RATIONALITY AND EXPONENTIAL GROWTH PROPERTIES OF THE BOUNDARY OPERATORS IN THE NOVIKOV COMPLEX

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§1. Introduction

The classical Morse-Thom-Smale construction associates to a Morse function $g : M \to \mathbb{R}$ on a closed manifold a chain complex $C_*(g)$ of free abelian groups, where the number of free generators of $C_p(g)$ equals the number of critical points of $g$ of index $p$ for each $p$. The homology of this complex is isomorphic to the homology of the manifold, and the boundary operator in this complex is defined in a geometric way, using the algebraic number of trajectories of a gradient of $g$, joining critical points of $g$ (see [5], [9], [13], [14], [15]).

In the early 80s S.P. Novikov generalized this construction to the case of maps $f : M \to S^1$ (see [6]). Here $M$ is a closed connected manifold, $f : M \to S^1$ is a Morse map, non-homotopic to zero. The corresponding analog of Morse complex is a free chain complex $C_*(f)$ over the ring $\mathbb{Z}((t))$ of the formal power series with integer coefficients and finite negative part (that is $\mathbb{Z}((t)) = \{ \sum a_n t^n \mid a_n \in \mathbb{Z} \text{ and } \exists N : a_n = 0 \text{ for } n < N \}$). The number of free generators of $C_p(f)$ equals the number of critical points of $f$ of index $p$, and the homology of $C_*(f)$ equals to the completed homology of the corresponding cyclic covering of $M$.

The boundary operator in this complex depends on the choice of a Riemannian metric on $M$ or of a gradient-like vector field $v$ for $f$. We prefer in our work the language of gradient-like vector fields (see §2 for precise definitions). To explicit the boundary operators, let $\bar{M} \xrightarrow{\mathcal{P}} M$ be the connected infinite cyclic covering for which $f \circ \mathcal{P}$ is homotopic to zero. Choose a lifting $F : \bar{M} \to \mathbb{R}$ of $f \circ \mathcal{P}$ and let $t$ be the generator of the structure group ($\approx \mathbb{Z}$) of $\mathcal{P}$ such that for every $x \in \bar{M}$ we have $F(xt) < F(x)$. For every critical point $x$ of $f$ choose a lifting $\bar{x}$ of $x$ to $\bar{M}$. Choose orientations of stable manifolds of critical points. Then for every critical points

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x, y of f with \( \text{ind} x = \text{ind} y + 1 \), and every \( k \in \mathbb{Z} \) the incidence coefficient \( n_k(x, y; v) \) is defined as the algebraic number of \((-v)\)-trajectories joining \( \bar{x} \) to \( \bar{y}t^k \), each trajectory being counted with the sign, arising from the choice of orientations. (Two trajectories which differ by a choice of parameter, are identified; \( v \) is supposed to satisfy the transversality assumption.)

Set \( n(x, y; v) = \sum_{k \in \mathbb{Z}} n_k(x, y; v)t^k \in \mathbb{Z}((t)) \). Now the boundary operator \( \partial : C_m(f) \to C_{m-1}(f) \) is defined by \( \partial x = \sum_y y \cdot n(x, y; v) \), where \( x \) is a critical point of \( f \) of index \( m \), and the sum ranges over critical points of \( f \) of index \( m - 1 \).

Since the beginning S. P. Novikov conjectured that the power series \( n(x, y; v) \) had some nice analytic properties. In particular he conjectured that

\[ (E) \text{Generically the coefficients } n_k(x, y; v) \text{ grow at most exponentially with } k. \]

This conjecture is stated in [7] for the case of analytic Morse maps \( f \). The present paper announces that generically the coefficients \( n(x, y; v) \) are rational functions in \( t \), which implies the exponential estimate (see §2 for precise statement).

Further, denote by \( N_k(x, y; v) \) the total number of \((-v)\)-trajectories joining \( \bar{x} \) to \( \bar{y}t^k \). The version of the conjecture \((E)\) presented in [1, p. 83] states that generically the numbers \( N_k(x, y; v) \) grow at most exponentially with \( k \). (There is also the corresponding statement for Morse forms.) Closely related to it is the question of V. I. Arnold about the asymptotic behaviour of \( A^nX \cap Y \) where \( A \) is a diffeomorphism of a manifold \( M \) to itself, \( X, Y \) are submanifolds of \( M \) and \( n \to \infty \) (see [1,2]).

We confirm the exponential estimate for \( N_k(x, y; v) \) as well; see §4 for the precise statement as well as for the statement of the corresponding result for Morse forms.

Using the universal covering \( \tilde{M} \to M \) instead of the cyclic covering above, one obtains a version of Novikov complex defined over corresponding completion of the group ring \( \mathbb{Z}\pi_1M \). We announce in §3 that the corresponding incidence coefficients belong to the (non commutative) localization in the sense of P. M. Cohn of the ring \( \mathbb{Z}\pi_1M \).

Instead of Morse maps \( M \to S^1 \) one can consider Morse 1-forms (recall that Morse form is a closed 1-form, which is locally a differential of a Morse function). To each Morse form \( \omega \) one associates the homomorphism of integration \( H(\omega) : H_1(M, \mathbb{Z}) \to \mathbb{R} \) and the free \( \mathbb{Z}^k \)-covering \( M' \to M \) corresponding to the kernel of the composition \( \pi_1M \to H_1(M, \mathbb{Z}) \xrightarrow{H(\omega)} \mathbb{R} \).
(where \( k = \text{rk } \text{Im } H(\omega) \)). The corresponding incidence coefficients belong to a suitable completion of the Laurent polynomial ring \( \mathbb{Z}[\mathbb{Z}^k] \). We announce that generically they belong to the corresponding localization of \( \mathbb{Z}[\mathbb{Z}^k] \) (see Theorem 3 of the present paper, which states a bit stronger assertion, concerning the maximal free abelian coverings).

The full proof of Theorem 1 is contained in \([11]\). The full proofs of the results of §3,4 are contained in \([12]\).

§2. Incidence coefficients with values in \( \mathbb{Z}((t)) \)

We need some definitions. Let \( f : M \to \mathbb{R}^1 \) be a Morse function on a manifold \( M \), \( \dim M = n \). The set of critical points of \( f \) will be denoted by \( S(f) \). Let \( v \) be a vector field on \( M \) (we assume that all objects are of class \( \mathcal{C}^\infty \)). We say that \( v \) is an \( f \)-gradient if for every non-critical point \( x \) of \( f \) we have \( df(v)(x) > 0 \) and for every critical point \( p \) of \( f \) there is a chart \( \Phi : W \to V \), where \( W \) is an open neighborhood of \( p \), and \( V \) is an open neighborhood of 0 in \( \mathbb{R}^n \), such that \( \Phi(p) = 0 \) and

\[
\begin{align*}
(1) \quad (f \circ \Phi^{-1})(x_1, \ldots, x_n) &= f(p) + \sum_{i=1}^{n} \alpha_i x_i^2 \\
&\quad \text{with } \alpha_m < 0 \text{ for } m \leq k \text{ and } \alpha_m > 0 \text{ for } m > k, \\
(2) \quad \Phi_*(v) &= (-x_1, \ldots, -x_k, x_{k+1}, \ldots, x_n).
\end{align*}
\]

(here \( k \) is the index of the critical point \( p \)). The set of all \( f \)-gradients will be denoted by \( \mathcal{G}(f) \), and the set of all \( f \)-gradients satisfying the transversality assumption will be denoted by \( \mathcal{G}_t(f) \). We assume similar terminology for maps \( f : M \to S^1 \).

**Theorem 1.** Let \( M \) be a closed connected manifold, \( f : M \to S^1 \) be a Morse map, non homotopic to zero. Then in the set \( \mathcal{G}_t(f) \) there is a subset \( \mathcal{G}_0(f) \) with the following properties:

1. \( \mathcal{G}_0(f) \) is open and dense in \( \mathcal{G}_t(f) \) with respect to \( \mathcal{C}^0 \) topology.
2. If \( v \in \mathcal{G}_0(f) \), \( x, y \in S(f) \) and \( \text{ind } x = \text{ind } y + 1 \), then \( \sum_{k \in \mathbb{Z}} n_k(x, y; v)t^k \) is a rational function of \( t \) of the form \( \frac{P(t)}{Q(t)} \), where \( P(t) \) and \( Q(t) \) are polynomials with integral coefficients, \( m \in \mathbb{N} \), and \( Q(0) = 1 \).
3. Let \( v \in \mathcal{G}_0(f) \). Let \( U \) be a neighborhood of \( S(f) \). Then for every \( w \in \mathcal{G}_0(f) \) such that \( w = v \) in \( U \) and \( w \) is sufficiently close to \( v \) in \( \mathcal{C}^0 \) topology we have: \( n_k(x, y; v) = n_k(x, y; w) \) for every \( x, y \in S(f) \) with \( \text{ind } x = \text{ind } y + 1 \), and every \( k \in \mathbb{Z} \).

**Remarks.**

1) The exponential estimate in (E) follow immediately, since the the Taylor series of every rational function of the form \( \frac{P(t)}{Q(t)} \) with \( Q(t) \neq 0 \) has a
non-zero radius of convergency.

2) From the main theorem of [10] it follows that for every finite sequence \( a_1, \ldots, a_k \) of integers there is a Morse map \( f : M \to S^1 \) on a manifold \( M \), an \( f \)-gradient \( v \) and critical points \( x, y \) of \( f \) with \( \text{ind}x = \text{ind}y + 1 \) such that \( n_0(x, y; v) = 1 \) and \( n_i(x, y; v) = a_i \) for \( 1 \leq i \leq k \).

§3. Incidence coefficients with values in completions of group rings

To state the results we recall some algebraic and Morse-theoretic definitions.

Let \( G \) be a group and \( \xi : G \to \mathbb{R} \) be a group homomorphism. We denote by \( (\mathbb{Z}G)^\sim \) the abelian group of all formal linear combinations \( \sum_{g \in G} n_g g \), infinite in general (where \( n_g \in \mathbb{Z} \)). Novikov ring \( \mathbb{Z}G_\xi^- \) is the ring of such \( \lambda \in (\mathbb{Z}G)^\sim \), that for every \( c \in \mathbb{R} \) the set \( \text{supp} \xi^{-1}([c, \infty[) \) is finite.

Let \( \omega \) be a closed 1-form on a manifold \( M \). The deRham cohomology class of \( \omega \) will be denoted by \( [\omega] \) and the corresponding homomorphism \( \pi_1 M \to \mathbb{R} \) will be denoted by \( \{\omega\} \). We say that \( \omega \) is a Morse form, if locally it is the differential of a Morse function. The set of zeros of \( \omega \) will be denoted by \( S(\omega) \). A Morse form \( \omega \) is the differential of a Morse map \( f : M \to S^1 \) if and only if \([\omega] \in \text{Im} (H^1(M, \mathbb{Z}) \to H^1(M, \mathbb{R})) \).

An obvious generalization of the definition of \( f \)-gradient for Morse maps to \( S^1 \) gives the notion of \( \omega \)-gradient of a Morse form \( \omega \). The set of all \( \omega \)-gradients will be denoted by \( \mathcal{G}(\omega) \) and the set of all \( \omega \)-gradients satisfying the transversality assumption will be denoted by \( \mathcal{G}t(\omega) \). Let \( \omega \) be a Morse form on a closed connected manifold \( M \) and let \( v \in \mathcal{G}t(\omega) \). For each zero \( x \) of \( \omega \) choose a lifting \( \tilde{x} \) of \( x \) to \( \tilde{M} \) and an orientation of the stable manifold of \( x \). Then for every \( x, y \in S(\omega) \) with \( \text{ind}x = \text{ind}y + 1 \) the incidence coefficient \( \tilde{n}(\tilde{x}, \tilde{y}; v) \in (\mathbb{Z}\pi_1 M)^\sim_{\xi} \) can be defined (the definition is similar to that of \( n(x, y; v) \) in §1; for the case of Morse maps \( M \to S^1 \) one can find it in [9].)

(Although we shall not use the notion of Novikov complex in this paper, we give the definition. Let \( C_p(\omega) \) be the free right \( \mathbb{Z}(\pi_1 M)^\sim_{\xi} \)-module, generated by \( S_p(\omega) \), where \( S_p(\omega) \) stands for the set of zeros of \( \omega \) of index \( p \). Define a homomorphism \( \partial_p : C_p(\omega) \to C_{p-1}(\omega) \) by \( \partial_p x = \sum_{y \in S_{p-1}(\omega)} y \cdot \tilde{n}(\tilde{x}, \tilde{y}; v) \),

where \( x \in S_p(\omega) \). One can prove that \( \partial_p \circ \partial_{p+1} = 0 \); the resulting complex is called Novikov complex.)

1. Morse maps \( M \to S^1 \)

Let \( \xi : G \to \mathbb{Z} \) be a group epimorphism. Denote \( \text{Ker} \xi \) by \( H \). For \( n \in \mathbb{Z} \) denote \( \xi^{-1}(n) \) by \( G(n) \) and \( \{ x \in \mathbb{Z}G \mid \text{supp} x \subset G(n) \} \) by \( \mathbb{Z}G(n) \). Denote
$\xi^{-1}([-\infty, -1])$ by $G_-$ and \{ $x \in \mathbb{Z}G$ | supp $x \subset G_-$ \} by $\mathbb{Z}G_-$. Choose $\theta \in \mathbb{Z}G_{(-1)}$. It is easy to see that $(\mathbb{Z}G)^{-}_\xi$ is identified with the ring of power series of the form $\sum_{i=-\infty}^{\infty} a_i \theta^i$, where $a_i \in \mathbb{Z}H$ and the negative part of the series is finite.

Set $\Sigma_n = \{ 1 + A \mid A \in \text{Mat}_n(\mathbb{Z}G_{(-1)}) \}$. Set $\Sigma = \bigcup_{n \geq 1} \Sigma_n$.

There is the corresponding localization ring $\mathbb{Z}G^{\Sigma}$ (see [4, p.255]). Every matrix in $\Sigma_n$ is invertible in $\text{Mat}_n(\mathbb{Z}G^{-}_\xi)$, the inverse of $1 + A$ being given by $\sum_{n=0}^{\infty} (-1)^n A^n$, therefore the localization map $\lambda : \mathbb{Z}G \rightarrow \mathbb{Z}G^{\Sigma}$ is injective and the inclusion $i : \mathbb{Z}G \rightarrow \mathbb{Z}G^{-}_\xi$ factors through a ring homomorphism $\ell : \mathbb{Z}G^{\Sigma} \rightarrow \mathbb{Z}G^{-}_\xi$.

Proceeding to the statement of the next result, let $M$ be a connected closed manifold and $f : M \rightarrow S^1$ be a Morse map, nonhomotopic to zero. Denote by $\xi$ the induced homomorphism $\pi_1 M \rightarrow \mathbb{Z}$. Assume that $\xi$ is epimorphic. Then by the previous discussion we have the localization $(\mathbb{Z}\pi_1 M)^{\Sigma}$ and the homomorphism $\ell : (\mathbb{Z}\pi_1 M)^{\Sigma} \rightarrow (\mathbb{Z}\pi_1 M)^{-}_\xi$.

**Theorem 2.** In $\mathcal{G}t(f)$ there is a subset $\mathcal{G}t_1(f)$ with the following properties:

1. $\mathcal{G}t_1(f)$ is open and dense in $\mathcal{G}t(f)$ with respect to $C^0$ topology.
2. If $v \in \mathcal{G}t_1(f)$ then for every $x, y \in S(f)$ with $\text{indx} = \text{indy} + 1$ we have $\tilde{n}(\tilde{x}, \tilde{y}; v) \in \text{Im } \ell$.
3. Let $v \in \mathcal{G}t_1(f)$. Let $U$ be a neighborhood of $S(f)$. Then for every $w \in \mathcal{G}t_1(f)$ such that $w = v$ in $U$ and $w$ is sufficiently close to $v$ in $C^0$ topology we have: $\tilde{n}(\tilde{x}, \tilde{y}; v) = \tilde{n}(\tilde{x}, \tilde{y}; w)$ for every $x, y \in S(f)$ with $\text{indx} = \text{indy} + 1$.

2. **Morse forms within arbitrary cohomology classes**

Let $\omega$ be a Morse form on a closed connected manifold $M$ and let $\phi : \overline{M} \rightarrow M$ be any connected regular covering with structure group $G$ such that $\phi^*([\omega]) = 0$. Then the homomorphism $\{ \omega \} : \pi_1 M \rightarrow \mathbb{R}$ factors as $\pi_1 M \rightarrow G \xrightarrow{\phi^*} \mathbb{R}$. Let $v \in \mathcal{G}t(\omega)$. For every $x \in S(\omega)$ choose a lifting $\overline{x}$ of $x$ to $\overline{M}$ and an orientation of the stable manifold of $x$. Then for every $x, y \in S(\omega)$ with $\text{indx} = \text{indy} + 1$ the incidence coefficient $\overline{n}(\overline{x}, \overline{y}; v) \in \mathbb{Z}G^{-}_{(\omega)}$ is defined (similarly to $\tilde{n}(\tilde{x}, \tilde{y}; v)$).

In particular it is the case for the maximal free abelian covering $\hat{M} \xrightarrow{p} M$ with the structure group $H_1(M, \mathbb{Z})/\text{Tors} \approx \mathbb{Z}^m$. The corresponding homomorphism $\mathbb{Z}^m \rightarrow \mathbb{R}$ is the one arising from the de Rham cohomology class
\[ \omega \]; it will be denoted by the same symbol \( \omega \). Set \( S_\omega = \{ P \in Z[Z^m] \mid P = 1 + Q \text{ and } \text{supp} \ Q \subset [\omega]^{-1}(-\infty, 0]\} \).

**Theorem 3.** Let \( \omega \) be a Morse form with \( [\omega] \neq 0 \). Then there is a subset \( \mathcal{G} t_1(\omega) \subset \mathcal{G} t(\omega) \) with the following properties:

1. \( \mathcal{G} t_1(\omega) \) is open and dense in \( \mathcal{G} t(\omega) \) with respect to \( C^0 \) topology.
2. For every \( v \in \mathcal{G} t_1(\omega) \) and every \( x, y \in S(\omega) \) with \( \text{ind} x = \text{ind} y + 1 \) we have: \( \tilde{n}(\tilde{x}, \tilde{y} ; v) \in S_{[\omega]}^{-1}Z[Z^m] \).
3. Let \( v \in \mathcal{G} t_1(\omega) \). Let \( U \) be a neighborhood of \( S(\omega) \). Then for every \( w \in \mathcal{G} t_1(\omega) \) such that \( w = v \) in \( U \) and \( w \) is sufficiently close to \( v \) in \( C^0 \) topology we have: \( \tilde{n}(\tilde{x}, \tilde{y} ; v) = \tilde{n}(\tilde{x}, \tilde{y} ; w) \) for every \( x, y \in S(\omega) \) with \( \text{ind} x = \text{ind} y + 1 \).

3. **Exponential growth estimates**

Let \( G \) be a group. For an element \( a = \sum n_g g \in ZG \) we denote by \( \|a\| \) the sum \( \sum |n_g| \).

Let \( \xi : G \rightarrow \mathbb{R} \) be a homomorphism. For \( \lambda = \sum n_g g \in ZG_\xi^{-} \) and \( c \in \mathbb{R} \) we denote by \( \lambda[c] \) the element \( \sum_{\xi(g) \geq c} n_g g \) of \( \mathbb{Z}[\pi_1 M] \) and we set \( N_c(\lambda) = ||\lambda[c]|| \). We shall say that \( \lambda \) is of exponential growth if there are \( A, B \geq 0 \) such that for every \( c \in \mathbb{R} \) we have \( N_c(\lambda) \leq Ae^{-cB} \). It is easy to prove that the elements of exponential growth form a subring of \( ZG_\xi^{-} \), which contains \( ZG \).

**Theorem 4.** Let \( \omega \) be a Morse form with \( [\omega] \neq 0 \), and let \( v \) be an \( \omega \)-gradient, belonging to \( \mathcal{G} t_1(\omega) \). Let \( x, y \in S(\omega) \), \( \text{ind} x = \text{ind} y + 1 \). Then \( \tilde{n}(\tilde{x}, \tilde{y} ; v) \) is of exponential growth.

§4. **Exponential growth estimate of the total number of trajectories**

1. **Morse maps** \( M \rightarrow S^1 \)

We assume here the terminology of §1. Recall from [10, §2B] that an \( f \)-gradient \( v \) is called good if for every \( p, q \in S(f) \) we have

\[
\left( \text{ind} p \leq \text{ind} q + 1 \right) \Rightarrow \left( D(p, v) \cap D(q, -v) \right).
\]

The set of all good \( f \)-gradients will be denoted by \( \mathcal{G} d(f) \). For \( p, q \in S(F) \) with \( \text{ind} p = \text{ind} q + 1 \) and for \( v \in \mathcal{G} d(f) \) the set of \((-v)\)-trajectories joining
p to q is finite. For \( x, y \in S(f) \) with \( \text{ind} x = \text{ind} y + 1 \) and \( k \in \mathbb{Z} \) denote by \( N_k(x, y; v) \) the number of \((-v)\)-trajectories joining \( \bar{x} \) to \( \bar{y}t^k \) (recall that \( \bar{x}, \bar{y} \) stand for the chosen liftings of \( x, y \) to \( M \)).

**Theorem 5.** In the set \( \mathcal{G}(f) \) there is a subset \( \mathcal{G}_0(f) \) with the following properties:

1. \( \mathcal{G}_0(f) \) is \( C^0 \) dense in \( \mathcal{G}(f) \) and \( \mathcal{G}_0(f) \subset \mathcal{G}(f) \).
2. Let \( v \in \mathcal{G}_0(f) \). Then there are constants \( C, D > 0 \) such that for every \( x, y \in S(f) \) with \( \text{ind} x = \text{ind} y + 1 \) and for every \( k \in \mathbb{Z} \) we have \( N_k(x, y; v) \leq C \cdot D^k \).

6. Morse forms within arbitrary cohomology classes

We assume here the terminology of §3. Further, for a Morse form \( \omega \) on \( M \) we denote by \( \mathcal{G}_d(\omega) \) the set of all the good \( \omega \)-gradients. (The definition of a good \( \omega \)-gradient is similar to that of good \( f \)-gradient from the above.)

For any \( v \in \mathcal{G}_d(\omega) \), any \( x, y \in S(\omega) \) with \( \text{ind} x = \text{ind} y + 1 \) and any \( g \in \pi_1 M \) the set of \((-v)\)-trajectories joining \( \bar{x} \) to \( \bar{y}g \) is finite and we denote its cardinality by \( N(\bar{x}, \bar{y}, g; v) \). For \( c \in \mathbb{R} \) we denote by \( N_{\geq c}(\bar{x}, \bar{y}, g; v) \) the sum \( \sum_{g \in \{ \omega \}(g) \geq c} N(\bar{x}, \bar{y}, g; v) \).

**Theorem 6.** In the set \( \mathcal{G}(\omega) \) there is a subset \( \mathcal{G}_0(\omega) \) with the following properties:

1. \( \mathcal{G}_0(\omega) \) is dense in \( \mathcal{G}(\omega) \) with respect to \( C^0 \) topology; \( \mathcal{G}_0(\omega) \subset \mathcal{G}_d(\omega) \).
2. Let \( v \in \mathcal{G}_0(\omega) \). There are constants \( C, D > 0 \) such that for every \( x, y \in S(\omega) \) with \( \text{ind} x = \text{ind} y + 1 \) and every \( \lambda \in \mathbb{R} \) we have \( N_{\geq \lambda}(x, y; v) \leq C \cdot D^{-\lambda} \).

**References**


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