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ON HEEGAARD DIAGRAMS

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1. Introduction

Given a compact orientable surface Σ_g of genus g, a *marking* $m = \cup_{i=1}^{3g-3} m_i$ is a disjoint union of $3g-3$ pairwise non-parallel, essential unoriented simple loops in Σ_g. In [HT], Hatcher and Thurston introduced two elementary moves on markings and showed that any two markings are related by a finite sequence of these moves. The type I (resp. type II) move on m produces a new marking $m' = \cup_{j=1}^{3g-3} m'_j$ where $m'_j = m_j$, $j \neq i$ and $m'_i \cap m_i$ consists of one point (resp. two points of different intersection signs) as in Figure 1. Suppose m, n are two markings. We say that marking n contains a *wave* with respect to m, if there is an arc α in n so that $\alpha \cap m = \alpha \cap m_i = \partial\alpha$ for some i and α approaches its end points from the same side of m_i (see [VKF]). The goal of the paper is to show the following theorem.

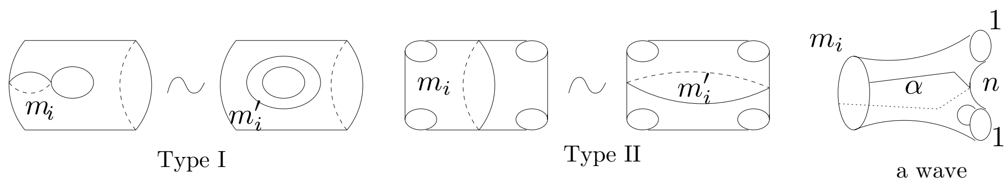

Figure 1

Theorem. *If m and n are two markings on Σ_g, then there is a marking m' obtained from m by a finite sequence of type II moves and isotopies so that n contains no waves with respect to m'. Furthermore, the new marking can be found algorithmically.*

Given a marking m in Σ_g, let $H(m)$ be the handlebody with boundary Σ_g obtained by attaching (thickened) 2-discs to Σ_g along m_i's and then attaching 3-balls. We say that two markings m, m' are *equivalent* (or determining the same handlebody structure) if the identity map from $\partial H(m)$ to $\partial H(m')$ extends to a homeomorphism from $H(m)$ to $H(m')$. Here is another way of characterizing the equivalence. Let $D(m)$ be the set of essential simple loops in Σ_g which bound discs in $H(m)$. Then m and m' are equivalent if and only if $D(m) = D(m')$ which is the same as: $m_i \in D(m')$, and $m'_i \in D(m)$ for all i. It follows that if

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m' is related to m by an elementary move of type II and an isotopy, denoted by $m \sim_{II} m'$, then $D(m) = D(m')$. One consequence of the theorem is the following.

Corollary 1. *If m, n are two markings which determine the same handlebody structure, then they are related by a finite sequence of type II moves.*

This is an analogy with the fact that any two ideal triangulations of a compact non-closed surface are related by two elementary moves (one of them is the diagonal switch). See Thurston [Th1], Harer [Har], Hatcher [Hat], and Mosher [Mo].

Given a simple loop s, let D_s be a Dehn twist on s.

Corollary 2. *If $\{s_1, ..., s_k\}$ is a collection of disjoint pairwise non-isotopic simple loops in the boundary of a handlebody V so that the composition $D_{s_1}^{a_1}...D_{s_k}^{a_k}$, $a_1...a_k \neq 0$, of Dehn twists extends to a homeomorphism of the handlebody, then each s_i is null homotopic in V.*

In [CG], Casson and Gordon introduced the *Heegaard diagram* as a pair of markings and established a nice criterion (the rectangle condition) on Heegaard diagrams so that the Heegaard splitting is irreducible. Hempel [He] has made more detailed study of 3-manifolds from the Heegaard diagram and the curve complex point of view. As a consequence of the theorem and corollary 1, one concludes that any 3-manifold has a special Heegaard diagram (n, m) so that n contains no waves. It is natural to ask if one can strengthen the result so that m contains no waves as well.

The organization of the paper is as follows. In section 2, we recall some basic notions. We prove the theorem in section 3. In section 4, we derive the corollaries and discuss some open questions.

2. Preliminaries

We work in the piecewise linear category. The interior and the boundary of a manifold M will be denoted by $\text{int}(M)$ and ∂M respectively. Given a finite set X, $|X|$ denotes the number of elements in X. A regular neighborhood of a submanifold c is denoted by $N(c)$. Regular neighborhoods are always assumed to be small.

Let $\Sigma_{g,r}$ be a compact orientable surface of genus g with r boundary components. A *curve system* in $\Sigma_{g,r}$ is a finite disjoint union of essential arcs and essential, non-boundary parallel simple loops. If c and c' are two isotopic submanifolds, we denote them by $c \cong c'$. The *geometric intersection number* $I(c, c')$ between two submanifolds c and c' is defined to be $\min\{|s \cap s'| : s \cong c$ and $s' \cong c'\}$.

If a, b are submanifolds intersecting transversely, a *wave* for a with respect to b is an arc α in a so that $\alpha \cap b = \alpha \cap b_i = \partial\alpha$ where b_i is a component of b and α approaches its end points from the same side of b_i. The set of all waves for a with respect b is denoted by $\mathrm{Wav}(a|b)$. We use $W(a|b)$ to denote $\min\{|\mathrm{Wav}(a'|b')| : a' \cong a, b' \cong b\}$. Note that if $|a \cap b| = I(a,b)$, then $W(a|b) = |\mathrm{Wav}(a|b)|$.

The following result is well known (see [Hat1]). See Figure 2.

Lemma 1. *Given $a_1, a_2, a_3 \in \mathbf{Z}_{\geq 0}$ so that $a_1 + a_2 + a_3$ is even, there exists a curve system c unique up to isotopy in $\Sigma_{0,3}$ so that $|c \cap b_i| = a_i$ where $\partial\Sigma_{0,3} = b_1 \cup b_2 \cup b_3$. Furthermore, $W(c|\partial\Sigma_{0,3}) = 0$ if and only if $a_i + a_j \geq a_k$ for $i \neq j \neq k \neq i$.*

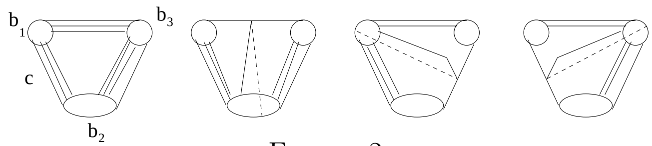

FIGURE 2

We say that two disjoint 0-spheres in S^1 are *unlinked* if they bound two disjoint intervals. As a consequence of lemma 1, we have the following result.

Lemma 2.

(a) *If a, b are curve systems in $\Sigma_{1,1}$ so that $\partial b = \emptyset$, then $W(a|b) = 0$.*
(b) *If a is an essential arc in $\Sigma_{0,4}$ with $a \in \mathrm{Wav}(a|\partial\Sigma_{0,4})$ and b is a curve system with $\partial b = \emptyset$ and $I(a,b) > 0$, then $W(a|b) > 0$.*
(c) *If a, b are essential arcs intersecting transversely in $\Sigma_{0,3}$ so that ∂a, ∂b lie in the same boundary component of $\Sigma_{0,3}$ and are unlinked in the component, then* $\mathrm{Wav}(a|b) \neq \emptyset$.

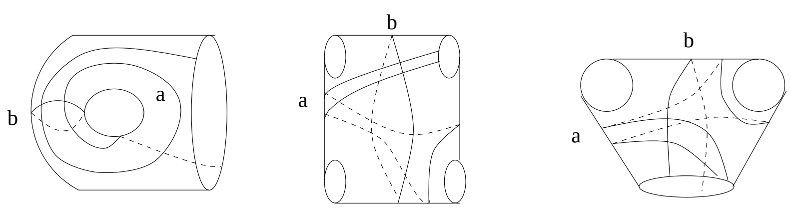

FIGURE 3

Indeed, to see part (a), we may assume that $|a \cap b| = I(a,b)$ after an isotopy. Now the result follows from lemma 1 by cutting the surface open along b. Parts (b) and (c) follow from the Jordan curve theorem and the outer most arc argument. See Figure 3.

3. Proof of the theorem

We prove the theorem by induction on the complexity $(W(n|m), I(n,m))$ in the lexicographic order.

Suppose that $W(n|m) > 0$. We isotopy m, n so that $|m \cap n| = I(m,n)$ and if $m_i \cong n_j$, then $m_i \subset \mathrm{int}(N(n))$, $n_j \subset \mathrm{int}(N(m))$. Let $\Sigma_g - \mathrm{int}(N(m)) = \cup_{i=1}^{2g-2} P_i$ where $P_i \cong \Sigma_{0,3}$. By the choice of $N(n)$ and $N(m)$, each intersection $n \cap P_i$ is a curve system in P_i. Take a wave $e \in \mathrm{Wav}(n|m)$, say, $e \subset n_1$ and $\partial e \subset m_1$.

We first note that $N(m_1)$ intersects two distinct 3-holed spheres, say P_1 and P_2. Indeed, if otherwise, say, $\partial N(m_1) \subset \partial P_1$, then $N(m_1) \cup P_1 = \Sigma_{1,1}$ and $n_1 \cap \Sigma_{1,1}$ is a curve system in $\Sigma_{1,1}$. Let e^* be the component of $n_1 \cap \Sigma_{1,1}$ which contains the wave e. Then $|e^* \cap m_1| = I(e^*, m_1)$ but $W(e^*|m_1) > 0$. This contradicts lemma 2(a).

Now let $\Sigma_{0,4} = P_1 \cup N(m_1) \cup P_2$ with $e \subset P_1 \cup N(m_1)$, and let e^* be the component of the curve system $n \cap \Sigma_{0,4}$ which contains the wave e as in Figure 4.

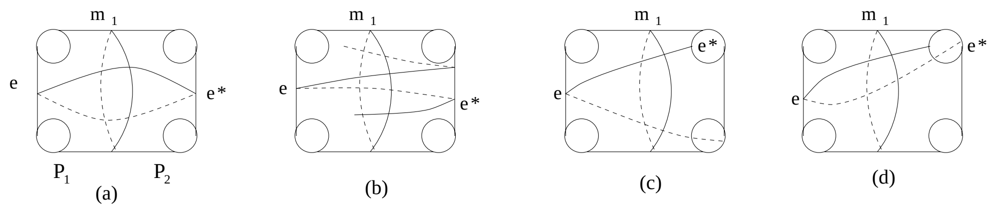

FIGURE 4

There are four cases we need to consider: case 1, $e^* = n_1$ and $|e^* \cap m_1| = 2$; case 2, $|e^* \cap m_1| > 2$; case 3, $|e^* \cap m_1| = 2$ and ∂e^* does not lie in a boundary component of $\Sigma_{0,4}$; case 4, $|e^* \cap m_1| = 2$ and ∂e^* lies in a boundary component of $\Sigma_{0,4}$. See Figure 4.

Case 1. $e^* = n_1$ and $|e^* \cap m_1| = 2$. Then n_1 intersects m_1 at two points of different signs and $n_1 \cap m_i = \emptyset$ for $i \geq 2$. Let $m'_1 = n_1$ and $m'_i = m_i$, $i \geq 2$. Then $m' = \cup m'_i$ is obtained from m by a type II move.

We claim that $W(n|m') < W(n|m)$. Take $b \in \mathrm{Wav}(n|m')$. If $b \cap \Sigma_{0,4} = \emptyset$, then $b \in \mathrm{Wav}(n|m)$. If $b \cap \Sigma_{0,4} \neq \emptyset$ and $\partial b \subset m_i$ for $i \geq 2$, then $b \cap m_1 \neq \emptyset$ (due to $b \cap m'_1 = \emptyset$). Thus by lemma 2(b) applied to $b \cap \Sigma_{0,4}$ and m_1 in $\Sigma_{0,4}$, we conclude that b contains a wave $b' \in \mathrm{Wav}(n|m)$ so that $\partial b' \subset m_1$. This produces an injective map from $\mathrm{Wav}(n|m')$ to $\mathrm{Wav}(n|m)$ whose image misses the wave e. Thus $W(n|m') < W(n|m)$. Note that we also have $W(m'|n) \leq W(m|n)$ in this case (see remark at the end of the proof).

Case 2. $|e^* \cap m_1| \geq 3$. Take three intersection points x, y, z in $e^* \cap m_1$ so that x, y (and y, z) are adjacent intersection points in e^*. Then the arcs xy and yz (in e^*) are in $\mathrm{Wav}(n|m)$ as in Figure 5(a).

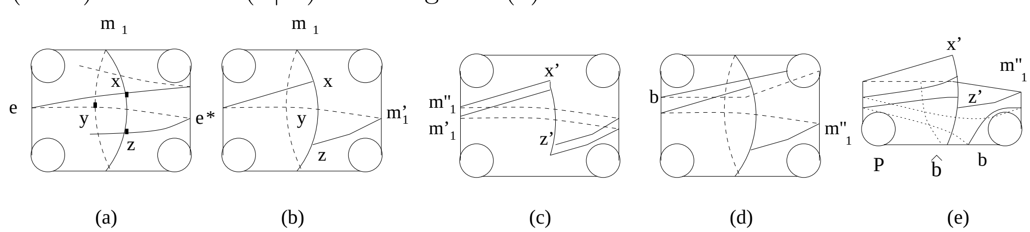

FIGURE 5

Let zx be the arc in m_1 so that $y \notin zx$. Let $m'_1 = xy \cup yx \cup zx$ and $m'_i = m_i$ for $i \geq 2$. Then $m' = \cup m'_i$ is obtained from m by a type II move and an isotopy. We claim that $W(n|m') < W(n|m)$. To see this, we choose a marking $m'' \cong m'$ so that $|m'' \cap n| = I(m'', n)$ as follows. Let $m''_i = m_i$ for $i \geq 2$ and $m''_1 = x'y' \cup y'z' \cup z'x'$ where $x'y' \cup y'z'$ is a parallel copy of $xy \cup yx$ $(x'y' \cup y'z' \subset \partial N(m'_1))$ and $z'x'$ is a slight translation of zx along m_1 as in Figure 5(c). Now take $b \in \mathrm{Wav}(n|m'')$. If $b \cap \Sigma_{0,4} = \emptyset$, then $b \in \mathrm{Wav}(n|m)$. If $b \cap \Sigma_{0,4} \neq \emptyset$ and $\partial b \subset m_i$ for $i \geq 2$ (see Figure 5(d)), then $b \cap m_1 \neq \emptyset$ (due to $b \cap m''_1 = \emptyset$). Thus, by lemma 2(b) applied to $b \cap \Sigma_{0,4}$ and m_1 in $\Sigma_{0,4}$, we conclude that b contains a wave b' in $\mathrm{Wav}(n|m)$ so that $\partial b' \subset m_1$. Finally, if $\partial b \subset m''_1$, then $\partial b \subset z'x'$. Let P be the 3-holed sphere in $\Sigma_{0,4}$ obtained by cutting $\Sigma_{0,4}$ open along m''_1 so that $b \subset P$, and let $\hat{b} = m_1 \cap P - z'x'$. Both b and $\hat{b}$ are waves for ∂P. If $\hat{b} \cap b = \emptyset$, then $b \in \mathrm{Wav}(n|m)$. If $\hat{b} \cap b \neq \emptyset$ (see Figure 5(e)), then by lemma 2(c), b contains a wave $b' \in \mathrm{Wav}(n|m)$ with $\partial b' \subset m_1$. Since the waves xy and yx in $\mathrm{Wav}(n|m)$ are eliminated, we have $W(n|m'') < W(n|m)$. Note that we have $W(m'|n) \leq W(m|n)$ in this case.

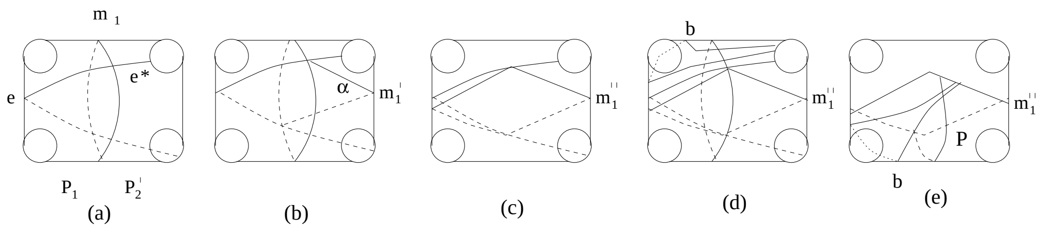

FIGURE 6

Case 3. $|e^* \cap m_1| = 2$ and ∂e^* lies in two boundary components, as in Figure 6. Let P'_2 be the 3-holed sphere in $\Sigma_{0,4}$ bounded by m_1 so that $P_2 \subset P'_2$. Let α be an essential arc in P'_2 with end points ∂e so that $\mathrm{int}(\alpha) \cap e^* = \emptyset$ as in Figure 6(b) (there are two choices of such arcs up to isotopy). Let $m'_1 = e \cup \alpha$ and $m'_i = m_i$ for $i \geq 2$. Then the marking $m' = \cup m'_i$ is obtained from m by a type II move. We claim that $W(n|m') < W(n|m)$. To see this we take $m'' \cong m'$ as follows $m''_i = m_i$ for $i \geq 2$ and m''_1 is as in Figure 6(c) so that $m''_1 \cap m'_1 = \alpha$ and $m''_1 \cap e^* = \partial e$. Take $b \in \mathrm{Wav}(n|m'')$. If $b \cap \Sigma_{0,4} = \emptyset$, then $b \in \mathrm{Wav}(n|m)$. If $b \cap \Sigma_{0,4} \neq \emptyset$ and $\partial b \subset m_i$ for $i \geq 2$ as in Figure 6(d), then b intersects m_1 since b is disjoint from m''_1. Thus by lemma 2(b) applied to $b \cap \Sigma_{0,4}$ and m_1 in $\Sigma_{0,4}$, we conclude that b contains a wave $b' \in \mathrm{Wav}(n|m)$ so that $\partial b' \subset m_1$. If $\partial b \subset m''_1$, then ∂b is in α. We consider the three-holed sphere P bounded by m''_1 so that $b \subset P$. Then by lemma 2(c) applied to b and $m_1 \cap P$ in P, since $b \cap m_1 \neq \emptyset$, we conclude that b contains a wave $b' \in \mathrm{Wav}(n|m)$ with $\partial b' \subset m_1$.

Case 4. $|e^* \cap m_1| = 2$ and ∂e^* lies in a boundary component b_2 of $\Sigma_{0,4}$. The end points of e^* decomposes b_2 into two arcs. One of the arc, denoted by α, makes the simple loop $e^* \cup \alpha$ non-boundary parallel in $\Sigma_{0,4}$. Let $m'_1 = e^* \cup \alpha$ and $m'_i = m_i$ for $i \geq 2$. Then the marking $m' = \cup m'_i$ is obtained from m by an isotopy and a type II move. We claim that $W(n|m') \leq W(n|m)$ and $I(n, m'_1) \leq I(n, m_2) - 2$ where m_2 is the component of m isotopic to the boundary component b_2.

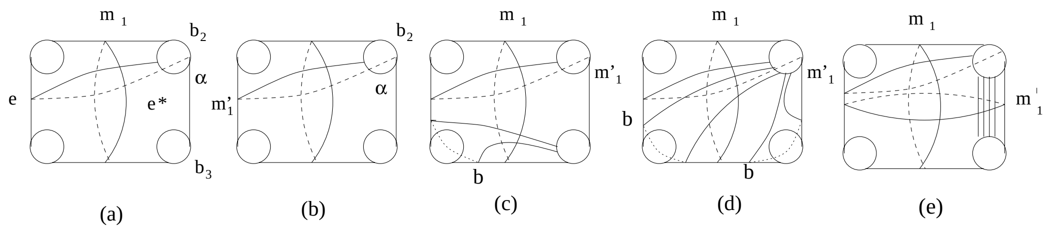

FIGURE 7

To see the claim, take $b \in \mathrm{Wav}(n|m')$. If $b \cap \Sigma_{0,4} = \emptyset$, then $b \in \mathrm{Wav}(n|m)$. If $b \cap \Sigma_{0,4} \neq \emptyset$ and $\partial b \subset m_i$ for $i \geq 2$, then $b \cap m_1 \neq \emptyset$ (due to $b \cap m_1' = \emptyset$). By lemma 2(b) applied to $b \cap \Sigma_{0,4}$ and m_1 in $\Sigma_{0,4}$, b contains a wave $b' \in \mathrm{Wav}(n|m)$ with $\partial b' \subset m_1$ (see Figure 7(c)). Finally, if $\partial b \subset m_1'$, then $\partial b \subset \alpha$. Now if $b \cap m_1 = \emptyset$, then b gives rise to a wave b' in $\mathrm{Wav}(n|m)$ so that $b \subset b'$ and $b' \subset b \cup N(m_2)$. If $b \cap m_1 \neq \emptyset$ and $b \neq e^*$, by lemma 2(c) (applied to the three-holed sphere P bounded by m_1' which contains b), we conclude that b contains a wave $b' \in \mathrm{Wav}(n|m)$ with $\partial b' \subset m_1$. Thus, $W(n|m') \leq W(n|m)$. The second statement $I(n, m_1') \leq I(n, m_2) - 2$ follows from the construction.

Since the intersection number $I(n, m)$ may increase during this process, the complexity is not reduced. However, the wave e^* gives rise to a wave e' for m' with $\partial e' \subset m_2$ and $e^* \subset e'$. We shall proceed at this new wave e' instead of any other waves in n. If any of the previous three cases occur for e', the complexity is reduced and we finish the proof by induction. Therefore, it remains to show that case 4 cannot occur indefinitely. To prove this, let us exam the change in the N-tuple of non-negative even integers $(I(n, m_1), ..., I(n, m_N)) = (a_1, ..., a_N)$, where $N = 3g - 3$. The algorithm states that at the first step, we replace one coordinate of the N-tuple $(a_1, ..., a_N)$, say a_{i_0}, by $a_{i_0}^{(1)}$, where $0 \leq a_{i_0}^{(1)} \leq a_{i_1} - 2$ for some $i_1 \neq i_0$. Let the new N-tuple be $(a_1^{(1)}, ..., a_N^{(1)})$. Now we replace $a_{i_1}^{(1)}$ by $a_{i_1}^{(2)}$ where $0 \leq a_{i_1}^{(2)} \leq a_{i_2}^{(1)} - 2$ for some $i_2 \neq i_1$. Suppose in the k-th step we obtain the N-tuple $(a_1^{(k)}, ..., a_N^{(k)})$ where $0 \leq a_{i_{k-1}}^{(k)} \leq a_{i_k}^{(k)} - 2$. Then in the $(k+1)-$th step, we replace $a_{i_k}^{(k)}$ by $a_{i_k}^{(k+1)}$ where $0 \leq a_{i_k}^{(k+1)} \leq a_{i_{k+1}}^{(k)} - 2$ for some $i_{k+1} \neq i_k$. We claim that after at most $1/2(\Sigma_{i=1}^N a_i +\max(a_1, ..., a_N))$ steps, the N-tuple is the zero vector $(0, ..., 0)$. Indeed, first of all $0 \leq a_i^{(k)} \leq \max(a_1, ..., a_N) - 2$ for all k and i by the construction. Furthermore,

$$\begin{aligned}\Sigma_{i=1}^N a_i^{(k+1)} &= \Sigma_{i=1}^N a_i^{(k)} - a_{i_k}^{(k)} + a_{i_k}^{(k+1)} \\ &\leq \Sigma_{i=1}^N a_i^{(k)} + a_{i_{k+1}}^{(k)} - a_{i_k}^{(k)} - 2 \\ &= \Sigma_{i=1}^N a_i^{(k)} + a_{i_{k+1}}^{(k+1)} - a_{i_k}^{(k)} - 2.\end{aligned}$$

Now consider the sum of all these inequalities from 0 to $k-1$. We obtain,

$$\begin{aligned}\Sigma_{i=1}^N a_i^{(k)} &\leq \Sigma_{i=1}^N a_i + a_{i_k}^{(k)} - 2k \\ &\leq \Sigma_{i=1}^N a_i + \max(a_1, ..., a_N) - 2k - 2.\end{aligned}$$

Thus the conclusion follows. This shows that after max $(a_1, ..., a_N)$ steps, the complexity is reduced. The proof of the theorem is complete by induction. □

Remarks. 1. In cases 3 and 4, it can be shown using lemma 2 that we have the following estimate of the number of waves: $W(m'|n) \le 2W(m|n) + 1$. However, this estimate is not good enough to eliminate all waves $\mathrm{Wav}(n|m)$ and $\mathrm{Wav}(m|n)$ using the type II moves. (The estimate $W(m'|n) \le 2W(m|n)$ will work).

2. H. Masur ([Ma], lemma 1.1) showed that given an essential simple loop c, then $c \in D(m)$ if and only if for each m' with $D(m') = D(m)$, either c is in $D(m)$ or $\mathrm{Wav}(c|m') \ne \emptyset$. This also follows from the proof above.

4. Proof of the corollaries and some questions

The proof of the corollaries is based on the following simple lemma.

Lemma 3. *Suppose c is an essential simple loop in Σ_g which is null homotopic in the handlebody $H(m)$ associated to a marking m and* $\mathrm{Wav}(c|m) = \emptyset$*. Then c is isotopic to a component of m.*

Proof. Since $\mathrm{Wav}(c|m) = \emptyset$, there are no bi-gons in $c \cup m$. Thus, $|c \cap m| = I(c, m)$. We claim that $|c \cap m| = 0$. If not, consider the intersections of the meridian discs bounded by c and m in the handlebody. The outer most arc of the intersection in the disc bounded by c gives rise to a wave in $\mathrm{Wav}(c|m)$ which contradicts the assumption. Now the only essential simple loop which is disjoint from m is isotopic to the components of m. Thus the result follows. □

Corollary 1 follows directly from lemma 3 and the theorem since if two markings n, m are equivalent and $\mathrm{Wav}(n|m) = \emptyset$, then $n \cong m$.

Proof of Corollary 2. By extending the set of simple loops $\{s_1, ..., s_k\}$ to a marking n and using the theorem, we construct a marking m which determines the handlebody V so that $\mathrm{Wav}(s_i|m) = \emptyset$ for each i. We claim that each s_i is isotopic to a component of m. To see this, consider the image of m under the composition $f = D_{s_1}^{a_1} ... D_{s_k}^{a_k}$. By the definition of the Dehn twist, the marking $f(m)$ has no waves with respect to m. But $f(m)$ is equivalent to m by the assumption on f. Thus by lemma 3, we conclude that $f(m)$ is isotopic to m. This is possible only if each s_i is isotopic to a component of m. □

Given a compact orientable 3-manifold M, a *Heegaard splitting* is a triple (F, V_1, V_2) where F is an embedded orientable surface of genus g in M and V_1 and V_2 are two handlebodies in M so that $V_1 \cap V_2 = F$ and $V_1 \cup V_2 = M$. Two Heegaard splittings (F, V_1, V_2) and (F', V_1', V_2') of M are *equivalent* if there is an orientation preserving homeomorphism $h : M \to M$ so that $h(V_i) = V_i'$ $(i = 1, 2)$. One translates the above setup into 2-dimensional setting as follows. Let $\phi : \Sigma_g \to F$ be a homeomorphism. Then the two handlebodies V_1 and V_2 are homeomorphic to $H(m^1)$ and $H(m^2)$ for some markings m^1 and m^2 on Σ_g by homeomorphisms $H(m^i) \to V_i$ which extends ϕ in the boundary (i=1,2). Thus a Heegaard splitting of a 3-manifold can be described as a pair of markings (m^1, m^2) on Σ_g. Now let $\phi' : \Sigma_g \to F'$ be a homeomorphism for the Heegaard surface F' and let (n^1, n^2) be a pair of markings on Σ_g corresponding to the Heegaard splitting (F', V_1', V_2'). Then the equivalence relation between (F, V_1, V_2) and (F', V_1', V_2') states that there is a self-homeomorphism f of Σ_g $(f = \phi\prime^{-1} h \phi)$ so that $D(f(m^i)) = D(n^i)$ $(i = 1, 2)$. This motivates the following definition.

Definition.

(a) Two Heegaard diagrams (m^1, m^2) and (n^1, n^2) are related by an elementary move if either $m^1 \sim_{II} n^1$, $m^2 = n^2$ or $m^1 = n^1$, and $m^2 \sim_{II} n^2$.

(b) Two Heegaard diagrams (m^1, m^2) and (n^1, n^2) are homeomorphic (resp. isotopic) if there is a homeomorphism h (resp. isotopy) of the surface so that $h(m^i) = n^i$, i=1,2 (h may not preserve the order of the indices of m^i_j).

(c) Two Heegaard diagrams are equivalent if they are related by a finite sequence of elementary moves and a homeomorphism.

One obvious question is whether there is an algorithm to decide if two Heegaard diagrams are equivalent. Solutions of this question has applications to the homeomorphism problem for irreducible non-Haken 3-manifolds in view of the work of Rubinstein and Scharlemann ([RS]) on the stabilization problem for Heegaard splittings.

Given a marking m and a simple loop c, there is an algorithm to decide if $c \in D(m)$ (Whitehead [Wh], see also [ER]). Thus, given two Heegaard diagrams, there is an algorithm to decide if they are related by elementary moves. On the other hand, given a marking m on Σ_g, Dehn-Thurston's theory gives a parametrization of the space of isotopy classes of curve systems $\mathrm{CS}(\Sigma_g)$ on Σ_g using the marking ([De], [FLP], [Th]). Thus each Heegaard diagram (m^1, m^2) has a Dehn-Thurston coordinate $(a_1, t_1, ..., a_{3g-3}, t_{3g-3})$ where $a_i = I(m^2, m^1_i)$ and t_i is the twisting number of m^2 at m^1_i. As a consequence, there is an algorithm to check if two Heegaard diagrams are homeomorphic. Also Dehn-Thurston theory gives an efficient way to list all isotopy classes of Heegaard diagrams.

Acknowledgments

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