7. Example 3

For this example, $\dim S = 10$ and $S \cong U \oplus E_8(2)$. Let $U = \mathbb{Z}c \oplus \mathbb{Z}e$ where $c^2 = 0$, $e^2 = -2$ and $c \cdot e = 1$. Then

$$\Delta^{\mathrm{ir}}_{-2} = \{\delta \in S \mid \delta^2 = -2,\ \delta \cdot c = 1\}.$$

For example, $e \in \Delta^{\mathrm{ir}}_{-2}$. This case is parabolic and $\rho = c$. For a K3 surface X with $S_X = S$, we have $|\rho| : X \to \mathbb{P}^1$ is elliptic fibration. All non-singular rational curves on X are sections of this fibration. Probably, this family of K3 surfaces had first appeared in [N3] (see also [N8]) where $X \in \mathcal{M}_S$ were described as follows. There exists an involution σ on X such that $H^2(X, \mathbb{Z})^\sigma = S$. This involution is unique on X and $\sigma^*\omega_X = -\omega_X$. The set of points of X fixed by this involution is union of two non-singular fibers (two elliptic curves) of the fibration $|\rho|$ above. Let Y be a K3 surface with involution σ on Y such that the set of points of Y fixed by this involution is union of two elliptic curves. Then $S \cong H^2(Y, \mathbb{Z})^\sigma \subset S_Y$ and Y belongs to $\mathcal{M}_S$.

We can interpret the results of [Bo3] as a construction of a function $F_3(Z)$ which gives the solution of Mirror Conjecture in §4 for $U(2)$ ($k = 2$). Thus, for this case, $T = U(2) \oplus S \cong U(2) \oplus U(1) \oplus E_8(2)$. Then $S' = T^\perp \cong U(2) \oplus E_8(2)$ and $\mathcal{M}_{S'}$ is the family of K3 surfaces which are universal coverings of Enriques surfaces ("Enriques family"). In other words, $X \in \mathcal{M}_{S'}$ has an involution τ without fixed points. Then $X/\{1, \tau\}$ is an Enriques surface.

8. Questions

It would be interesting to formulate the Mirror Conjecture of §4 for hyperbolic lattices S which do not have a lattice Weyl vector for $\Delta^{\mathrm{ir}}_{-2}$. It is certainly possible for some cases. Is this possible for arbitrary hyperbolic lattices S? What is an analog of the Mirror Conjecture in §4 for Calabi–Yau threefolds?

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