RADIAL VARIATION OF BLOCH FUNCTIONS

PETER W. JONES AND PAUL F. X. MÜLLER

0. The result

In 1971 J. M. Anderson [A] conjectured that for any conformal map \( \varphi \) in the unit disc there exists \( \beta, 0 \leq \beta \leq 2\pi \) such that

\[
\int_0^1 |\varphi''(re^{i\beta})|dr < \infty.
\]

More recently this problem has been posed in the works of N. Makarov [M], Ch. Pommerenke [P] and D. Gnuschke - Ch. Pommerenke [G-P]. The purpose of the present note is to prove Anderson’s conjecture. This will be done by showing the following theorem about the associated Bloch function \( b = \log |\varphi'| \).

**Theorem 1.** There exists \( \beta, 0 \leq \beta \leq 2\pi \) such that

\[
b(re^{i\beta}) \leq -\delta \int_0^r |\nabla b(\rho e^{i\beta})|d\rho + \frac{1}{\delta}, \quad \text{for } 0 < r < 1,
\]

where \( \delta > 0 \) is independent of \( r < 1 \).

The proof of Theorem 1 is given in section 3 where we also discuss how Anderson’s conjecture follows.

1. Preliminary inequalities

In this section we recall three estimates due to J. Bourgain, Ch. Pommerenke and A. Beurling respectively. In section 2 the construction of stopping time Lipschitz domains is based on Pommerenke’s inequality. In section 3 the selection of good directions \( e^{i\beta} \) is based on the result of J. Bourgain and estimates for harmonic measure due to A. Beurling.

We first discuss Bourgain’s inequality from [B]. For \( e^{i\alpha} \in \mathbb{T} \) we let \( \Gamma_\alpha \) be the collection of curves \( \gamma \) which admit the following parametrization. For \( 0 < r < 1 \),

\[
\gamma(r) = re^{i\alpha}e^{i\theta(r)} \quad \text{where } |\theta(r)| < C(1-r) \text{ and } |\theta'(r)| < C.
\]

We fix a non-negative, harmonic function \( h \) in \( \mathbb{D} \) and we let \( K \) be an interval in \( \mathbb{T} \). Then the following result was proven in [B].
Theorem 2. There exists $e^{i\alpha} \in K$ so that for each curve $\gamma \in \Gamma_\alpha$,
\[
\delta_0 \int_\gamma |\nabla h(\zeta)| d\zeta \leq h(0),
\]
where $\delta_0 > 0$ depends only on $C$ and $|K|$.

The Bloch space $B$ consists of those harmonic functions $b : \mathbb{D} \to \mathbb{R}$ for which $\|b\|_B = \sup_{z \in \mathbb{D}} |\nabla b(z)|(1 - |z|)$ is finite. Pommerenke's theorem (see [P, p. 78]) is the following.

Theorem 3. Let $b : \mathbb{D} \to \mathbb{R}$ be in the Bloch space $B$ and $\|b\|_B \leq 1$. Let $I \subset \mathbb{T}$ be an interval. Then there exists $e^{i\alpha} \in I$ so that for each $z \in \{re^{i\alpha} : 0 < r < 1\}$, the estimate $|b(z) - b(0)| \leq 22|I|^{-1}$ holds.

For a conformal map $\varphi : \mathbb{D} \to \Omega$ the function $b = \log |\varphi'|$ belongs to the Bloch space $B$ with Bloch norm $\leq 6$. This is a consequence of classical distortion theorems. (See [P].) Bloch functions are related to $|\varphi''|$ as follows. Let $g = \log \varphi'$, $g' = \varphi''/\varphi'$, hence $|\varphi''| = |g'||\varphi'|$. Let $b = \log |\varphi'|$, then $b$ is the real part of $g$. Hence by the Cauchy-Riemann equations $2|g'| = |\nabla b|$, and
\[2|\varphi''| = |\nabla b||\varphi'| = |\nabla b|e^b.
\]
This identity provides a link between estimates for the variation of Bloch functions and estimates for the $L^1$ norm of $\varphi''$. This has been exploited in [G-P].

Next we recall a minorization for harmonic measure due to A. Beurling. Fix $e \in \mathbb{D}$ and $0 < \delta < 1$. Then we define the Stolz angle $C(e, \delta)$ to be the convex hull of $\{z \in \mathbb{D} : |z| < (1 - \delta)|e|\}$ and $e$. If $1/4 < \delta < 1$, then we write simply $C(e)$ for $C(e, \delta)$. Let $W \subset \mathbb{D}$ be a Lipschitz domain, $e \in \partial W$, and suppose that the cone $C(e, \delta/2)$ is contained in $W$. Let $D$ be a connected subset of $\partial W$, so that $\text{dist}(D, e) < rC_1$ and $\text{diam}(D) > rC_2$. Then the following estimate holds.

Theorem 4. For each $z \in C(e, \delta)$, with $|e - z| < r$, the harmonic measure satisfies $\omega(z, D, W) > \eta_0$, where $\eta_0$ depends only on $C_1, C_2$, and $\delta$.

Below we will be concerned with the question whether a given curve in the unit disc remains in a fixed Stolz angle or not. To decide this the following criterion which is a folk theorem uses lower estimates for harmonic measure. We fix $e^{i\alpha} \in \mathbb{T}$, and let $I_\alpha = \{e^{i\theta} : \alpha - \pi/2 < \theta < \alpha \pi/2\}$ and $J_\alpha = \{e^{i\theta} : \alpha - \pi/2 < \theta < \alpha\}$. For $z \in \mathbb{D}$ we denote by $\omega(z, I)$ the harmonic measure of $I$ with respect to $\mathbb{D}$ evaluated at $z \in \mathbb{D}$.

Theorem 5. For a path $\Gamma$ in $\mathbb{D}$ the following conditions are equivalent.

(i) There exists $\delta$ such that $\Gamma \subset C(e^{i\alpha}, \delta)$.

(ii) There exists $\eta > 0$ such that for each $z \in \Gamma$, there hold the lower estimates for harmonic measure $\omega(z, I_\alpha) > \eta$ and $\omega(z, J_\alpha) > \eta$. 
We combine this criterion and Beurling’s estimates. Let \( f : \mathbb{D} \to W \) be the conformal map from the unit disc to the Lipschitz domain \( W \). Fix \( e \in \partial W \), and let \( e^{i\alpha} = f^{-1}(e) \). Suppose that \( C(e, \delta/2) \subset W \), and let \( \Gamma \) be a path in \( C(e, \delta) \). Then the following holds.

**Theorem 6.** There exists \( \delta_0 > 0 \), depending only on \( \delta \), such that \( f^{-1}(\Gamma) \) is contained in \( C(e^{i\alpha}, \delta_0) \).

## 2. Stopping time Lipschitz domains

In this section we define the stopping time Lipschitz domain \( W(z_0) \), and we collect some of its basic properties. We let \( b : \mathbb{D} \to \mathbb{R} \) be a Bloch function, and we fix it throughout this section. We also fix \( z \in \mathbb{D} \) with \( |z| > 15/16 \). First we construct an auxiliary domain \( V(z) \).

Let \( I = \{ \zeta \in \mathbb{T} : |z - \zeta| \leq 8(1 - |z|) \} \). The intervals \( I_1, I_2 \) have length \( 1 - |z| \), they are attached to the left respectively right endpoint of \( I \). Let \( r_1 = 2|z| - 1 \), and let \( S_0 = \{ w \in \mathbb{D} : |w| = r_1 \) and \( |w - z| \leq 1 - |z| \} \). The left respectively right endpoint of \( S_0 \) are \( s_1 \) resp. \( s_0 \). By Theorem 3 there are line segments \( S_i \) connecting \( s_i \) to \( I_i \) such that

\[
|b(w) - b(s_i)| \leq 25||b||/\omega(s_i, I_i),
\]

whenever \( w \in S_i \), \( i \in \{1,2\} \). We let \( V(z) \) be the domain in \( \mathbb{D} \) which is bounded by \( S_0 \cup S_1 \cup S_2 \).

The domain \( V(z) \) satisfies

\[
\sup |b(w) - b(z)| \leq ||b||_B/\omega_0,
\]

where the supremum is taken over \( S_0 \cup S_1 \cup S_2 \), and where \( \omega_0 = \min\{\omega(s_1, I_1), \omega(s_2, I_2)\}/30 \). The boundary of \( V(z) \) intersects \( \mathbb{T} \) in an interval \( J \). For the harmonic measure of \( J \) we have the lower estimate \( w(z,J,V(z)) \geq 1/3 \). Moreover we observe the following.

\[(2.1) \quad \text{The angle formed by} \ S_1 \text{ and} \ J, \text{ resp.} \ S_2 \text{ and} \ J \text{ is less than} \ \pi/5.\]

Next fix \( z_0 \in \mathbb{D} \) with \( |z_0| \geq 15/16 \), and \( M \in \mathbb{N} \) large enough. We now turn to the construction of \( W(z_0) \). Using a stopping time \( W(z_0) \) is obtained as a subdomain of \( V(z_0) \). For an interval \( I \subseteq \mathbb{T} \) we let \( T(I) = \{ w \in \mathbb{D} : w/|w| \in I \text{ and } |I|/2 < 1 - |w| < |I| \} \). Now we let \( \{I_i : i \in \mathbb{N}\} \) be the collection of maximal dyadic intervals with the property that there exists \( z_i \in T(I_i) \) with

\[(2.2) \quad b(z_i) - b(z_0) \leq -M.\]

The stopping time Lipschitz domain is defined as

\[
W(z_0) = V(z_0) \setminus \bigcup_{i=1}^{\infty} V(z_i).
\]
For \( z_0 = 0 \) we define the points \( z_i \) using the stopping time condition (2.2). By (2.4), each point \( z_i \) satisfying (2.2) must have modulus \( \geq 15/16 \). Hence \( V(z_i) \) is well defined and we put \( W(0) = \mathbb{D} \setminus \bigcup_{i=1}^{\infty} V(z_i) \).

The following list of remarks collects the basic properties of \( W(z_0) \).

**Remarks.**

1. It follows from (2.1) that \( W(z_0) \) is a Lipschitz domain with starcenter \( z_0 \). The Lipschitz constant is independent of \( z_0 \).
2. Suppose that \( w \in \partial V(z_i) \) for some \( i > 0 \), and suppose also that \( |w| < 1 \). Then for \( M \) large enough we have that
   \[
   -2M < b(w) - b(z_0) < -M/2.
   \]
3. For \( M \) large enough we have \( V(z_i) \subset V(z_0) \). Moreover the estimate \( b(z_i) - b(z_0) < -M \) implies that
   \[
   1 - |z_i| \leq (1 - |z_0|)/16.
   \]
4. Let \( I(z_0) = \{ \zeta \in \mathbb{T} : |\zeta - z_0/|z_0|| \leq (1 - |z_0|)/4 \} \). Pick \( \zeta_0 \in I(z_0) \). Let \( R \) be the ray connecting \( 0 \) to \( \zeta_0 \), and let \( L = R \setminus V(z_0) \). Then \( L \) is contained in the Stolz angle \( C(z_0) \).
5. Let \( K(z_0) \) be the convex hull of \( z_0 \) and \( I(z_0) \), and let \( D = K(z_0) \cap \partial W(z_0) \), then by Theorem 4, \( \omega(z_0, D, W(z_0)) > \omega_0 \), where \( \omega_0 > 0 \) is a universal constant.

3. The selection of a good ray

In this section we fix a conformal map \( \varphi \) in the unit disc, and we will select \( e^{i\beta} \in \mathbb{T} \) such that

\[
\int_{0}^{1} |\varphi''(re^{i\beta})|dr < \infty.
\]

Let \( b = \log |\varphi'| \) be the Bloch function associated to the Riemann map \( \varphi \). The ray \( L = \{ re^{i\beta} : 0 < r < 1 \} \) will be chosen so that on \( L \) there are points \( Q_k \) satisfying

\[
b(Q_k) - b(Q_{k-1}) < -M/3,
\]
and

\[
\int_{l_k} |\nabla b(\zeta)|d|\zeta| \leq C_0 M,
\]
where \( l_k \) is the line segment connecting \( Q_k \) and \( Q_{k-1} \). Note that (3.2) and (3.3) imply Theorem 1 and (3.1). Indeed, summing (3.2) gives

\[
b(Q_k) < -kM/3.
\]

Using (3.3) we obtain from (3.4),

\[
b(\zeta) < -kM/3 + C_0 M, \text{ for } \zeta \in l_k.
\]
Clearly (3.3) and (3.5) imply the conclusion in Theorem 1. Now recall that 2|φ''| = |∇b|e^b, together with (3.3) and (3.5) this identity gives the following estimate.

\[
\int_0^1 |φ''(re^{iβ})|dr = \int_0^1 |∇b(re^{iβ})|e^{b(re^{iβ})}dr \leq \sum_{k=0}^{∞} \int_{I_k} |∇b(ζ)|e^{b(ζ)}dζ
\]

\[
\leq e^{C_0M} \sum_{k=0}^{∞} e^{-kM/3} \int_{I_k} |∇b(ζ)||dζ| \leq C_0e^{C_0M}M.
\]

Now we begin the proof of Theorem 1. First we give an inductive definition of an auxiliary sequence of points \( e_k \in \mathbb{D} \). Their limit will be the point \( e^{iβ} \) satisfying the required properties (3.2) and (3.3).

Fix \( M \in \mathbb{N} \) large enough and assume \( b(0) = 0 \). Let \( W(0) \) be the stopping time Lipschitz domain constructed in section 2. Clearly we have \( 0 \in W(0) \). Let \( f : \mathbb{D} \to W(0) \) be the Riemann map normalized such that \( f(0) = 0 \).

We use the conformal map \( f \) to pull back \( b \) from \( W(0) \) to the unit disc \( \mathbb{D} \). The composition \( h = b \circ f \) is harmonic and satisfies \( h > -2M \) in \( \mathbb{D} \), and \( h(0) = 0 \). By Bourgain’s theorem there exists \( e^{iα} \in \mathbb{T} \) such that

\[
δ_0 \int_γ |∇h(ζ)||dζ| \leq M, \text{ for } γ \in Γ_α.
\]

Now we let \( e_1 = f(e^{iα}) \). We distinguish between the cases \( |e_1| = 1 \) and \( |e_1| < 1 \). If we have \( |e_1| < 1 \), then by (2.3) we have \( b(e_1) < -M/2 \). If \( |e_1| = 1 \), then we stop the construction of the points \( \{e_k\} \).

Next we give the induction step in the construction of the points \( \{e_k\} \). We are given \( e_1, \ldots, e_l \), points in \( \mathbb{D} \), so that \( I(e_{k+1}) \subset I(e_k) \) and \( |I(e_{k+1})| < |I(e_k)|/4 \), for \( 1 \leq k \leq l-1 \). Let \( D = ∂W(e_l) \cap K(e_l) \). \( D \) is connected, and by Remark 5 in Section 2, for the harmonic measure we have the estimate \( w(e_l, D, W(e_l)) \geq ω_0 \).

Let \( f : \mathbb{D} \to W(e_l) \) with \( f(0) = e_l \) be the Riemann map for the domain \( W(e_l) \), then \( K = f^{-1}(D) \) is an interval and \( |K| \geq ω_0 \).

Again, the conformal map \( f \) is used to pull back \( b \) from \( W(e_l) \) to the unit disc. The composition \( h = b \circ f - b \circ f(0) \) is harmonic and satisfies \( h > -2M \) in \( \mathbb{D} \) and \( h(0) = 0 \). By Bourgain’s theorem there exists \( e^{iα} \in K \) such that

\[
(3.6) \quad δ_0 \int_γ |∇h(ζ)||dζ| \leq M, \text{ for } γ \in Γ_α.
\]

Let \( e_{l+1} = f(e^{iα}) \). As \( e_{l+1} \) is a point in \( D = ∂W(e_l) \cap K(e_l) \) it follows from (2.4) that

\[
(3.7) \quad I(e_{l+1}) \subset I(e_l) \text{ and } |I(e_{l+1})| < |I(e_l)|/4.
\]

If \( |e_{l+1}| < 1 \), then by (2.3),

\[
(3.8) \quad b(e_{l+1}) - b(e_l) < -M/2.
\]

Otherwise, i.e., when \( |e_{l+1}| = 1 \) we stop the construction of the points \( \{e_k\} \).
Having completed the construction of the points \( \{e_k\} \) we let \( e^{i\beta} = \lim e_k \). By (3.7) the limit exists and lies in \( T \). Moreover it follows that \( e^{i\beta} = \bigcap I(e_k) \). We let \( L \) be the ray connecting 0 and \( e^{i\beta} \). By (3.7) the limit exists and lies in \( T \). Moreover it follows that \( e^{i\beta} = \bigcap I(e_k) \). We let \( L \) be the ray connecting 0 and \( e^{i\beta} \). Notice that \( L \) intersects the boundary \( \partial W(e_k) \) at least once and at most twice. We let \( Q_k \) be the point in \( L \cap \partial W(e_k) \) that has the smaller modulus. If we let \( l_k \) be the line segment connecting \( Q_k \) and \( Q_{k+1} \) then clearly \( l_k \) coincides with \( L \cap \{W(e_k) \setminus V(e_{k+1})\} \). Moreover by Remark 4,

\[
(3.9) \quad l_k \text{ is a subset of } C(e_{k+1}).
\]

Now we wish to verify (3.2) and (3.3). We fix \( k \in \mathbb{N} \) and let \( f : \mathbb{D} \to W(e_k) \) be the conformal map that was used in the definition of \( e_{k+1} \in \partial W(e_k) \). Then we had determined \( \alpha \) by \( f(e^{i\alpha}) = e_{k+1} \). Now (3.9) and Theorem 6 imply that the curve

\[
f^{-1}(l_k) \text{ is contained in } C(e^{i\alpha}).
\]

Moreover by the distortion theorem, the curve \( f^{-1}(l_k) \) can be decomposed into say \( F_1 \cup \cdots \cup F_{m_0} \), with a universal \( m_0 \), so that each of the \( F_i \) is contained in a curve \( \gamma \in \Gamma_\alpha \). By (3.6) this implies that

\[
\int_{f^{-1}(l_k)} |\nabla h(\zeta)||d\zeta| \leq C_0 M,
\]

where \( h = b \circ f - b \circ f(e_k) \). A change of variables gives the desired

\[
\int_{l_k} |\nabla b(\zeta)||d\zeta| \leq C_0 M.
\]

To obtain (3.2) from (3.8) we observe that \( L \) hits the boundary \( \partial W(e_k) \) near \( e_k \). More precisely, the hyperbolic distance between \( Q_k \) and \( e_k \) is bounded independent of \( k \). Hence from (3.8) we obtain

\[
b(Q_{k+1}) - b(Q_k) < -M/3,
\]

provided that \( M \) is large enough.

References


Department of Mathematics, Yale University, New Haven, CT 06520

E-mail address: jones@math.yale.edu, muller@math.yale.edu