EXOTIC 4-MANIFOLDS WITH $b_2^+ = 1$

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1. Introduction

In this paper we present a new family of simply-connected smooth closed 4-manifolds with $b_2^+ = 1$.

The first examples of simply-connected smooth closed 4-manifolds that are homeomorphic but not diffeomorphic were found by Donaldson, see [D1]. Later hordes of such examples were found, see for example [FM1], [D2], [GM], [FS1], [FS3], [Ko1], [Sz1], [MSz]. While the smooth structures of simply-connected smooth closed 4-manifolds turned out to be very rich, we know much less of the $b_2^+ = 1$ case. The previously studied simply-connected smooth closed 4-manifolds with $b_2^+ = 1$ were all Kähler surfaces: $S^2 \times S^2$, $CP^2 \# n\overline{CP}^2$, $B \# n\overline{CP}^2$ and $E_{p,q} \# n\overline{CP}^2$, where $B$ is the Barlow surface, $E_{p,q}$ is an elliptic surface with geometric genus $p_g = 0$ and two multiple fibers with multiplicity $p$, $q$, where $p > 1$, $q > 1$ and $(p,q) = 1$. These 4-manifolds all have different smooth structures, see [D1], [FM1], [Ko1], [Ko2], [Fr], [FM2].

Our first result is the following:

**Theorem 1.1.** There exists a family of smooth closed simply-connected 4-manifolds $Y_n$, parametrized by $n \geq 2$, with $b_2^+(Y_n) = 1$, $b_2^-(Y_n) = 9$ such that

(i) $Y_n$ is irreducible.

(ii) If $k \geq 0$ and $n \neq m$ then $Y_n \# k\overline{CP}^2$ is not diffeomorphic to $Y_m \# k\overline{CP}^2$.

(iii) If $k \geq 0$, then $Y_n \# k\overline{CP}^2$ is not diffeomorphic to any Kähler surface.

It follows that $Y_n$ form a new family of simply-connected smooth closed 4-manifolds with $b_2^+ = 1$. The construction of $Y_n$ is presented in Section 2. We prove Theorem 1.1 in Section 3 by using Seiberg-Witten invariants in the $b_2^+ = 1$ case.

Using results of Taubes on symplectic 4-manifolds, see [T1], [T2], we can strengthen Theorem 1.1:

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Theorem 1.2. For all \( n \geq 2 \) neither \( Y_n \) nor \( \overline{Y}_n \) have symplectic structure.

It follows that the 4-manifolds \( Y_n \) provide new counter-examples to the Minimal Conjecture. Counter-examples with \( b_2^+ > 1 \) were given in [Sz2] using a related construction.

2. Construction of \( Y_n \)

Let us start by recalling the Kodaira-Thurston manifold [Th], which we denote by \( W \). Let \( \phi : T^2 \to T^2 \) be an orientation preserving self-diffeomorphism satisfying \( \phi_*(a_1) = a_1 + a_2, \phi_*(a_2) = a_2 \), where \( a_1, a_2 \in H_1(T^2, \mathbb{Z}) \) form a basis. Let \( Z_\phi \) denote the mapping torus of \( \phi \). Then \( W \) is defined as \( W = Z_\phi \times S^1 \).

The definition of \( Z_\phi \) gives a fibration \( T^2 \to Z_\phi \to S^1 \). We can assume that \( \phi \) has fixpoints. Let the circle \( \gamma \hookrightarrow Z_\phi \) be a section corresponding to a fixpoint. Let us fix another circle \( \delta \hookrightarrow Z_\phi \) that lies in a fiber and represents \( a_1 \). Now we define smoothly embedded 2-tori \( T_1 = \gamma \times S^1 \hookrightarrow W \) and \( T_2 = \delta \times S^1 \hookrightarrow W \). The self-intersections of \( T_1 \) and \( T_2 \) are equal to 0. It follows from [Th] that \( W \) has a symplectic structure for which \( T_1 \) is a symplectic submanifold. By fixing such a symplectic form on \( W \) we get an induced orientation on \( T_1 \).

Now take a rational elliptic surface \( E(1) = CP^2 \# 9\overline{CP^2} \). Fix a generic fiber \( F \hookrightarrow E(1) \) of the elliptic fibration of \( E(1) \). Then \( F \) is a smoothly embedded torus of self-intersection 0, and the complex structure of \( E(1) \) induces an orientation on \( F \). Fix an orientation preserving diffeomorphism \( f : F \to T_1 \) and lift it to an orientation reversing diffeomorphism \( g \) between the closed tubular neighborhoods. Using \( g \) we get the fiber sum of \( E(1) \) and \( W \):

\[
M = (E(1) \setminus nd(F)) \cup_g (W \setminus nd(T_1)),
\]

where \( nd \) denotes the open tubular neighborhood.

Now \( T_2 \hookrightarrow M \) is a smoothly embedded torus of self-intersection 0. We define the family \( Y_n \) by performing logarithmic transformations along \( T_2 \):

Let us fix a circle \( \delta' \hookrightarrow \partial(Z_\phi \setminus nd(\delta)) \) that lies in a fiber of \( Z_\phi \) and represents \( a_1 \). In other words \( \delta' \) is a parallel copy of \( \delta \). Let \( \alpha \in H_1(\partial(M \setminus nd(T_2)), \mathbb{Z}) \) be the homology class of \( \delta' \times p \hookrightarrow \partial(M \setminus nd(T_2)) = \partial(W \setminus nd(T_2)) \), where \( p \in S^1 \). Let \( \beta \in H_1(\partial(M \setminus nd(T_2)), \mathbb{Z}) \) represent the homology class of the meridian around \( T_2 \).

For each \( n \geq 0 \) let us fix an orientation reversing diffeomorphism \( \phi_n : \partial(D^2 \times T^2) \to \partial(M \setminus nd(T_2)) \) that satisfies

\[
(\phi_n)_*(e) = \alpha + n\beta,
\]
where \( e \in H_1(\partial(D^2 \times T^2), \mathbb{Z}) \) is defined by \( e = [\partial(D^2) \times q] \), where \( q \in T^2 \).
Now we define $Y_n$:

$$Y_n = (M \setminus nd(T_2)) \cup_{\phi_n} (D^2 \times T^2).$$

**Lemma 2.1.** For all $n \geq 0$ the smooth closed 4-manifolds $Y_n$ are simply-connected, $b_2^+(Y_n) = 1$ and $b_2^-(Y_n) = 9$.

**Proof.** First note that

$$\pi_1(Z_{\phi}) = \langle g_1, g_2, g_3 | [g_1, g_2] = [g_2, g_3] = 1, g_3^{-1}g_1g_3 = g_1g_2 \rangle,$$

where $g_1, g_2$ correspond to $a_1, a_2$ and $g_3$ corresponds to $\gamma$. Since $\pi_1(E(1) \setminus nd(F)) = 1$, it follows that $\pi_1(M) = \pi_1(Z_{\phi})/(g_3 = 1)$. So we get $\pi_1(M) = Z$ where the generator is $g_1$. It is not hard to see that $\pi_1(M \setminus nd(T_2)) = Z$ and the generator is represented by $\delta' \times p \mapsto \partial(M \setminus nd(T_2))$. Let $i : \partial(M \setminus nd(T_2)) \rightarrow M \setminus nd(T_2)$ be the inclusion. Since $i_*(\beta) = 0$ and $\alpha = [\delta' \times p]$, it follows that $H_1(Y_n, Z) = 0$ for all $n$. On the other hand $\pi_1(M \setminus nd(T_2)) = Z$ shows that $\pi_1(Y_n)$ is abelian. It follows that $\pi_1(Y_n) = 1$. The rest of the lemma is trivial. \hfill $\square$

**3. Proof of Theorem 1.1 and Theorem 1.2**

In this section we use Seiberg-Witten invariants for smooth closed oriented 4-manifolds with $b_2^+ = 1$. Let us recall that the usual Seiberg-Witten invariant for a smooth closed oriented 4-manifold $X$ with $b_2^+(X) > 1$ is an integer valued function defined on the set of spin$^c$ structures over $X$. In case $H_1(X, Z)$ has no 2-torsion it is convenient to use the one-to-one correspondence between the set of spin$^c$ structures over $X$ and set of characteristic elements in $H^2(X, Z)$. After fixing a homology orientation, i.e an orientation on $detH^2_2(X, R) \otimes detH^1(X, R)$, we have

$$SW_X : \{K \in H^2(X, Z) | K \equiv w_2(TX) \pmod{2} \} \rightarrow \mathbb{Z}.$$ 

$K$ is called a basic class of $X$ if $SW_X(K) \neq 0$.

In the $b_2^+(X) = 1$ case however $SW_X$ depends on other parameters as well. Let us recall, see [Wi], [KM], [M], that the perturbed Seiberg-Witten moduli space $M_X(K, g, h)$ is defined as the solution space of the Seiberg-Witten equations

$$F_A^+ = q(\phi) + ih, \quad D_A \phi = 0$$

divided by the gauge-group. Here $q$ is a riemannian metric on $X$, $A$ is an $S^1$ connection on the line bundle $L$ with $c_1(L) = K$, $\phi$ is a section of the positive spin bundle corresponding to the spin$^c$ structure determined by $K$, $F_A^+$ is the self-dual part of the curvature of $A$, $q$ is a certain quadratic map, $D_A$ is the Dirac operator coupled with $A$, and $h$ is an arbitrary closed real-valued self-dual 2-form on $X$. 

EXOTIC 4-MANIFOLDS WITH $b_2^+ = 1$
If \(b^+_2(X) \geq 1\) and \(h\) is generic then the moduli space \(\mathcal{M}_X(K,g,h)\) is a closed manifold with formal dimension \(d = (K^2 - 2e(X) - 3\text{sign}(X))/4\), where \(d < 0\) implies that \(\mathcal{M}_X(K,g,h)\) is empty. If \(d < 0\) then \(SW_X(K) = 0\) by definition. In the \(d \geq 0\) case one defines

\[
SW_X(K, g, h) = ([\mathcal{M}_X(K, g, h)], \mu^{d/2}),
\]

where \(\mu \in H^2(\mathcal{M}_X(K, g, h), \mathbb{Z})\) is the Euler-class of the base fibration.

In the \(b^+_2(X) = 1\) case \(SW_X(K, g, h)\) depends on \(g\) and \(h\), since if one varies the metric \(g\) and the perturbing 2-form \(h\) in a generic one-parameter family then the corresponding cobordism could contain singularities (where \(\phi \equiv 0\)).

In this paper we work with the \(b^+_2(X) = 1\), \(H_1(X, Z) = 0\) case, where the dependence is as follows.

**Lemma 3.1.** (See [KM], [M, p105]) Let \(X\) be a smooth closed oriented 4-manifold with \(b^+_2(X) = 1\) and \(H_1(X, Z) = 0\). Fix a homology orientation of \(H^2_+(X, \mathbb{R})\). For each riemannian metric \(g\) let \(\omega^+(g)\) be the unique \(g\)-harmonic self-dual 2-form that has norm 1 and is compatible with the orientation of \(H^2_+(X, \mathbb{R})\). Then for each characteristic elements \(K \in H^2(X, Z)\) with \(d = (K^2 - 2e(X) - 3\text{sign}(X))/4 \geq 0\) we have

- If \((2\pi K + h_1) \cdot \omega^+(g_1)\) and \((2\pi K + h_2) \cdot \omega^+(g_2)\) are not zero and have the same signs then
  \[
  SW_X(K, g_1, h_1) = SW_X(K, g_2, h_2)
  \]

- If \((2\pi K + h_1) \cdot \omega^+(g_1) < 0 < (2\pi K + h_2) \cdot \omega^+(g_2)\), then
  \[
  SW_X(K, g_1, h_1) = SW_X(K, g_2, h_2) + (-1)^{d/2}.
  \]

It follows that if furthermore \(b^-_2(X) \leq 9\) then we have a preferred Seiberg-Witten invariant.

**Lemma 3.2.** Let \(X\) be a smooth closed oriented 4-manifold with \(H_1(X, Z) = 0\), \(b^+_2(X) = 1\) and \(b^-_2(X) \leq 9\). Then for every characteristic element \(K \in H^2(X, Z)\), pair of riemannian metrics \(g_1, g_2\) and small enough perturbing 2-forms \(h_1, h_2\) we have

\[
SW_X(K, g_1, h_1) = SW_X(K, g_2, h_2).
\]

**Proof.** Let \(K \in H^2(X, Z)\) be a characteristic element for which \(d \geq 0\). Then \(2e(X) + 3\text{sign}(X) = 4 + 5b^+_2(X) - b^-_2(X) \geq 0\), implies \(K^2 \geq 0\). As a corollary we have that \(K \cdot \omega^+(g_1)\), \(K \cdot \omega^+(g_2)\) are non-zero and have the same signs. Now Lemma 3.2 follows from Lemma 3.1. \(\square\)

From now on we denote the invariant described in Lemma 3.2 by \(SW_X(K)\). Our first result in this section is the following.
**Theorem 3.3.** Let \( Y_n \), for \( n \geq 0 \), be defined as in Section 2. Let \( SW_{Y_n} \) be defined according to Lemma 3.2. Then we have

- \( SW_{Y_n}(\pm L) = \pm n \), where \( L = PD[T_1] \)
- \( SW_{Y_n}(L') = 0 \) for all \( L' \neq \pm L \).

The main input in the proof of Theorem 3.3 is a surgery formula that relates \( SW_M \), \( SW_{Y_0} \) and \( SW_{Y_n} \). This result is a special case of the more general surgery formulas in [MMSz].

**Lemma 3.4.** (See [MMSz], cf. also [Sz2]). For a characteristic element \( K \in H^2(M, \mathbb{Z}) \) that satisfies \( \langle K, [T_2] \rangle = 0 \), let \( \overline{K} \) denote the corresponding characteristic element in \( Y_n \). Then we have

\[
SW_{Y_n}(\overline{K}) = SW_{Y_0}(\overline{K}) + n \sum_{i=-\infty}^{\infty} SW_M(K + 2iF),
\]

where \( F = PD[T_2] \), \( SW_{Y_n} \) is defined according to Lemma 3.2 and \( SW_M \) is well-defined since \( b^+_2(M) = 2 \).

**Proof of Theorem 3.3.** We compute \( SW_M \), \( SW_{Y_0} \) and then apply Lemma 3.4. Note first that the symplectic sum construction of Gompf, see [G], implies that \( M \) has a symplectic structure where the canonical class of the symplectic structure is equal to \( PD[T_1] \). It follows from [T1] that

\[
SW_M(\pm PD[T_1]) = \pm 1.
\]

On the other hand using the generalized adjunction formula, see [KM], [MMSz], it is an easy exercise to show that \( SW_M(L') = 0 \) for all \( L' \neq \pm PD[T_1] \).

It is not hard to show, cf. [Sz2], that \( Y_0 \) contains a smoothly embedded torus with self-intersection 1. Applying the generalized adjunction formula to the \( b^+_2 = 1 \) case, it follows that \( SW_{Y_0} \) vanishes.

Now applying Lemma 3.4, we get

\[
SW_{Y_n}(\pm PD[T_1]) = \pm n
\]

and \( SW_{Y_n}(L') = 0 \) for all \( L \neq \pm PD[T_1] \).

**Proof of Theorem 1.1.** Suppose that there exists \( n \geq 2 \) such that \( Y_n \) is not irreducible, i.e. \( Y_n = X \# Z \) with neither \( X \) nor \( Z \) being a homotopy \( S^4 \). Since \( \pi_1(Y_n) = 1, b^+_2(Y_n) = 1 \), \( X \) or \( Z \) is negative definite with \( b_2 > 0 \). Now Lemma 3.2 and the blow-up formula of [FS2] for Seiberg-Witten invariants contradicts Theorem 3.3 and this proves (i).

In order to prove (ii), (iii) we need to study the chamber structure of \( Y_n \# k\overline{CP^2} \). For a smooth closed oriented 4-manifold \( X \) with \( b^+_2(X) = 1 \) and \( H_1(X, \mathbb{Z}) = 0 \) we define the set of chambers in the following way.
Fix an orientation of $H^2_+(X, R)$. Let $\Omega = \{ x \in H^2(X, R) | x^2 = 1 \}$. Let $\Omega^+$ denote the positive component of $\Omega$. If $K \in H^2(X, Z)$ is a characteristic element, i.e $K \equiv w_2(TX) \pmod 2$, and $K^2 \geq 2e(X) + 3\text{sign}(X)$ then we define a wall

$$w(K) = \{ x \in \Omega^+ | x \cdot K = 0 \}.$$ 

The union of these walls $W$ is locally compact in $\Omega^+$. We define the set of chambers of $X$ as the set of connected components of $\Omega^+ \setminus W$. Note that the chambers are open.

For every chamber $C$ we define $SW^C_X(K)$ to be equal to $SW_X(K, g, h)$ where $[\omega^+(g)] \in C$ and $h$ is small enough. It follows from Lemma 3.1, that if $K \cdot C_1, K \cdot C_2$ have opposite signs then

$$SW^{C_1}_X(K) = SW^{C_2}_X(K) \pm 1$$

and if $K \cdot C_1, K \cdot C_2$ have the same signs then

$$SW^{C_1}_X(K) = SW^{C_2}_X(K).$$

$K$ is called a basic class of $C$ if $SW^C_X(K) \neq 0$. Let $\text{dist}(C)$ denote the maximum of $A \cdot B$ where $A, B$ are basic classes of $C$. We claim the following.

**Lemma 3.5.** Let $n \geq 1$ and $k \geq 0$. Then every chamber $C$ of $Y_n \# k\mathbb{CP}^2$ has at least one basic class $K$ with $SW^C_{Y_n \# k\mathbb{CP}^2}(K) = \pm n$, and there exists a chamber $C_0$ satisfying that

$$SW^C_{Y_n \# k\mathbb{CP}^2}(K') = \pm n$$

for all basic classes $K'$ of $C_0$. Furthermore if a chamber $C$ of $Y_n \# k\mathbb{CP}^2$ have $\text{dist}(C) = k$, then all basic classes $A$ of $C$ satisfies

$$A = (2l + 1)L + \sum_{i=1}^{k} (-1)^{\delta_i} E_i$$

with some $l \in \mathbb{Z}$, $\delta_i = 0, 1$ for $i = 1, \ldots, k$, where $L = PD[T_1]$ and $E_i$ is the exceptional class of the $i$-th copy of $\mathbb{CP}^2$.

**Proof.** Let us fix the orientation of $H^2_+(Y_n \# k\mathbb{CP}^2, R)$ in such a way that $L \cdot \omega > 0$ for all $\omega \in \Omega^+$. There is a unique chamber $C_0$ of $Y_n \# k\mathbb{CP}^2$ for which $C_0 \cap \text{Im}(i)$ is not empty, where $i : H^2(Y_n, R) \to H^2(Y_n \# k\mathbb{CP}^2, R)$ is the obvious inclusion. Let us fix $\omega_0 \in C_0 \cap \text{Im}(i)$. It follows from
Theorem 3.3 and the blow-up formula that all basic classes of $C_0$ are given by $\pm L \pm E_1 \cdots \pm E_k$ and

$$SW^{C_0}_{Y_n \# k\mathbb{C}P^2}(\pm L \pm E_1 \cdots \pm E_k) = \pm n.$$

Now let $C$ be another chamber of $Y_n \# k\mathbb{C}P^2$ and fix $\omega \in C$. Then $\omega$ decomposes as $\omega = \omega_1 + \sum_{i=1}^{k} (-1)^{\epsilon_i} l_i E_i$, where $\omega_1$ lies in $Im(i)$, $\epsilon_i = 0, 1$ and $l_i \geq 0$. Let $K = L + \sum_{i=1}^{k} (-1)^{\epsilon_i} t_i E_i$. It is easy to see that $K \cdot \omega > 0$, $K \cdot \omega_0 > 0$. It follows that

$$SW^C_{Y_n \# k\mathbb{C}P^2}(K) = SW^{C_0}_{Y_n \# k\mathbb{C}P^2}(K) = \pm n.$$

Now suppose $dist(C) = k$ and there is a basic class $A$ of $C$ that is not a basic class of $C_0$. $A$ decomposes as

$$A = A_0 + \sum_{i=1}^{k} (-1)^{\delta_i} l_i E_i,$$

where $A_0 \in Im(i)$, $\delta_i = 0, 1$, $l_i \geq 1$ and odd. After multiplying $A$ by $-1$ if necessary, we can assume that $A \cdot \omega_0 > 0$. Note that since $A_0^2 \geq 0$ we have $A_0 \cdot L \geq 0$, where equality implies that $A_0$ is an odd multiple of $L$.

Since $A$ is a basic class of $C$ but not a basic class of $C_0$, it follows that $A \cdot \omega_1 < 0$. Let

$$A' = A_0 + \sum_{i=1}^{k} (-1)^{\epsilon_i} t_i E_i.$$

It is easy to see, that $A' \cdot \omega_1 < 0 < A' \cdot \omega_0$ and so $A'$ is a basic class of $C$. Now

$$A' \cdot K = A_0 \cdot L + \sum_{i=1}^{k} t_i \geq A_0 \cdot L + k \geq k,$$

where $A' \cdot K = k$ implies that $t_i = 1$ for all $i = 1, \ldots, k$ and $A_0$ is an odd multiple of $L$. This finishes the proof of Lemma 3.5. \(\Box\)

Now suppose that contrary to (ii) of Theorem 1.1 there is a diffeomorphism $f : Y_n \# k\mathbb{C}P^2 \to Y_m \# k\mathbb{C}P^2$, with $n \neq m$. It is clear that $f$ has to be orientation preserving. Let us fix the chamber $C_0$ of $Y_n \# k\mathbb{C}P^2$ as in Lemma 3.5, and let $C$ be the pullback of $C_0$ under $f^*$. Then for all characteristic elements $K$ of $Y_n \# k\mathbb{C}P^2$ we have

$$SW^{C_0}_{Y_n \# k\mathbb{C}P^2}(K) = SW^{C}_{Y_m \# k\mathbb{C}P^2}(f^* K).$$

This contradicts the first part of Lemma 3.5 and the contradiction proves (ii).
Note that a simply-connected Kähler surface with $b_2^+ = 1$ is either rational, a surface of general type or a non-rational elliptic surface, in which case it is equal to one of $E_{p,q} \# k\overline{\mathbb{C}P}^2$ where $p > 1$, $q > 1$, $(p,q) = 1$ and $k \geq 0$.

Since $\mathbb{C}P^2 \# k\overline{\mathbb{C}P}^2$ has a chamber where the Seiberg-Witten invariant vanishes, it follows from Lemma 3.5, that $Y_n \# k\overline{\mathbb{C}P}^2$ with $n \geq 1$ is not diffeomorphic to any rational surface.

The Seiberg-Witten invariants of surfaces of general type are known. It is proved for example in [M], that any surface of general type $S$ with $b_2^+(S) = 1$ has a chamber $C$, in which $SW^C_X(K) = \pm 1$ for all basic classes of $C$. It follows now from Lemma 3.5, that if $n \geq 2$, then $Y_n \# k\overline{\mathbb{C}P}^2$ is not diffeomorphic to $S$.

Now we deal with $E_{p,q}$, where $p > 1$, $q > 1$, $(p,q) = 1$. Since $b_2^+(E_{p,q}) = 1$, $b_2^-(E_{p,q}) = 9$, it follows from Lemma 3.2 that $E_{p,q}$ has a unique chamber. Let $K$ denote the canonical class of $E_{p,q}$. It is proved in [M], that $SW_{E_{p,q}}(\pm K) = \pm 1$ and all basic classes $K'$ of $E_{p,q}$ satisfy $K' = tK$, where $|t| \leq 1$.

Suppose that there is a diffeomorphism $f : Y_n \# k\overline{\mathbb{C}P}^2 \to E_{p,q} \# k\overline{\mathbb{C}P}^2$, with $n \geq 2$. After fixing a homology orientation for $Y_n \# k\overline{\mathbb{C}P}^2$, $f$ induces an orientation on $H^2_+(E_{p,q} \# k\overline{\mathbb{C}P}^2, R)$. Let $C_1$ be the unique chamber of $E_{p,q} \# k\overline{\mathbb{C}P}^2$ for which $C_1 \cap Im(i)$ is not empty, where $i : H^2(E_{p,q}, R) \to H^2(E_{p,q} \# k\overline{\mathbb{C}P}^2, R)$ is the obvious inclusion. The blow-up formula shows that every basic class of $C_1$ can be written as $tK + \sum_{i=1}^k (-1)^{\delta_i} D_i$ with some $|t| \leq 1$, $\delta_i = 0, 1$, where $D_i$ denotes the exceptional class of the $i$-th copy of $\overline{\mathbb{C}P}^2$. Furthermore

$$SW^C_{E_{p,q} \# k\overline{\mathbb{C}P}^2}(\pm K \pm D_1 \cdots \pm D_k) = \pm 1.$$  

It follows that $\text{dist}(C_1) = k$.

Let $C$ denote the image of $C_1$ under $f^*$. Then $C$ is a chamber of $Y_n \# k\overline{\mathbb{C}P}^2$ with $\text{dist}(C) = k$. It follows then from the second part of Lemma 3.5, that $f^*(V_i) = V_0$, where $V_0 = \langle L, E_1, \ldots, E_k \rangle$ and $V_1 = \langle K, D_1, \ldots, D_k \rangle$. Since $L^2 = K^2 = 0$, it follows that $f^*(K)$ is a multiple of $L$. Just as in the proof of Lemma 3.5, we have a basic class $K'$ of $C$ such that $K' = L + \sum_{i=1}^k (-1)^{\delta_i} E_i$ with $SW^C_{Y_n \# k\overline{\mathbb{C}P}^2}(\pm K') = \pm n$, and for all $j > 0$ we have

$$1) \quad SW^C_{Y_n \# k\overline{\mathbb{C}P}^2}(K' + 2jL) = 0.$$  

Let $A$ be the unique characteristic element of $E_{p,q} \# k\overline{\mathbb{C}P}^2$ with $f^*(A) = \ldots$
$K'$. It follows that

$$SW^{C_1}_{E_p,q \neq kCP^2}(A) = \pm n,$$

which implies $A = tK + \sum_{i=1}^{k} (-1)^{i}E_i$, with $|t|$ strictly less than 1. Now $A + (1 - t)K, A - (1 + t)K$ are basic classes of $C_1$ and consequently $f^*(A + (1 - t)K) = K' + 2j_1L, f^*(A - (1 + t)K) = K' + 2j_2L$ are basic classes of $C$. Since one of $j_1, j_2$ is positive, this contradicts (1). This finishes the proof of Theorem 1.1.

**Proof of Theorem 1.2.** We first need a result of Taubes on symplectic 4-manifolds.

**Lemma 3.6.** (See [T1], [T2]). Let $X$ be an oriented symplectic 4-manifold with $H_1(X,Z) = 0$ and $b_2^+(X) = 1$. For all characteristic element $L$ of $X$ and symplectic form $\omega$ with $\omega^2 = 1$ we define

$$SW^\omega_X(L) = SW_X(L, g, -r\omega),$$

where $\omega^+(g) = \omega$ and $r$ is large enough. Then we have

$$SW^\omega_X(-K) = \pm 1,$$

where $K$ is the canonical class of the symplectic structure. Furthermore for all characteristic element $K'$ with $SW^\omega_X(K') \neq 0$ we have

$$-K \cdot \omega \leq K' \cdot \omega,$$

where equality implies $-K = K'$.

Now suppose that there exists an $n \geq 2$ for which $Y_n$ has a symplectic structure. By multiplying with $(-1)$ if necessary, we can assume that $L \cdot \omega > 0$, where $L = PD[T_1]$. Lemma 3.1 shows that if a characteristic element $K'$ of $Y_n$ satisfies $K' \cdot \omega < 0$, then we have

$$SW_{Y_n}(K') = SW_{Y_n}^\omega(K').$$

Now it follows from Theorem 3.3 that

$$(2) \quad SW_{Y_n}^\omega(-L) = \pm n$$

and for all $K'$ with $SW_{Y_n}^\omega(K') \neq 0$ we have

$$-L \cdot \omega \leq K' \cdot \omega.$$
Final remark

Starting with any smooth closed four-manifold $X$ that contains a smoothly embedded torus $T \hookrightarrow X$ with self-intersection 0 and satisfies $\pi_1(X \setminus nd(T)) = 1$, one can define a family of simply-connected 4-manifolds $Z_n$ by making the fiber sum of $X$ and the Kodaira-Thurston manifold $W$ along $T$, $T_1$ and then using $\phi_n$ to make a logarithmic transformation along $T_2$. In this way one can construct interesting simply-connected 4-manifolds. For example one can start with the $K3$ surface which contains three disjoint Gompf nuclei, see [GM]. By using the above construction repeatedly along the three fibers contained in the different nuclei we get a three parameter family of homotopy $K3$ surfaces, $Z_{n,m,k}$. It easily follows from Theorem 3.3 and [Sz2] that if $n \geq 2, m \geq 2, k \geq 2$, then $Z_{n,m,k}$ is non-symplectic.

As another generalization of [Sz2] Fintushel and Stern recently constructed a surprisingly rich family of non-symplectic homotopy $K3$ surfaces, and also proved that $Z_{n,m,k}$ arise as a special case of their construction.

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References


EXOTIC 4-MANIFOLDS WITH $b_2^+ = 1$


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