Mathematical Research Letters **4**, 413–425 (1997)

FACTORIZATION OF DIFFERENTIAL OPERATORS, QUASIDETERMINANTS, AND NONABELIAN TODA FIELD EQUATIONS

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Abstract. We integrate nonabelian Toda field equations [Kr] for root systems of types A, B, C, for functions with values in any associative algebra. The solution is expressed via quasideterminants introduced in [GR1],[GR2], [GR4]. In the appendix we review some results concerning noncommutative versions of other classical integrable equations.

Introduction

Nonabelian Toda equations are equations with respect to n unknowns $\phi = (\phi_1, ..., \phi_n) \in A[[u, v]]$, where A is some associative (not necessarily commutative) algebra with unit:

$$\frac{\partial}{\partial u}\left(\frac{\partial \phi_j}{\partial v}\phi_j^{-1}\right) = \begin{cases} \phi_2\phi_1^{-1}, & j = 1 \\ \phi_{j+1}\phi_j^{-1} - \phi_j\phi_{j-1}^{-1}, & 2 \le j \le n-1 \\ -\phi_n\phi_{n-1}^{-1}, & j = n \end{cases} \tag{1}$$

Suppose that A is a $*$-algebra, i.e. it is equipped with an involutive antiautomorphism $* : A \to A$. Then, setting in (1) $\phi_{n+1-i} = (\phi_i^*)^{-1}$, we obtain a new system of equations. If $n = 2k$, we get the nonabelian Toda system for root system C_k:

$$\frac{\partial}{\partial u}\left(\frac{\partial \phi_j}{\partial v}\phi_j^{-1}\right) = \begin{cases} \phi_2\phi_1^{-1}, & j = 1 \\ \phi_{j+1}\phi_j^{-1} - \phi_j\phi_{j-1}^{-1}, & 2 \le j \le k-1 \\ (\phi_k^*)^{-1}\phi_k^{-1} - \phi_k\phi_{k-1}^{-1}, & j = k \end{cases} \tag{2}$$

If $n = 2k+1$, we get the nonabelian Toda system for root system B_k:

$$\frac{\partial}{\partial u}\left(\frac{\partial \phi_j}{\partial v}\phi_j^{-1}\right) = \begin{cases} \phi_2\phi_1^{-1}, & j = 1 \\ \phi_{j+1}\phi_j^{-1} - \phi_j\phi_{j-1}^{-1}, & 2 \le j \le k \\ (\phi_k^*)^{-1}\phi_{k+1}^* - \phi_{k+1}\phi_k^{-1}, & j = k+1, \end{cases} \tag{3}$$

Received January 8, 1997.

where $\phi^*_{k+1} = \phi^{-1}_{k+1}$.

Quasideterminants were introduced in [GR1], as follows. Let X be an $m \times m$-matrix over A. For any $1 \le i, j \le m$, let $r_i(X)$, $c_j(X)$ be the i-th row and the j-th column of X. Let X^{ij} be the submatrix of X obtained by removing the i-th row and the j-th column from X. For a row vector r let $r^{(j)}$ be r without the j-th entry. For a column vector c let $c^{(i)}$ be c without the i-th entry. Assume that X^{ij} is invertible. Then the quasideterminant $|X|_{ij} \in A$ is defined by the formula

$$|X|_{ij} = x_{ij} - r_i(X)^{(j)}(X^{ij})^{-1}c_j(X)^{(i)}, \tag{4}$$

where x_{ij} is the ij-th entry of X.

In this paper we will use quasideterminants to integrate nonabelian Toda equations for root systems of types A, B, C. We do not know yet how to generalize these results to root systems of types D-G, and to affine root systems.

Our method of integration, which is based on interpreting the Toda flow as a flow on the space of factorizations of a fixed ordinary differential operator. This method and explicit solutions of Toda equations are well known in the commutative case (see [LS]; see also [FF] and references therein; for the one variable case see [Ko]).

Remark. As it was mentioned in [AP], some version of abelian Toda equations was essentially considered by J.J. Sylvester [S] and G. Darboux [Da]. Nonabelian Toda equations for the root system A_{n-1} were introduced by Polyakov (see [Kr]). Nonabelian Toda lattice for functions of one variable appeared in [PC], [P]. I.M. Krichever [Kr] constructed algebraic-geometric solutions for the periodic two-dimensional nonabelian Toda lattice (affine A_n root system).

1. Factorization of differential operators

Let R be an associative algebra over a field k of characteristic zero, and $D : R \to R$ be a k-linear derivation.

Let $f_1, ..., f_m$ be elements of R. By definition, the Wronski matrix $W(f_1, ..., f_m)$ is

$$W(f_1, ..., f_m) = \begin{pmatrix} f_1 & ... & f_m \\ Df_1 & ... & Df_m \\ .. & ... & .. \\ D^{m-1}f_1 & ... & D^{m-1}f_m \end{pmatrix}$$

We call a set of elements $f_1, ..., f_m \in R$ nondegenerate if $W(f_1, ..., f_m)$ is invertible.

Denote by $R[D]$ the space of polynomials of the form $a_0D^n + a_1D^{n-1} + ... + a_n$, $a_i \in R$. It is clear that any element of $R[D]$ defines a linear operator on R.

Example. $R = C^\infty(\mathbb{R}), D = \frac{d}{dt}$. In this case, $R[D]$ is the algebra of differential operators on the line.

By analogy with this example, we will call elements of $R[D]$ differential operators.

We will consider operators of the form $L = D^n + a_1 D^{n-1} + ... + a_n$. We will call such L an operator of order n with highest coefficient 1. Denote the space of all such operators by $R_n(D)$.

Theorem 1.1.

(i) *Let $f_1, ..., f_n \in R$ be a nondegenerate set of elements. Then there exists a unique differential operator $L \in R[D]$ of order n with highest coefficient 1, such that $Lf_i = 0$ for $i = 1, ..., n$. It is given by the formula*

$$Lf = |W(f_1, ..., f_n, f)|_{n+1,n+1}. \tag{1.1}$$

(ii) *Let L be of order n with highest coefficient 1, and $f_1, ..., f_n$ be a set of solutions of the equation $Lf = 0$, such that for any $m \leq n$ the set of elements $f_1, ..., f_m$ is nondegenerate. Then L admits a factorization $L = (D - b_n)...(D - b_1)$, where*

$$b_i = (DW_i)W_i^{-1}, \ W_i = |W(f_1, ..., f_i)|_{ii}. \tag{1.2}$$

Proof.

(i) We look for L in the form $L = D^n + a_1 D^{n-1} + ... + a_n$. From the equations $Lf_i = 0$ it follows that

$$(a_n, ..., a_1) = -(D^n f_1, ..., D^n f_n)W(f_1, ..., f_n)^{-1}.$$

By definition,

$$|W(f_1, ..., f_n, f)|_{n+1,n+1} = \\ D^n f - (D^n f_1, ..., D^n f_n)W(f_1, ..., f_n)^{-1}(f, Df, ..., D^{n-1}f)^T = \\ D^n f + (a_n, ..., a_1)(f, Df, ..., D^{n-1}f)^T = Lf.$$

(ii) We will prove the statement by induction in n. For $n = 1$, the statement is obvious. Suppose it is valid for the differential operator L_{n-1} of order $n - 1$ with highest coefficient 1, which annihilates $f_1, ..., f_{n-1}$ (by (i), it exists and is unique). Set $b_n = (DW_n)W_n^{-1}$, and consider the operator $\tilde{L} = (D - b_n)L_{n-1}$. It is obvious that $\tilde{L} f_i = 0$ for $i = 1, ..., n - 1$. Also, by (i)

$$\tilde{L} f_n = (D - b_n)L_{n-1} f_n = (D - b_n)W_n = 0.$$

Therefore, by (i), $\tilde{L} = L$. □

Now consider the special case: $R = A[[t]]$, where A is an associative algebra over k, and $D = \frac{d}{dt}$ (here t commutes with everything). In this case, it is easy to show that nondegenerate sets of solutions of $Lf = 0$ exist, and are in 1-1 correspondence with elements of the group $GL_n(A)$, via $\mathbf{f} = (f_1, ..., f_n) \to W(\mathbf{f})(0)$.

It is clear that two different sets of solutions of the equation $Lf = 0$ can define the same factorization of L. However, to each factorization γ of L we can assign a set $\mathbf{f}_\gamma = (f_1, ..., f_n)$ of solutions of $Lf = 0$, which gives back the factorization γ under the correspondence of Theorem 1.1(ii). This set is uniquely defined by the condition that the matrix $W(\mathbf{f}_\gamma)(0)$ is lower triangular with 1-s on the diagonal.

Here is a formula for computing $\mathbf{f}_\gamma$, which is well known in the commutative case.

Proposition 1.2. *If γ has the form*

$$L = (D - (Dg_n)g_n^{-1})...(D - (Dg_1)g_1^{-1}),$$

where $g_i(0) = 1$, then $\mathbf{f}_\gamma = (f_1, ..., f_n)$, where

$$f_j(t) =$$

$$\int_0^t \int_0^{t_1} ... \int_0^{t_{j-2}} g_1(t)g_1(t_1)^{-1}g_2(t_1)g_2(t_2)^{-1}...g_j(t_{j-1})dt_{j-1}...dt_2dt_1, \tag{1.3}$$

where $\int_0^u (\sum a_i t^i)dt := \sum a_i \frac{u^{i+1}}{i+1}$.

Proof. It is easy to see that if $\mathbf{f} = (f_1, ..., f_n)$, with f_j given by (1.3), then $W(\mathbf{f})(0)$ is strictly lower triangular. So it remains to show that f_j is a solution of the equation $L_j f = 0$, where $L_j = (D - (Dg_j)g_j^{-1})...(D - (Dg_1)g_1^{-1})$.

We prove this by induction in j. The base of induction is clear, since from (1.3) we get $f_1 = g_1$. Let us perform the induction step. By the induction assumption, from (1.3), we have

$$f_j(t) = g_1(t) \int_0^t g_1(s)^{-1}h(s)ds,$$

where h obeys the equation $(D - (Dg_j)g_j^{-1})...(D - (Dg_2)g_2^{-1})h = 0$. Thus, we get

$$(D - (Dg_1)g_1^{-1})f_j = h.$$

This proves that $L_j f_j = 0$. □

For matrix-valued functions a more general version of proposition 1.2 for the periodic case was proved in [Kr3].

Now consider an application of these results to the noncommutative Vieta theorem [GR3]. Let A be an associative algebra. Call a set of elements $x_1, ..., x_n \in A$ generic if their Vandermonde matrix $V(x_1, ..., x_n)$ ($V_{ij} := x_j^{i-1}$) is invertible.

Consider an algebraic equation

$$x^n + a_1x^{n-1} + ... + a_n = 0. \tag{1.4}$$

with $a_i \in A$. Let $x_1, ..., x_n \in A$ be solutions of (1.4) such that $x_1, ..., x_i$ form a generic set for each i. Let $V(i) = V(x_1, ..., x_i)$, and $y_i = |V(i)|_{ii}x_i|V(i)|_{ii}^{-1}$.

Theorem 1.3. **[GR3]** (*Noncommutative Vieta theorem*)

$$a_r = (-1)^r \sum_{i_1 < ... < i_r} y_{i_r} ... y_{i_1}. \tag{1.5}$$

Proof. (Using differential equations.) Consider the differential operator with constant coefficients in $R = A[[t]]$:

$$L = D^n + a_1 D^{n-1} + ... + a_n.$$

We have solutions $f_i = e^{tx_i}$ of the equation $Lf = 0$, and for any i the set $f_1, ..., f_i$ is nondegenerate, since its Wronski matrix $W(i)$ is of the form

$$W(i) = V(i)\mathrm{diag}(e^{tx_1}, ..., e^{tx_i}). \tag{1.6}$$

Thus, by Theorem 1(ii), the operator L admits a factorization

$$L = (D - b_n)...(D - b_1),$$

where $b_i = (D|W(i)|_{ii})|W(i)|_{ii}^{-1}$. Substituting (1.6) into this equation, we get

$$b_i = |V(i)|_{ii} x_i |V(i)|_{ii}^{-1} = y_i. \tag{1.7}$$

Since $[D, b_i] = 0$, we obtain the theorem.

Note, that the connection between matrix algebraic equations and matrix differential equations appeared in [CS].

2. Toda equations

In this section we will solve equations (1),(2),(3).

Let $M = \{(g, h) \in A[[t]] \oplus A[[t]] : g(0) = h(0) \in A^*\}$, where A^* is the set of invertible elements of A. We have the following obvious proposition, which says that solutions of Toda equations are uniquely determined by initial conditions.

Proposition 2.1. *The assignment $\phi(u, v) \to (\phi(u, 0), \phi(0, v))$ is a bijection between the set of solutions of the Toda equations* (1) *and the set M.*

Let B be an algebra over k, $R = B[[v]]$, $D = \frac{d}{dv}$. For $\phi = (\phi_1, ..., \phi_n)$, where $\phi_i \in B[[v]]$ are invertible, define $L_i^\phi \in R[D]$ by

$$L_i^\phi = (D - (D\phi_i)\phi_i^{-1})...(D - (D\phi_1)\phi_1^{-1}). \tag{2.1}$$

Now set $B = A[[u]]$. Then $B[[v]] = A[[u, v]]$, so for any $\phi = (\phi_1, ..., \phi_n)$, with all $\phi_i \in A[[u, v]]$ invertible, we can define L_i^ϕ.

Proposition 2.2. *A vector-function $\phi(u,v)$ is a solution of Toda equations* (1) *if and only if*

$$\frac{\partial L_i^{\phi}}{\partial u} = -\phi_{i+1}\phi_i^{-1}L_{i-1}^{\phi}, i \le n-1; \ \frac{\partial L_n^{\phi}}{\partial u} = 0. \tag{2.2}$$

Proof. Let ϕ be a solution of the Toda equations. Set $b_i = (D\phi_i)\phi_i^{-1}$. We have $L_i^{\phi} = L_i = (D-b_i)...(D-b_1)$. Therefore, for $i \le n-1$ we have

$$\begin{aligned} \frac{\partial L_i}{\partial u} = &-\sum_{j=1}^{i}(D-b_i)...(D-b_{j+1})\phi_{j+1}\phi_j^{-1}(D-b_{j-1})...(D-b_1) \\ &+\sum_{j=1}^{i-1}(D-b_i)...(D-b_{j+2})\phi_{j+1}\phi_j^{-1}(D-b_j)...(D-b_1). \end{aligned} \tag{2.3}$$

But

$$(D-b_{j+1})\circ\phi_{j+1}\phi_j^{-1} = \phi_{j+1}\phi_j^{-1}(D-b_j).$$

Therefore, the right hand side of (2.3) equals $-\phi_{i+1}\phi_i^{-1}L_{i-1}$. The same argument shows that for $i=n$ the derivative $\frac{\partial L_n}{\partial u}$ vanishes.

Conversely, it is easy to see that equations (2.2) imply (1). □

Remark. This proposition is just a reformulation of the well-known statement that the two-dimensional Toda lattice is equivalent to a compatibility condition for the linear system

$$\partial_v\xi_j = \xi_{j+1} + (\partial_v\phi_j)\phi_j^{-1}\xi_j$$
$$\partial_u\xi_j = -(\phi_j\phi_{j-1}^{-1})\xi_{j-1}$$

with $\xi_{n+1} = \xi_0 = 0$. (Here $\xi_j = L_{j-1}f$, where $Lf = 0$).

Now we will compute the solutions of Toda equations explicitly. Let $\eta_1,...,\eta_n,\psi_1,...,\psi_n \in A[[t]]$ be such that $\eta_i(0) = \psi_i(0) \in A^*$. We will find the solution of the following initial value problem for Toda equations:

$$\phi_i(u,0) = \psi_i(u), \phi_i(0,v) = \eta_i(v) \tag{2.4}$$

Proposition 2.1 states that this solution exists and is unique.

Let $g_i(v) = \eta_i(v)\eta_i(0)^{-1}$. Let $\mathbf{f} = (f_1,...,f_n)$ be given by formula (1.3) in terms of g_i. Define the lower triangular matrix $\Delta(u)$ whose entries are given by the formula

$$\Delta_{ij}(u) = \int_0^u\int_0^{t_1}...\int_0^{t_{i-j-1}}\psi_i(t_{i-j})\psi_{i-1}^{-1}(t_{i-j})\psi_{i-1}(t_{i-j-1})...\psi_j^{-1}(t_1)\psi_j(u)dt_{i-j}...dt_1.$$

Let $\mathbf{f}^u = (f_1^u,...,f_n^u)$ be defined by the formula

$$\mathbf{f}^u = \mathbf{f}\Delta(u).$$

Then we have

Theorem 2.3. *The solution of the problem* (2.3) *is given by the formula*

$$\phi_i(u,v) = |W(f_1^u, ..., f_i^u)|_{ii}. \tag{2.5}$$

Proof. Let $L_i^u = L_i^{\phi(u,v)}$, where $\phi(u,v)$ is defined by (2.5). By Theorem 1.1(ii), we have $L_i^u f_j^u = 0$ for $j \leq i$. Differentiating this equation with respect to u, we get

$$L_i^u \frac{\partial f_j^u}{\partial u} + \frac{\partial L_i^u}{\partial u} f_j^u = 0. \tag{2.6}$$

On the other hand, it is easy to see that

$$\Delta'(u) = \Delta(u)\Theta(u),$$

where

$$\Theta_{ij}(u) = \begin{cases} \psi_i^{-1}(u)\psi_i'(u) & i = j \\ 1 & i = j+1 \\ 0 \text{ otherwise} \end{cases}$$

This implies that

$$\frac{\partial f_i^u}{\partial u} = f_{i+1}^u + f_i^u \psi_i^{-1} \psi_i'.$$

Therefore, $L_i^u \frac{\partial f_j^u}{\partial u} = 0$ for $i \geq j+1$, and

$$L_i^u \frac{\partial f_j^u}{\partial u} = L_i^u f_{i+1}^u = |W(f_1^u, ..., f_{i+1}^u)|_{i+1 i+1} = \phi_{i+1}.$$

Thus, from (2.6) we get

$$\frac{\partial L_i^u}{\partial u} f_j^u = -\phi_{i+1}\phi_i^{-1} L_{i-1} f_j^u, \; i \geq j.$$

By Theorem 1.1 (i), this implies equations (2.2), which are equivalent to Toda equations.

It is obvious that $\phi(u,v)$ satisfies the required initial conditions. The theorem is proved. □

If initial conditions (2.4) satisfy the symmetry property $\phi_{n+1-i} = (\phi_i^*)^{-1}$, then we obtain a solution of the initial value problem for equations (2) (for even n), and (3) (for odd n). This gives a complete description of solutions of systems of equations (1),(2),(3).

In the case $\phi(u,0) = 1$, the Toda flow can be interpreted as a flow on the space of factorizations of a differential operator. Namely, let L be a differential operator of order n with highest coefficient 1. Let $F(L)$ be the space of factorizations of L. Let N_n^- be the group of strictly lower triangular matrices over A. It is easy to see that the map $\pi : F(L) \to N_n^-$, given by $\pi(\gamma) = W(\mathbf{f}_\gamma)(0)$, is a bijection. We will identify $F(L)$ with N_n^- using π.

Let $\gamma(u) = \gamma(0)e^{uJ_n}$ be a curve on N_n^- generated by the 1-parameter subgroup e^{uJ_n}, where J_n is the lower triangular nilpotent Jordan matrix ($(J_n)_{ij} = \delta_{i,j+1}$). Let $L = (D - b_n^u)...(D - b_1^u)$ be the factorization of L corresponding to the point $\gamma(u)$. Let $\phi_i(u,v)$ be such that $(D\phi_i)\phi_i^{-1} = b_i^u$, and $\phi_i(u,0) = 1$. (here $D = \frac{\partial}{\partial v}$). Then we have the following Corollary from Theorem 2.3.

Corollary 2.4. *$\phi = (\phi_1, ..., \phi_n)$ is a solution of Toda equations (1), and all solutions with $\phi(u, 0) = 1$ are obtained in this way.*

Proof. For the proof it is enough to observe that if $\psi_i(u) = 1$ then $\Theta(u) = J_n$, and $\Delta(u) = e^{uJ_n}$. □

Remark. An analogous statement can be made for equations (2) and (3). In this case, instead of an arbitrary differential operator $L \in R_n(D)$ one should consider a selfadjoint (respectively, skew-adjoint) operators, instead of the group N_n^- – a maximal nilpotent subgroup in $Sp_n(A)$ (respectively, $O_n(A)$), and instead of J_n the sum of simple root elements in the Lie algebra of this group. In the commutative case, such a picture of the Toda flow is known for all simple Lie groups [FF].

Consider now the statement of Theorem 2.3 for $i = 1$. In this case we have

$$\phi_1(u, v) = f_1^u(v) = \sum_{i=1}^{n} f_i^0(v)\Delta_{i1}(u). \tag{2.7}$$

Thus, we get

Corollary 2.5. **[RS]** *If ϕ is a solution of the Toda equations then $\phi_1(u, v) = \sum_{i=1}^{n} p_i(u)q_i(v)$, where p_i, q_i are some formal series.*

Now let us discuss infinite Toda equations. These are equations (1) with $n = \infty$. These equations allow to express ϕ_i recursively in terms of $f = \phi_1$, which can be done using quasideterminants:

$$\phi_i = |Y_i(f)|_{ii},\ Y_i(f) := W\left(f, \frac{\partial f}{\partial u}, ..., \frac{\partial^{i-1} f}{\partial u^{i-1}}\right), \tag{2.8}$$

where $\frac{\partial^i f}{\partial u^i}$ are regarded as functions of v for a fixed u. This formula is easily proved by induction. It appears in [GR2], section 4.5 in the case $u = v$, and in [RS] in the 2-variable case.

Formula (2.8) can be used to give another expression for the general solution of finite Toda equations, which appears in [RS]. Indeed, we have the following easy proposition.

Proposition 2.6. *Let $f \in A[[u, v]]$ be such that $Y_1(f), ..., Y_n(f)$ are invertible matrices. Then $|Y_{n+1}(f)|_{n+1n+1} = 0$ if and only if f is "kernel of rank n", i.e.*

$$f = \sum_{i=1}^{n} p_i(u)q_i(v). \tag{2.9}$$

Thus, taking $f = \phi_1$ of the form (2.9), with $Y_1, ..., Y_n$ invertible, and using formula (2.6) we will get a solution $(\phi_1, ..., \phi_n)$ of the finite Toda system of length n. It is not difficult to show that in this way one gets all possible solutions.

Example. Consider the nonabelian Liouville equation

$$\frac{\partial}{\partial u}\left(\frac{\partial \phi}{\partial v}\phi^{-1}\right) = (\phi^*)^{-1}\phi^{-1}, \tag{2.10}$$

which is a special case of (2) for $k = 1$. Consider the initial value problem

$$\phi(0, v) = \eta(v), \phi(u, 0) = \psi(u), \eta(0) = \psi(0) = a. \tag{2.11}$$

By Theorem 2.3, we get the following formula for the solution:
(2.12)

$$\phi(u, v) = \eta(v)\left(a^{-1} + \int_0^v \int_0^u \eta^{-1}(t)(\eta^*)^{-1}(t)a^*(\psi^*)^{-1}(s)\psi^{-1}(s)dsdt\right)\psi(u).$$

For example, if $\psi(u) = 1$, we get

$$\phi(u, v) = \eta(v)\left(1 + u\int_0^v \eta^{-1}(t)(\eta^*)^{-1}(t)dt\right). \tag{2.13}$$

In the commutative case, these formulas coincide with the standard formulas for solutions of the Liouville equation.

Appendix: Noncommutative soliton equations

In this appendix we will review some results about noncommutative versions of classical soliton hierarchies (KdV,KP). These results are mostly known or can be obtained by a trivial generalization of the corresponding commutative results, but they have never been exposed systematically.

We will follow Dickey's book [D].

A1. Nonabelian KP hierarchy.

Nonabelian KP hierarchy is defined in the same way as the usual KP hierarchy [D]. Let

$$L = \partial + w_0\partial^{-1} + w_1\partial^{-2} + ... \tag{A1}$$

be a formal pseudodifferential operator. Here $\partial = \frac{d}{dx}$, $w_i \in A[[x, t_1, t_2, ...]]$, where A is an associative (not necessarily commutative) algebra with 1. Consider the following infinite system of differential equations:

$$\frac{\partial L}{\partial t_m} = [B_m, L], B_m = (L^m)_+, \tag{A2}$$

where for a pseudodifferential operator M, we denote by M_+, M_- the differential and the integral parts of M (i.e. all terms with nonnegative, respectively negative, powers of ∂). For brevity we will write $L^m_\pm$ instead of $(L^m)_\pm$.

Each of the differential equations $(A2)$ defines a formal flow on the space P of pseudodifferential operators of the form (A1). Indeed, $[L^m_+, L] = -[L^m_-, L]$, and the order of $[L^m_-, L]$ is at most zero.

Proposition A1. *The flows defined by equations* (A2) *commute with each other.*

Proof. The proof is the same as the proof of Proposition (5.2.3) in [D]. Namely, one needs to check the zero curvature condition

$$\frac{\partial B_m}{\partial t_n} - \frac{\partial B_n}{\partial t_m} - [B_m, B_n] = 0, \tag{A3}$$

which is done by a direct calculation given in [D].

The hierarchy of flows defined by (A2) for $m = 1, 2, 3, ...$ is called the noncommutative KP hierarchy.

A2. Noncommutative KdV and nKdV hierarchies.

As in the commutative case, we can restrict the KP hierarchy to the subspace $P_n \subset P$ of all operators L such that L^n is a differential operator, i.e. $L^n_- = 0$ [D]. This subspace is invariant under the KP flows, as $\frac{\partial L^n_-}{\partial t_m} = [L^m_+, L^n_-]_-$. The space P_n can be identified with the space D_n of differential operators of the form

$$M = \partial^n + u_2\partial^{n-2} + ... + u_n, \tag{A4}$$

by the map $L \to M = L^n$. The KP hierarchy induces a hierarchy of flows on D_n called the nKdV hierarchy:

$$\frac{\partial M}{\partial t_m} = [M^{m/n}_+, M]. \tag{A5}$$

Among these flows, the flows corresponding to $m = nl$, $l \in \mathbb{Z}_+$, are trivial, but the other flows are nontrivial.

For $n = 2$ the nKdV hierarchy is the usual KdV hierarchy. The first two nontrivial equations of this hierarchy are

$$u_{t_1} = u_x,\ u_{t_3} = \frac{1}{4}(u_{xxx} + 3u_x u + 3uu_x). \tag{A6}$$

The second equation is the noncommutative KdV equation.

A3. Finite-zone solutions of the nKdV hierarchy.

Denote the vector fields of the noncommutative nKdV hierarchy by V_m, $m \neq nl$. Let $V = \sum_{i=1}^p a_i V_i$, $a_p \neq 0$, be a finite linear combination of V_i with coefficients $a_i \in l$. Let $S(a)$, $a = (a_1, ..., a_p)$ be the set of stationary points of V in $D_n = A[[x]]^{n-1}$.

The set $S(a)$ can be identified with $A^{(n-1)p}$. Indeed, $S(a)$ is the subset of D_n defined by the differential equations $V(M) = 0$, which have the form

$$\frac{d^p u_i}{dx^p} = F_i(u_j, u_j', ..., u_j^{(p-1)}). \tag{A7}$$

So u_i are uniquely determined by their first p derivatives at 0.

Since the vector field V commutes with the nKdV hierarchy, $S(a)$ is invariant under the flows of this hierarchy. Using the identification $S(a) \to A^{(n-1)p}$, we can rewrite each of the nKdV flows as a system of ordinary differential equations on $A^{(n-1)p}$. Thus, solutions of nKdV equations belonging to $S(a)$ can be computed by solving ordinary differential equations. Such solutions are called finite-zone solutions.

Remark. If $M \in S(a)$ then $[Q(M^{1/n})_+, M] = 0$, where $Q(x) = \sum a_i x^i$. Thus, we have two commuting differential operators M and $N = Q(M^{1/n})_+$ in 1 variable. If the algebra A is finite dimensional over its center then there exists a nonzero polynomial $R(x, y)$ with coefficients in the center of A, such that $R(M, N) = 0$. This polynomial defines an algebraic curve, called the spectral curve. The operator M can then be computed explicitly using the method of Krichever [Kr2]. Similarly, all nKdV equations resticted to the space of such operators can be solved explicitly in quadratures. However, if A is infinite-dimensional over its center, the polynomial R does not necessarily exist, and we do not know any way of computing M explicitly.

A4. Multisoliton solutions of the noncommutative KP hierarchy.

Here we will construct N-soliton solutions of the KP and KdV hierarchies in the noncommutative case. We will use the dressing method. In the exposition we will closely follow Dickey's book [D].

We will now construct a solution of the KP hierarchy. Let $t = (t_1, t_2, ...)$ Consider the formal series

$$\xi(x, t, \alpha) = (x + t_1)\alpha + t_2\alpha^2 + ... + t_r\alpha^r + ..., \alpha \in A. \tag{A8}$$

Fix $\alpha_1, ..., \alpha_N, \beta_1, ..., \beta_N, a_1, ..., a_N \in A$, and set

$$y_s(x, t) = e^{\xi(x,t,\alpha_s)} + a_s e^{\xi(x,t,\beta_s)}. \tag{A9}$$

Define the differential operator of order N, with highest coefficient 1, by

$$\Phi f(x) = |W(y_1, ..., y_N, f)|_{N+1,N+1}(x), \tag{A10}$$

where the derivatives in the Wronski matrix are taken with respect to x, and we assume that the functions $y_1, ..., y_N$ are a generic set (in the sense of Chapter 1). Set

$$L = \Phi\partial\Phi^{-1}. \tag{A11}$$

Proposition A2. *The operator-valued series $L(x, t)$ is a solution of the KP hierarchy.*

Proof. The proof is the same as the proof of Proposition 5.3.6 in [D].

Such solutions are called N-soliton solutions.

Proposition A2 can be used to construct N-soliton solutions of the nKdV hierarchy. Namely, we should restrict the above construction to the case when $\beta_k = \varepsilon_k \alpha_k$, where ε_k is an n-th root of unity. In this case it can be shown as in [D] that the operator $M = L^n = \Phi \partial^n \Phi^{-1}$ is a differential operator of order n, and it is a solution of the nKdV hierarchy.

As an example, let us consider the N-soliton solutions of the KdV hierarchy. In this case, we may set $t_{2k} = 0$, and we have

(A12) $$y_s = e^{\xi(x,t,\alpha_s)} + a_s e^{-\xi(x,t,\alpha_s)}.$$

Proposition A3. *Let*

(A13) $$b_i = (\partial W_i) W_i^{-1},\ W_i := |W(y_1, ..., y_i)|_{ii}.$$

Then the function

(A14a) $$u(x,t) = 2\partial(\sum_{i=1}^{N} b_i).$$

is a solution of the noncommutative KdV hierarchy.

Proof. Let $\Phi = \partial^N + v_1 \partial^{N-1} + ...$ Then from the equation $(\partial^2 + u)\Phi = \Phi \partial^2$ we obtain $u = -2\partial v_1$, and from Theorem 1.1(ii) $v_1 = -\sum_i b_i$, where b_i are given by (A13). This implies (A14a).

Another formula for u, which is equivalent to (A14a), is the following. Let $Y(y_1, ..., y_N)$ be the matrix which coincides with the Wronski matrix $W(y_1, ..., y_N)$ except at the last row, where it has $y_i^{(N)}$ instead of $y_i^{(N-1)}$. Let $Y_N = |Y(y_1, ..., y_N)|_{NN}$. Then the function u given by (A14a) can be written as

(A14b.) $$u = 2\partial(Y_N W_N^{-1}),$$

In the commutative case formulas (A14a),(A14b) reduce to the classical formula

(A15) $$u = 2\partial^2 \ln[\det W(y_1, ..., y_N)].$$

In particular, we can obtain N-soliton solutions of the noncommutative KdV equation $u_t = \frac{1}{4}(u_{xxx} + 3u_x u + 3u u_x)$. For this purpose set $t_i = 0$, $i \neq 3$, and $t_3 = t$. Then we get

(A16) $$y_s = e^{\alpha_s x + \alpha_s^3 t} + a_s e^{-\alpha_s x - \alpha_s^3 t},$$

and the solution $u(x,t)$ is given by (A14a),(A14b).

For example, consider the 1-soliton solution. According to (A14a), it has the form

(A17) $$u = 2\frac{\partial}{\partial x}[(e^{\alpha x + \alpha^3 t} - a e^{-\alpha x - \alpha^3 t})\alpha(e^{\alpha x + \alpha^3 t} + a e^{-\alpha x - \alpha^3 t})^{-1}].$$

In the commutative case, it reduces to the well known solution

(A18) $$u = \frac{2\alpha^2}{\cosh^2(\alpha x + \alpha^3 t - c)},\ c = \frac{1}{2}\ln a,$$

– the solution corresponding to the solitary wave which was observed by J. S. Russell in August of 1834.

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