DESINGULARIZATION OF SINGULAR
HYPERKÄHLER VARIETIES I

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Abstract. Let $M$ be a singular hyperkähler variety, obtained as a moduli space of stable holomorphic bundles on a compact hyperkähler manifold (alg-geom/9307008). Consider $M$ as a complex variety in one of the complex structures induced by the hyperkähler structure. We show that normalization of $M$ is smooth, hyperkähler and does not depend on the choice of induced complex structure.

0. Introduction

The structure of this paper is as follows.

- In the first section, we give a compendium of definitions and results from hyperkähler geometry, all known from literature.
- Section 2 deals with the real analytic varieties underlying complex varieties. We define almost complex structures on a real analytic variety. This notion is used in order to define hypercomplex varieties. We show that a hyperkähler manifold is always hypercomplex.
- In Section 3, we give a definition of a singular hyperkähler variety, following [V-bun] and [V3]. We cite basic properties and list the examples of such manifolds.
- In Section 4, we define locally homogeneous singularities. A space with locally homogeneous singularities (SLHS) is an analytic space $X$ such that for all $x \in X$, the $x$-completion of a local ring $\mathcal{O}_x X$ is isomorphic to an $x$-completion of associated graded ring $(\mathcal{O}_x X)_{gr}$. We show that a complex variety is SLHS if and only if the underlying real analytic variety is SLHS. This allows us to define invariantly the notion of a hyperkähler SLHS. The natural examples of hyperkähler SLHS include the moduli spaces of stable holomorphic bundles, considered in [V-bun]. We conjecture that every hyperkähler variety is a space with locally homogeneous singularities.
- In Section 5, we study the tangent cone of a singular hyperkähler manifold $M$ in the point $x \in M$. We show that its reduction, which is a

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1In [V-bun], we proved that the moduli of stable bundles over a compact hyperkähler manifold is a hyperkähler variety, if we assume certain numerical restrictions on the bundle’s Chern classes. The stable bundles satisfying these restrictions are called hyperholomorphic.
closed subvariety of $T_xM$, is a union of linear subspaces $L_i \subset T_xM$. These subspaces are invariant under the natural quaternion action in $T_xM$. This implies that a normalization of $(M,I)$ is smooth. Here, as usually, $(M,I)$ denotes $M$ considered as a complex variety, with $I$ a complex structure induced by the singular hyperkähler structure on $M$.

- In Section 6, we formulate and prove the desingularization theorem for hyperkähler varieties with locally homogeneous singularities. For each such variety $M$ we construct a finite surjective morphism $\tilde{M} \to M$ of hyperkähler varieties, such that $\tilde{M}$ is smooth and $n$ is an isomorphism outside of singularities of $M$. The $\tilde{M}$ is obtained as a normalization of $M$; thus, our construction is canonical and functorial.

1. Hyperkähler manifolds

1.1. Definitions. This subsection contains a compression of the basic definitions from hyperkähler geometry, found, for instance, in [Bes] or in [Beau].

Definition 1.1. ([Bes]) A hyperkähler manifold is a Riemannian manifold $M$ endowed with three complex structures $I$, $J$ and $K$, such that the following holds.

(i): The metric on $M$ is Kähler with respect to these complex structures and
(ii): $I$, $J$ and $K$, considered as endomorphisms of a real tangent bundle, satisfy the relation $I \circ J = -J \circ I = K$.

The notion of a hyperkähler manifold was introduced by E. Calabi ([C]).

Clearly, hyperkähler manifold has the natural action of quaternion algebra $\mathbb{H}$ in its real tangent bundle $TM$. Therefore its complex dimension is even. For each quaternion $L \in \mathbb{H}$, $L^2 = -1$, the corresponding automorphism of $TM$ is an almost complex structure. It is easy to check that this almost complex structure is integrable ([Bes]).

Definition 1.2. Let $M$ be a hyperkähler manifold, $L$ a quaternion satisfying $L^2 = -1$. The corresponding complex structure on $M$ is called an induced complex structure. The $M$ considered as a complex manifold is denoted by $(M,L)$.

Let $M$ be a hyperkähler manifold. We identify the group $SU(2)$ with the group of unitary quaternions. This gives a canonical action of $SU(2)$ on the tangent bundle, and all its tensor powers. In particular, we obtain a natural action of $SU(2)$ on the bundle of differential forms.

Lemma 1.3. The action of $SU(2)$ on differential forms commutes with the Laplacian.

Proof. This is Proposition 1.1 of [V-bun].

Thus, for compact $M$, we may speak of the natural action of $SU(2)$ in cohomology.
1.2. Trianalytic subvarieties in compact hyperkähler manifolds. In this subsection, we give a definition and a few basic properties of trianalytic subvarieties of hyperkähler manifolds. We follow [V2].

Let $M$ be a compact hyperkähler manifold, $\dim_{\mathbb{R}} M = 2m$.

**Definition 1.4.** Let $N \subset M$ be a closed subset of $M$. Then $N$ is called **trianalytic** if $N$ is a complex analytic subset of $(M,L)$ for any induced complex structure $L$.

Let $I$ be an induced complex structure on $M$, and $N \subset (M,I)$ be a closed analytic subvariety of $(M,I)$, $\dim_{\mathbb{C}} N = n$. Denote by $[N] \in H^{2n}(M)$ the homology class represented by $N$. Let $(N) \in H^{2m-2n}(M)$ denote the Poincare dual cohomology class. Recall that the hyperkähler structure induces the action of the group $SU(2)$ on the space $H^{2m-2n}(M)$.

**Theorem 1.5.** Assume that $(N) \in H^{2m-2n}(M)$ is invariant with respect to the action of $SU(2)$ on $H^{2m-2n}(M)$. Then $N$ is trianalytic.

**Proof.** This is Theorem 4.1 of [V2].

**Remark 1.6.** Trianalytic subvarieties have an action of quaternion algebra in the tangent bundle. In particular, the real dimension of such subvarieties is divisible by 4.

1.3. Totally geodesic submanifolds.

**Proposition 1.7.** Let $X \stackrel{\phi}{\hookrightarrow} M$ be an embedding of Riemannian manifolds (not necessarily compact) compatible with the Riemannian structure. Then the following conditions are equivalent.

(i): Every geodesic line in $X$ is geodesic in $M$.

(ii): Consider the Levi-Civita connection $\nabla$ on $TM$, and restriction of $\nabla$ to $TM|_{X}$. Consider the orthogonal decomposition

\[ TM|_{X} = TX \oplus TX^\perp. \]  

Then, this decomposition is preserved by the connection $\nabla$.

**Proof.** Well known; see, for instance, [Bes].

**Proposition 1.8.** Let $X \subset M$ be a trianalytic submanifold of a hyperkähler manifold $M$, where $M$ is not necessarily compact. Then $X$ is totally geodesic.

**Proof:** This is [V3], Corollary 5.4.

2. Real analytic varieties

For the reference and results about real analytic varieties and spaces, see [GMT].

Let $X$ be a complex analytic variety. The “real analytic space underlying $X$” (denoted by $X_\mathbb{R}$) is the following object. By definition, $X_\mathbb{R}$ is a ringed space
with the same topology as $X$, but with a different structure sheaf, denoted by $\mathcal{O}_{X_r}$. Let $i : U \hookrightarrow B^n$ be a closed embedding of an open subset $U \subset X$ to an open ball $B^n \subset \mathbb{C}^n$, and $I$ be an ideal defining $i(U)$. Then $\mathcal{O}_{X_r}|_{U}$ is a quotient sheaf $\mathcal{O}_{B^n}/Re(I)$ of the sheaf of real analytic functions on $B^n$ by the ideal $Re(I)$ generated by the real parts of the functions $f \in I$.

Note that the real analytic space underlying $X$ needs not be reduced.

Consider the sheaf $\mathcal{O}_X$ of holomorphic functions on $X$ as a subsheaf of the sheaf $\mathcal{C}(X, \mathbb{C})$ of continuous $\mathbb{C}$-valued functions on $X$. The sheaf $\mathcal{C}(X, \mathbb{C})$ has a natural automorphism $f \mapsto \bar{f}$, where $\bar{f}$ is complex conjugation. By definition, the section $f$ of $\mathcal{C}(X, \mathbb{C})$ is called antiholomorphic if $\bar{f}$ is holomorphic. Let $\mathcal{O}_X$ be the sheaf of holomorphic functions, and $\mathcal{O}_{X_r}$ be the sheaf of antiholomorphic functions on $X$. Let $\mathcal{O}_X \otimes \mathcal{C} \to \mathcal{O}_{X_r} \otimes \mathcal{C}$ be the natural multiplication map.

**Claim 2.1.** The sheaf homomorphism $i : \mathcal{O}_X \otimes \mathcal{C} \to \mathcal{O}_{X_r} \otimes \mathcal{C}$ is injective. For each point $x \in X$, $i$ induces an isomorphism on $x$-completions of $\mathcal{O}_X \otimes \mathcal{C} \overline{\mathcal{O}}_X$ and $\mathcal{O}_{X_r} \otimes \mathcal{C}$.

**Proof.** Clear from the definition.

Let $\Omega^1(\mathcal{O}_{X_r})$, $\Omega^1(\mathcal{O}_X \otimes \mathcal{C} \overline{\mathcal{O}}_X)$, $\Omega^1(\mathcal{O}_{X_r} \otimes \mathcal{C})$ be the sheaves of continuous differentials associated with the corresponding ring sheaves. There are natural sheaf maps

\[ \Omega^1(\mathcal{O}_{X_r}) \otimes \mathcal{C} \to \Omega^1(\mathcal{O}_{X_r} \otimes \mathcal{C}) \tag{2.1} \]

and

\[ \Omega^1(\mathcal{O}_{X_r} \otimes \mathcal{C}) \to \Omega^1(\mathcal{O}_X \otimes \mathcal{C} \overline{\mathcal{O}}_X), \tag{2.2} \]

corresponding to the monomorphisms

\[ \mathcal{O}_{X_r} \hookrightarrow \mathcal{O}_{X_r} \otimes \mathcal{C}, \quad \mathcal{O}_X \otimes \mathcal{C} \overline{\mathcal{O}}_X \hookrightarrow \mathcal{O}_{X_r} \otimes \mathcal{C}. \]

**Claim 2.2.** The map (2.1) is an isomorphism. Tensoring both sides of (2.2) by $\mathcal{O}_{X_r} \otimes \mathcal{C}$ produces an isomorphism

\[ \Omega^1(\mathcal{O}_X \otimes \mathcal{C} \overline{\mathcal{O}}_X) \otimes_{\mathcal{O}_X \otimes \mathcal{C} \overline{\mathcal{O}}_X} (\mathcal{O}_{X_r} \otimes \mathcal{C}) = \Omega^1(\mathcal{O}_{X_r} \otimes \mathcal{C}). \]

**Proof.** Clear.

According to the general results about differentials (see, for example, [H], Chapter II, Ex. 8.3), the sheaf $\Omega^1(\mathcal{O}_X \otimes \mathcal{C} \overline{\mathcal{O}}_X)$ admits a canonical decomposition:

\[ \Omega^1(\mathcal{O}_X \otimes \mathcal{C} \overline{\mathcal{O}}_X) = \Omega^1(\mathcal{O}_X) \otimes \mathcal{C} \overline{\mathcal{O}}_X \otimes \mathcal{O}_X \otimes \mathcal{C} \Omega^1(\overline{\mathcal{O}}_X). \]
Let $\tilde{I}$ be an endomorphism of $\Omega^1(\mathcal{O}_X \otimes \mathcal{O}_X)$ which acts as a multiplication by $\sqrt{-1}$ on
\[ \Omega^1(\mathcal{O}_X) \otimes \mathcal{O}_X \subset \Omega^1(\mathcal{O}_X \otimes \mathcal{O}_X), \]
and as a multiplication by $-\sqrt{-1}$ on
\[ \mathcal{O}_X \otimes \Omega^1(\mathcal{O}_X) \subset \Omega^1(\mathcal{O}_X \otimes \mathcal{O}_X). \]

Let $I$ be the corresponding $\mathcal{O}_X \otimes \mathbb{C}$-linear endomorphism of $\Omega^1(\mathcal{O}_X \otimes \mathcal{O}_X) = \Omega^1(\mathcal{O}_X)$.

As easy check ensures that $I$ is real, that is, comes from the $\mathcal{O}_X \otimes \mathbb{C}$-linear endomorphism of $\Omega^1(\mathcal{O}_X \otimes \mathcal{O}_X)$.

Definition 2.3. Let $X, Y$ be complex analytic varieties, and

\[ f : X_R \longrightarrow Y_R \]

be a morphism of underlying real analytic spaces. Let $f^*\Omega^1_{Y_R} \xrightarrow{P} \Omega^1_{X_R}$ be the natural map of sheaves of differentials associated with $f$. Let

\[ I_X : \Omega^1_{X_R} \longrightarrow \Omega^1_{X_R}, \quad I_Y : \Omega^1_{Y_R} \longrightarrow \Omega^1_{Y_R} \]

be the complex structure operators, and

\[ f^*I_Y : f^*\Omega^1_{Y_R} \longrightarrow f^*\Omega^1_{Y_R} \]

be $\mathcal{O}_{X_R}$-linear automorphism of $f^*\Omega^1_{Y_R}$ defined as a pullback of $I_Y$. We say that $f$ commutes with the complex structure if

\[ P \circ f^*I_Y = I_X \circ P. \]

Theorem 2.4. Let $X, Y$ be complex analytic varieties, and

\[ f_R : X_R \longrightarrow Y_R \]

be a morphism of underlying real analytic spaces, which commutes with the complex structure. Then there exist a morphism $f : X \longrightarrow Y$ of complex analytic varieties, such that $f_R$ is its underlying morphism.

Proof. By Corollary 9.4, [V3], the map $f$, defined on the sets of points of $X$ and $Y$, is meromorphic; to prove Theorem 2.4, we need to show it is holomorphic. Let $\Gamma \subset X \times Y$ be the graph of $f$. Since $f$ is meromorphic, $\Gamma$ is a complex subvariety of $X \times Y$. It will suffice to show that the natural projections $\pi_1 : \Gamma \longrightarrow X$,
\[ \pi_2 : \Gamma \longrightarrow Y \] are isomorphisms. By [V3], Lemma 9.12, the morphisms \( \pi_i \) are flat. Since \( f_\mathbb{R} \) induces isomorphism of Zariski tangent spaces, same is true of \( \pi_i \). Thus, \( \pi_i \) are unramified. Therefore, the maps \( \pi_i \) are étale. Since they are one-to-one on points, \( \pi_i \) étale implies \( \pi_i \) is an isomorphism.

**Definition 2.5.** For a topological space \( X \), denote by \( C(X, \mathbb{R}) \) the sheaf of continuous \( \mathbb{R} \)-values functions on \( X \). For \( X \) a real analytic space, consider the evaluation map \( ev : \mathcal{O}_X \rightarrow C(X, \mathbb{R}) \). The kernel \( I \) of this map is an ideal sheaf in \( \mathcal{O}_X \). Consider the ringed space with the same topology as \( X \) and with structure sheaf \( \mathcal{O}_X/I \). This object is called the reduction of \( X \), denoted by \( X^r \). A real analytic space which coincides with its reduction is called a real analytic variety.

Consider the reduction morphism \( X^r \rightarrow X \) It is easy to define the functor \( r^* : \text{Sh}(X^r) \rightarrow \text{Sh}(X) \) of sheaves of \( \mathcal{O}_X \)-modules. Clearly, \( \Omega^1(X^r) = r^*\Omega^1(X) \). Thus, the almost complex structure \( I \), if given on \( X \), automatically carries over to \( X^r \). We obtain that a reduction of an almost complex space is an almost complex variety.

**Definition 2.6.** Let \( M \) be a real analytic space, and

\[ I : \Omega^1(\mathcal{O}_M) \longrightarrow \Omega^1(\mathcal{O}_M) \]

be an endomorphism satisfying \( I^2 = -1 \). Then \( I \) is called an almost complex structure on \( M \). If there exist a structure \( \mathcal{C} \) of complex variety on \( M \) such that \( I \) appears as the complex structure operator associated with \( \mathcal{C} \), we say that \( I \) is integrable. Theorem 2.4 implies that this complex structure is unique if it exists.

For a real analytic variety \( M^r \), and an automorphism \( I : \Omega^1(\mathcal{O}_{M^r}) \rightarrow \Omega^1(\mathcal{O}_{M^r}) \), we say that \( I \) is integrable if \( M^r \) appears as a reduction of some real analytic space with an integrable complex structure.

**Definition 2.7.** (Hypercomplex variety) Let \( M \) be a real analytic variety equipped with almost complex structures \( I, J \) and \( K \), such that \( I \circ J = -J \circ I = K \). Assume that for all \( a, b, c \in \mathbb{R} \), such that \( a^2 + b^2 + c^2 = 1 \), the almost complex structure \( aI + bJ + cK \) is integrable. Then \( M \) is called a hypercomplex variety.

**Claim 2.8.** Let \( M \) be a hyperkähler manifold. Then \( M \) is hypercomplex.

**Proof.** Let \( I, J \) be induced complex structures. We need to identify \( (M, I)_{\mathbb{R}} \) and \( (M, J)_{\mathbb{R}} \) in a natural way. These varieties are canonically identified as \( C^\infty \)-manifolds; we need only to show that this identification is real analytic. This is [V3], Proposition 6.5.

The following proposition will be used further on in this paper.

**Proposition 2.9.** Let \( M \) be a complex variety, \( x \in X \) a point, and \( Z_x M \subset T_x M \) be the reduction of the Zariski tangent cone to \( M \) in \( x \), considered as a closed subvariety of the Zariski tangent space \( T_x M \). Let \( Z_x M_{\mathbb{R}} \subset T_x M_{\mathbb{R}} \) be the Zariski tangent cone for the underlying real analytic space \( M_{\mathbb{R}} \). Then \( (Z_x M)_{\mathbb{R}} \subset (T_x M)_{\mathbb{R}} = T_x M_{\mathbb{R}} \) coinsides with \( Z_x M_{\mathbb{R}} \).
Proof. For each \( v \in T_x M \), the point \( v \) belongs to \( Z_x M \) if and only if there exist a real analytic path \( \gamma : [0, 1] \to M \), \( \gamma(0) = x \) satisfying \( \frac{d\gamma}{dt} = v \). The same holds true for \( Z_x M_{\mathbb{R}} \). Thus, \( v \in Z_x M \) if and only if \( v \in Z_x M_{\mathbb{R}} \). \( \square \)


In this section, we follow \([V3]\), Section 10. For more examples, motivations and reference, the reader is advised to check \([V3]\).

Definition 3.1. ([V-bun], Definition 6.5) Let \( M \) be a hypercomplex variety (Definition 2.7). The following data define a structure of hyperkähler variety on \( M \).

\( (i) \): For every \( x \in M \), we have an \( \mathbb{R} \)-linear symmetric positively defined bilinear form \( s_x : T_x M \times T_x M \to \mathbb{R} \) on the corresponding real Zariski tangent space.

\( (ii) \): For each triple of induced complex structures \( I, J, K \), such that \( I \circ J = K \), we have a holomorphic differential 2-form \( \Omega \in \Omega^2(M, I) \).

\( (iii) \): Fix a triple of induced complex structure \( I, J, K \), such that \( I \circ J = K \). Consider the corresponding differential 2-form \( \Omega \) of (ii). Let \( J : T_x M \to T_x M \) be an endomorphism of the real Zariski tangent spaces defined by \( J \), and \( \text{Re}\Omega \) the real part of \( \Omega \), considered as a bilinear form on \( T_x M \). Let \( r_x \) be a bilinear form \( r_x : T_x M \times T_x M \to \mathbb{R} \) defined by \( r_x(a, b) = -\text{Re}\Omega\big|_x(a, J(b)) \). Then \( r_x \) is equal to the form \( s_x \) of (i). In particular, \( r_x \) is independent from the choice of \( I, J, K \).

Remark 3.2.

(a): It is clear how to define a morphism of hyperkähler varieties.

(b): For \( M \) non-singular, Definition 3.1 is equivalent to the usual one (Definition 1.1). If \( M \) is non-singular, the form \( s_x \) becomes the usual Riemann form, and \( \Omega \) becomes the standard holomorphically symplectic form.

(c): It is easy to check the following. Let \( X \) be a hypercomplex subvariety of a hyperkähler variety \( M \). Then, restricting the forms \( s_x \) and \( \Omega \) to \( X \), we obtain a hyperkähler structure on \( X \). In particular, trianalytic subvarieties of hyperkähler manifolds are always hyperkähler, in the sense of Definition 3.1.

Caution. Not everything which looks hyperkähler satisfies the conditions of Definition 3.1. Take a quotient \( M/G \) of a hyperkähler manifold by an action of finite group \( G \), acting in accordance with hyperkähler structure. Then \( M/G \) is, generally speaking, not hyperkähler (see \([V3]\), Section 10 for details).

The following theorem, proven in [V-bun] (Theorem 6.3), gives a convenient way to construct examples of hyperkähler varieties.

Theorem 3.3. Let \( M \) be a compact hyperkähler manifold, \( I \) an induced complex structure and \( B \) a stable holomorphic bundle over \((M, I)\). Let \( \text{Def}(B) \) be
a reduction\(^1\) of the deformation space of stable holomorphic structures on \(B\). Assume that \(c_1(B), c_2(B)\) are \(SU(2)\)-invariant, with respect to the standard action of \(SU(2)\) on \(H^*(M)\). Then \(\text{Def}(B)\) has a natural structure of a hyperkähler variety.

4. Spaces with locally homogeneous singularities

**Definition 4.1.** (local rings with LHS) Let \(A\) be a local ring. Denote by \(\mathfrak{m}\) its maximal ideal. Let \(A_{gr}\) be the corresponding associated graded ring. Let \(\hat{A}, \hat{A}_{gr}\) be the \(\mathfrak{m}\)-adic completion of \(A, A_{gr}\). Let \((\hat{A})_{gr}, (\hat{A}_{gr})_{gr}\) be the associated graded rings, which are naturally isomorphic to \(A_{gr}\). We say that \(A\) has locally homogeneous singularities (LHS) if there exists an isomorphism \(\rho : \hat{A} \rightarrow \hat{A}_{gr}\) which induces the standard isomorphism \(i : (\hat{A})_{gr} \rightarrow (\hat{A}_{gr})_{gr}\) on associated graded rings.

**Definition 4.2.** (SLHS) Let \(X\) be a complex or real analytic space. Then \(X\) is called be a space with locally homogeneous singularities (SLHS) if for each \(x \in M\), the local ring \(O_x M\) has locally homogeneous singularities.

By system of coordinates on a complex space \(X\), defined in a neighbourhood \(U\) of \(x \in X\), we understand a closed embedding \(U \hookrightarrow B\) where \(B\) is an open subset of \(\mathbb{C}^n\). Clearly, a system of coordinates can be considered as a set of functions \(f_1, ..., f_n\) on \(U\). Then \(U \subset B\) is defined by a system of equations on \(f_1, ..., f_n\).

**Remark 4.3.** Let \(X\) be a complex space. Assume that for each \(x \in X\), there exist a system of coordinates \(f_1, ..., f_n\) in a neighbourhood \(U\) of \(x\), such that \(U \subset B\) is defined by a system of homogeneous polynomial equations. Then \(X\) is a space with locally homogeneous singularities. This explains the term.

**Claim 4.4.** Let \(X\) be a complex or real analytic space with locally homogeneous singularities, and \(X_r\) its reduction Then \(X_r\) is also a space with locally homogeneous singularities.

**Proof.** Clear.

**Lemma 4.5.** Let \(A_1, A_2\) be local rings over \(\mathbb{C}\), with \(A_i/\mathfrak{m}_i = \mathbb{C}\), where \(\mathfrak{m}_i\) is the maximal ideal of \(A_i\). Then \(A_1 \otimes_{\mathbb{C}} A_2\) is LHS if and only if \(A_1\) and \(A_2\) are LHS.

**Proof ("if" part).** Let \(\rho_1 : \hat{A}_1 \rightarrow (\hat{A}_1)_{gr}\) be the maps given by LHS condition. Consider the map
\[
(4.1) \quad \rho_1 \otimes \rho_2 : \hat{A}_1 \otimes_{\mathbb{C}} \hat{A}_2 \rightarrow (\hat{A}_1)_{gr} \otimes_{\mathbb{C}} (\hat{A}_2)_{gr}.
\]
Denote the functor of adic completions of local rings by \(B \rightarrow \hat{B}\). Clearly, \(\hat{A}_1 \otimes_{\mathbb{C}} \hat{A}_2 = A_1 \otimes_{\mathbb{C}} A_2\), and \((\hat{A}_1)_{gr} \otimes_{\mathbb{C}} (\hat{A}_2)_{gr} = (A_1)_{gr} \otimes_{\mathbb{C}} (A_2)_{gr}\). Plugging

\(^1\)The deformation space might have nilpotents in the structure sheaf. We take its reduction to avoid this.
these isomorphisms into the completion of both sides of (4.1), we obtain that a completion of $\rho_1 \otimes \rho_2$ provides an LHS map for $A_1 \otimes_C A_2$.

(“only if” part). Let $\rho : A_1 \otimes_C A_2 \rightarrow ((A_1) \otimes_C (A_2))_{gr}$ be the LHS map for $A_1 \otimes_C A_2$. There are natural maps $u : \hat{A}_1 \rightarrow A_1 \otimes_C A_2$ and $v : ((A_1) \otimes_C (A_2))_{gr} \rightarrow (\hat{A}_1)_{gr}$. The $u$ comes from the natural embedding $a \rightarrow a \otimes 1 \in A_1 \otimes_C A_2$ and $v$ from the natural surjection $a \otimes b \rightarrow a \otimes \pi(b) \in A_1 \otimes_C \mathbb{C}$, where $\pi : A_2 \rightarrow \mathbb{C}$ is the standard quotient map. It is clear that $u \circ v$ induces identity on the associated graded ring of $A_1$. Lemma 4.5 is proven. □

Proposition 4.6. Let $M$ be a complex variety, $M_\mathbb{R}$ the underlying real analytic space. Then $M_\mathbb{R}$ is a space with locally homogeneous singularities (SLHS) if and only if $M$ is SLHS.

Proof. By Claim 2.1, $(\mathcal{O}_x M_\mathbb{R}) \otimes \mathbb{C} = \mathcal{O}_x M \otimes \mathcal{O}_x M$. Thus, Proposition 4.6 is implied immediately by Lemma 4.5. □

Corollary 4.7. Let $M$ be a hyperkähler (or hypercomplex) variety, $I_1$, $I_2$ induced complex structures. Then $(M, I_1)$ is a space with locally homogeneous singularities if and only is $(M, I_2)$ is SLHS.

Proof. The real analytic space underlying $(M, I_1)$ coincides with that underlying $(M, I_2)$. Applying Proposition 4.6, we immediately obtain Corollary 4.7. □

Definition 4.8. Let $M$ be a hyperkähler variety. Then $M$ is called a space with locally homogeneous singularities (SLHS) if the underlying real analytic space is SLHS or, equivalently, for some induced complex structure $I$ the $(M, I)$ is SLHS.

Theorem 4.9. Let $M$ be a compact hyperkähler manifold, $I$ an induced complex structure and $B$ a stable holomorphic bundle over $(M, I)$. Assume that $c_1(B)$, $c_2(B)$ are $SU(2)$-invariant, with respect to the standard action of $SU(2)$ on $H^*(M)$. Let $\text{Def}(B)$ be a reduction of a deformation space of stable holomorphic structures on $B$, which is a hyperkähler variety by Theorem 3.3. Then $\text{Def}(B)$ is a space with locally homogeneous singularities (SLHS).

Proof. Let $x$ be a point of $\text{Def}(B)$, corresponding to a stable holomorphic bundle $B$. In [V-bun], Section 7, the neighbourhood $U$ of $x$ in $\text{Def}(B)$ was described explicitly as follows. We constructed a locally closed holomorphic embedding $U \hookrightarrow H^1(\text{End}(B))$. We proved that $v \in H^1(\text{End}(B))$ belongs to the image of $\varphi$ if and only if $v^2 = 0$. Here $v^2 \in H^2(\text{End}(B))$ is the square of $v$, taken with respect to the product $H^1(\text{End}(B)) \times H^1(\text{End}(B)) \rightarrow H^2(\text{End}(B))$. 

associated with the algebraic structure on \( \text{End}(B) \). Clearly, the relation \( v^2 = 0 \) is homogeneous. This relation defines a locally closed SLHS subspace \( Y \) of \( H^1(\text{End}(B)) \), such that \( \varphi(U) \) is its reduction. Applying Claim 4.4, we obtain that \( \varphi(U) \) is also a space with locally homogeneous singularities.

**Theorem 4.10.** Let \( M \) be a hyperkähler variety. Then \( M \) is a space with locally homogeneous singularities.

**Proof.** The paper [V-ne], which is a second part of the present paper, is fully taken by the proof of Theorem 4.10.

We don’t use Theorem 4.10 in the present paper.

5. Tangent cone of a hyperkähler variety

Let \( M \) be a hyperkähler variety, \( I \) an induced complex structure and \( Z_x(M, I) \) be a reduction of a Zariski tangent cone to \((M, I)\) in \( x \in M \). Consider \( Z_x(M, I) \) as a closed subvariety in the Zariski tangent space \( T_xM \). The space \( T_xM \) has a natural metric and quaternionic structure. This makes \( T_xM \) into a hyperkähler manifold, isomorphic to \( \mathbb{H}^n \).

**Theorem 5.1.** Under these assumptions, the following assertions hold:

(i): The subvariety \( Z_x(M, I) \subset T_xM \) is independent from the choice of induced complex structure \( I \).

(ii): Moreover, \( Z_x(M, I) \) is a trianalytic subvariety of \( T_xM \).

**Proof.** Theorem 5.1 (i) is implied by Proposition 2.9. By Theorem 5.1 (i), the Zariski tangent cone \( Z_x(M, I) \) is a hypercomplex subvariety of \( TM \). According to Remark 3.2 (c), this implies that \( Z_x(M) \) is hyperkähler. 

Further on, we denote the Zariski tangent cone to a hyperkähler variety by \( Z_xM \). The Zariski tangent cone is equipped with a natural hyperkähler structure.

The following theorem shows that the Zariski tangent cone \( Z_xM \subset T_xM \) is a union of planes \( L_i \subset T_xM \).

**Theorem 5.2.** Let \( M \) be a hyperkähler variety, \( I \) an induced complex structure and \( x \in M \) a point. Consider the reduction of the Zariski tangent cone (denoted by \( Z_xM \)) as a subvariety of the quaternionic space \( T_xM \). Let \( Z_x(M, I) = \cup L_i \) be the irreducible decomposition of the complex variety \( Z_x(M, I) \). Then

(i): The decomposition \( Z_x(M, I) = \cup L_i \) is independent from the choice of induced complex structure \( I \).

(ii): For every \( i \), the variety \( L_i \) is a linear subspace of \( T_xM \), invariant under quaternion action.

**Proof.** Let \( L_i \) be an irreducible component of \( Z_x(M, I) \), \( Z_x^{ns}(M, I) \) be the non-singular part of \( Z_x(M, I) \), and \( L_i^{ns} := Z_x^{ns}(M, I) \cap L_i \). Then \( L_i \) is a closure of \( L_i^{ns} \) in \( T_xM \). Clearly from Theorem 5.1, \( L_i^{ns}(M) \) is a hyperkähler submanifold in \( T_xM \). By Proposition 1.8, \( L_i^{ns} \) is totally geodesic. A totally geodesic submanifold
of a flat manifold is again flat. Therefore, $L_i^{ns}$ is an open subset of a linear
subspace $\bar{L}_i \subset T_x M$. Since $L_i^{ns}$ is a hyperkähler submanifold, $\bar{L}_i$ is invariant with
respect to quaternions. The closure $L_i$ of $L_i^{ns}$ is a complex analytic subvariety
of $T_x (M, I)$. Therefore, $\bar{L}_i = L_i$. This proves Theorem 5.2 (ii). From the
above argument, it is clear that $Z_x^{ns} (M, I) = \bigsqcup L_i^{ns}$ (disconnected sum). Taking
connected components of $Z_x^{ns} M$ for each induced complex structure, we obtain
the same decomposition $Z_x (M, I) = \cup L_i$, with $L_i$ being closure of connected
components. This proves Theorem 5.2 (ii).

**Corollary 5.3.** Let $M$ be a hyperkähler (or hypercomplex) variety, and $I$ an
induced complex structure. Assume that $M$ is a space with locally homogeneous
singularities. Then the normalization of $(M, I)$ is smooth.

**Proof.** The normalization of $Z_x M$ is smooth by Theorem 5.2. The normalization
is compatible with the adic completions ([M], Chapter 9, Proposition 24.E).
Therefore, the integral closure of the completion of $O_{Z_x M}$ is a regular ring.
Now, from the definition of locally homogeneous intersections, it follows that the
integral closure of $O_x M^*$ is also a regular ring, where $O_x M^*$ is an adic completion
of the local ring of holomorphic functions on $(M, I)$ in a neighbourhood of $x$.
Applying [M], Chapter 9, Proposition 24.E again, we obtain that the integral
closure of $O_x M$ is regular. This proves Corollary 5.3.

6. Desingularization of hyperkähler varieties

**Theorem 6.1.** Let $M$ be a hyperkähler or a hypercomplex variety. Assume that
$M$ is a space with locally homogeneous singularities, and $I$ an induced complex
structure. Let

\[
(\widetilde{M}, I) \xrightarrow{n} (M, I)
\]

be the normalization of $(M, I)$. Then $(\widetilde{M}, I)$ is smooth and has a natural hy-
perkähler (respectively, hypercomplex) structure $\mathcal{H}$, such that the associated map
$n : (\widetilde{M}, I) \rightarrow (M, I)$ agrees with $\mathcal{H}$. Moreover, the hyperkähler manifold
$\widetilde{M} := (M, I)$ is independent from the choice of induced complex structure $I$.

**Proof.** The variety $(\widetilde{M}, I)$ is smooth by Corollary 5.3. Let $x \in M$, and $U \subset M$
be a neighbourhood of $x$. Let $\mathfrak{R}_x (U)$ be the set of irreducible components of $U$
which contain $x$. There is a natural map $\tau : \mathfrak{R}_x (U) \rightarrow \text{Irr}(\text{Spec} O_x M^*)$, where
$Irr(\text{Spec} O_x M^*)$ is a set of irreducible components of $\text{Spec} O_x M^*$, where $O_x M^*$
is a completion of $O_x M$ in $x$. Since $O_x M$ is Henselian ([R], VII.4), there exist
a neighbourhood $U$ of $x$ such that $\tau : \mathfrak{R}_x (U) \rightarrow \text{Irr}(\text{Spec} O_x M^*)$ is a bijection.
Fix such an $U$. Since $M$ is a space locally with locally homogeneous singularities,
the irreducible decomposition of $U$ coincides with the irreducible decomposition
of the tangent cone $Z_x M$.

Let $\bigsqcup U_i \xrightarrow{u} U$ be the morphism mapping a disjoint union of irreducible
components of $U$ to $U$. By Theorem 5.2, the $x$-completion of $O_{U_i}$ is regular.
Shrinking $U_i$ if necessary, we may assume that $U_i$ is smooth. Then, the morphism $u$ coincides with the normalization of $U$.

For each variety $X$, we denote by $X^{\text{ns}} \subset X$ the set of non-singular points of $X$. Clearly, $u(U_i) \cap U^{\text{ns}}$ is a connected component of $U^{\text{ns}}$. Therefore, $u(U_i)$ is tri-analytic in $U$. By Remark 3.2 (c), $U_i$ has a natural hyperkähler structure, which agrees with the map $u$. This gives a hyperkähler structure on the normalization $\hat{U} := \bigsqcup U_i$. Gluing these hyperkähler structures, we obtain a hyperkähler structure $\mathcal{H}$ on the smooth manifold $\hat{M}$. Consider the normalization map $n : (\hat{M}, I) \longrightarrow M$, and let $\hat{M}^n := n^{-1}(M^{\text{ns}})$. Then, $n\big|_{\hat{M}^n} : \hat{M}^n \longrightarrow M^{\text{ns}}$ is a finite covering which is compatible with the hyperkähler structure. Thus, $\mathcal{H}\big|_{\hat{M}^n}$ can be obtained as a pullback from $M$. Clearly, a hyperkähler structure on a manifold is uniquely defined by its restriction to an open dense subset. We obtain that $\mathcal{H}$ is independent from the choice of $I$.

\begin{remark}
\text{The desingularization argument works well for hypercomplex varieties. The word “hyperkähler” in this article can be in most cases replaced by “hypercomplex”, because we never use the metric structure.}
\end{remark}

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