EQUIVARIANT RESOLUTION OF SINGULARITIES IN CHARACTERISTIC 0

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0. Introduction

We work over an algebraically closed field $k$ of characteristic 0.

0.1. Statement. In this paper, we use techniques of toric geometry to reprove the following theorem:

**Theorem 0.1.** Let $X$ be a projective variety of finite type over $k$, and let $Z \subset X$ be a proper closed subset. Let $G \subset \text{Aut}_k(Z \subset X)$ be a finite group. Then there is a $G$-equivariant modification $r : X_1 \to X$ such that $X_1$ is nonsingular projective variety, and $r^{-1}(Z_{\text{red}})$ is a $G$-strict divisor of normal crossings.

This theorem is a weak version of the equivariant case of Hironaka’s well known theorem on resolution of singularities. It was announced by Hironaka, but a complete proof was not easily accessible for a long time. The situation was remedied by E. Bierstone and P. Milman [B-M2], and by O. Villamayor [V]. They gave constructions of completely canonical resolution of singularities. These constructions are based on a thorough understanding of the effect of blowing up - one carefully build up an invariant pointing to the next blowup.

The proof we give in this paper takes a completely different approach. It uses two ingredients: first, we assume that we know the existence of resolution of singularities without group actions. The method of resolution is not important: any of [H], [B-M1], [V] [R-dJ] or [B-P] would do. Second, we use equivariant toroidal resolution of singularities. Unfortunately, in [KKMS] the authors do not treat the equivariant case. But proving this turns out to be straightforward given the methods of [KKMS]. (For a similar argument in the toric case see [B].)

To this end, section 2 of this paper is devoted to proving the following:

**Theorem 0.2.** Let $U \subset X$ be a strict toroidal embedding, and let $G \subset \text{Aut}(U \subset X)$ be a finite group acting toroidally. Then there is a $G$-equivariant toroidal ideal sheaf $\mathcal{I}$ such that the normalized blowup of $X$ along $\mathcal{I}$ is a nonsingular $G$-strict toroidal embedding.
1. Preliminaries

First recall some definitions. We restrict ourselves to the case of varieties over $k$. A large portion of the terminology is borrowed from [8-dJ].

A modification is a proper birational morphism of irreducible varieties.

Let a finite group $G$ act on a (possibly reducible) variety $Z$. Let $Z = \bigcup Z_i$ be the decomposition of $Z$ into irreducible components. We say that $Z$ is $G$-strict if the union of translates $\bigcup_{g \in G} g(Z_i)$ of each component $Z_i$ is a normal variety. We simply say that $Z$ is strict if it is $G$-strict for the trivial group, namely every $Z_i$ is normal.

A divisor $D \subset X$ is called a divisor of normal crossings if étale locally at every point it is the zero set of $u_1 \cdots u_k$ where $u_1, \ldots, u_k$ is part of a regular system of parameters. Thus, in a strict divisor of normal crossings $D$, all components of $D$ are nonsingular.

An open embedding $U \hookrightarrow X$ is called a toroidal embedding if locally in the étale topology it is isomorphic to a torus embedding $T \hookrightarrow V$, (see [KKMS], II §1). One may replace “étale locally” by “complex analytically” in case $k = \mathbb{C}$, or “formally”, obtaining the same class of embeddings. Let $E_i, i \in I$ be the irreducible components of $X \setminus U$. A finite group action $G \subset \text{Aut}(U \hookrightarrow X)$ is said to be toroidal if the stabilizer of every point can be identified on the appropriate neighborhood with a subgroup of the torus $T$. We say that a toroidal action is $G$-strict if $X \setminus U$ is $G$-strict. In particular the toroidal embedding itself is said to be strict if $X \setminus U$ is strict. This is the same as the notion of toroidal embedding without self-intersections in [KKMS]. For any subset $J$ of $I$, the components of the sets $\bigcap_{i \in J} E_i - \bigcup_{i \notin J} E_i$ define a stratification of $X$. Each component is called a stratum.

Recall that in [KKMS], p. 69-70 one defines the notion of a conical polyhedral complex with integral structure. As in [KKMS], p. 71, to every strict toroidal embedding $U \subset X$ one canonically associates a conical polyhedral complex with integral structure. In the sequel, when we refer to a conical polyhedral complex, it is understood that it is endowed with an integral structure.

In [KKMS], p. 86 (Definition 2) one defines a rational finite partial polyhedral decomposition $\Delta'$ of a conical polyhedral complex $\Delta$. We will restrict attention to the case where $|\Delta'| = |\Delta|$, and we will call this simply a polyhedral decomposition or subdivision.

The utility of polyhedral decompositions is given in Theorem 6* of [KKMS] (page 90), which establishes a correspondence between allowable modifications of a given strict toroidal embedding (which in our terminology are proper), and polyhedral decompositions of the associated conical polyhedral complex.

In order to guarantee that a modification is projective, one needs a bit more. Following [KKMS], p. 91, a function $\text{ord} : \Delta \rightarrow \mathbb{R}$ defined on a conical polyhedral complex with integral structure is called an order function if:
(1) $\text{ord}(\lambda x) = \lambda \cdot \text{ord}(x), \lambda \in \mathbb{R}^+$

(2) ord is continuous, piecewise-linear

(*)

(3) $\text{ord}(N_Y \cap \sigma^Y) \subset \mathbb{Z}$ for all strata $Y$.

(4) ord is convex on each cone $\sigma \subset \Delta$

For an order function on the conical polyhedral complex corresponding to $X$, we can define canonically a coherent sheaf of fractional ideals on $X$, and vice versa (see [KKMS], I §2). The order function is positive if and only if the corresponding sheaf is a a genuine ideal sheaf. We have the following important theorem [KKMS]:

**Theorem 1.1.** Let $F$ be a coherent sheaf of ideals corresponding to a positive order function $\text{ord}_F$, and let $B_F(X)$ be the normalized blowup of $X$ along $F$. Then $B_F(X) \rightarrow X$ is an allowable modification of $X$, described by the decomposition of $|\Delta|$ obtained by subdividing the cones into the biggest subcones on which $\text{ord}_F$ is linear.

A polyhedral decomposition is said to be projective if it is obtained in such a way from an order function. The corresponding modification is indeed a projective morphism.

Given a cone $\sigma$ and a rational ray $\tau \subset \sigma$, it is natural to define the decomposition of $\sigma$ centered at $\tau$, whose cones are of the form $\sigma' + \tau$, where $\sigma'$ runs over faces of $\sigma$ disjoint from $\tau$. Given a polyhedral complex $\Delta$ and a rational ray $\tau$, we can take the subdivision of all cones containing $\tau$ centered at $\tau$, and again call the resulting decomposition of $\Delta$, the subdivision centered at $\tau$.

From [KKMS] I §2, lemmas 1-3, p. 33-35 it follows that the subdivision centered at $\tau$ is projective.

One can also obtain the barycentric subdivision inductively the other way: the barycentric subdivision of an $m$-dimensional cone $\delta$ is formed by first taking the barycentric subdivision of all its faces, and for each one of the resulting cones $\sigma$, including also the cone $\sigma + b(\delta)$. This way it is clear that $B(\Delta)$ is a simplicial subdivision.

2. Equivariant toroidal modifications

**Lemma 2.1.** Let $U \subset X$ be a strict toroidal embedding, $G \subset \text{Aut}(U \subset X)$ a finite group action. Then
(1) The group $G$ acts linearly on $\Delta(X)$.

(2) Assume that the action of $G$ is strict toroidal. Let $g \in G$, and let $\delta \subset \Delta(X)$ be a cone, such that $g(\delta) = \delta$. Then $g_\delta = id$.

Proof.

(1) Clearly, $G$ acts on the stratification of $U \subset X$. Note that, from Definition 3 of [KKMS], page 59, $\Delta(X)$ is built up from the groups $M^Y$ of Cartier divisors on $\text{Star}(Y)$ supported on $\text{Star}(Y) \setminus U$, as $Y$ runs through the strata. As $g \in G$ canonically transforms $M^Y$ to $M^{g^{-1}Y}$ linearly, our claim follows.

(2) Assume $g : \delta \to \delta$, and $g_\delta \neq id$, then there exists an edge $e_1 \in \delta$, s.t $g(e_1) \neq e_1$. Denote $g(e_1) = e_2$. Assume $e_1$ corresponds to a divisor $E_1$, and $e_2$ corresponds to a divisor $E_2$. Since $g(e_1) = e_2$ we have $g(E_1) = E_2$. As $e_1, e_2$ are both edges of $\delta$, $E_1 \cap E_2 \neq \phi$. So $\cup g(E_1)$ cannot be normal since it has two intersecting components. This is a contradiction to the fact that $G$ acts strictly on $X$.

Lemma 2.2. Let $G \subset \text{Aut}(U \subset X)$ act toroidally. Let $\Delta_1$ be a $G$-equivariant subdivision of $\Delta$, with corresponding modification $X_1 \to X$. Then $G$ acts toroidally on $X_1$. Moreover, if $G$ acts strictly on $X$, it also acts strictly on $X_1$.

Proof. The fact that $G$ acts on $X_1$ follows from the canonical manner in which $X_1$ is constructed from the decomposition $\Delta_1$, see Theorems 6* and 7* of [KKMS], II §2, p. 90.

Now for any point $a \in X_1$ and $g \in \text{Stab}_a$, we have $g \circ f(a) = f \circ g(a) = f(a)$ hence $g \in \text{Stab}_{f(a)}$. Thus $\text{Stab}_a$ is a subgroup of $\text{Stab}_{f(a)}$, which is identified with a subgroup of the torus in a neighbourhood of $f(a)$. This proved that $\text{Stab}_a$ is identified with a subgroup of the torus.

We are left with showing that if $G$ acts strictly on $X$, then it acts strictly on $X_1$. Assume it is not the case. There exist two edges $\tau_1, \tau_2$ in $\Delta_1$, which are both edges of a cone, $\delta'$, and $g(\tau_1) = \tau_2$. We choose the cone $\delta'$ of minimal dimension. Clearly, $\tau_1$ and $\tau_2$ cannot be both edges in $\Delta$, since $G$ acts strictly on $X$. Let us assume $\tau_2$ is not an edge in $\Delta$. So $\tau_2$ must be in the interior of a cone $\delta$ in $\Delta$, which contains $\delta'$. Now since $\delta' \cap g(\delta') \supset \tau_2 \subset \text{interior of } \delta$, we conclude: interior of $\delta \cap g(\delta) : \neq \phi$, which means that $g(\delta) = \delta$. From the previous lemma, $g_\delta = id$, so $g_{\delta'} = id$ too, contradiction.

Proposition 2.3.

(1) There is a one to one correspondence between edges $\tau_i$ in the barycentric subdivision $B(\Delta)$ and positive dimensional cones $\delta_i$ in $\Delta$. We denote this by $\tau \mapsto \delta_{\tau}$.

(2) Let $\tau_i \neq \tau_j$ be edges of a cone $\hat{\delta} \in B(\Delta)$. Then $\dim \delta_{\tau_i} \neq \delta_{\tau_j}$.
(3) If $G$ is a finite group acting toroidally on a strict toroidal embedding $U \subset X$ with corresponding polyhedral complex $\Delta$, then the action of $G$ on $X_B(\Delta)$ is strict.

Remark. Using this proposition, the argument at the end of [8-dJ] can be significantly simplified: there is no need to show $G$-strictness of the toroidal embedding obtained there, since the barycentric subdivision automatically gives a $G$ strict modification.

Proof.  

1. Define a map $b : \text{positive dimensional cones in } \Delta \to \text{edges in } B(\Delta)$ by  
   
   \[ b(\delta) = \text{the barycenter of } (\delta) \]

   and define $\delta : \text{edges in } B(\Delta) \to \text{cones in } \Delta$ by  

   \[ \delta_\tau = \text{the unique cone whose interior contains } \tau \]

   then it is easy to see that $b$ and $\delta$ are inverses of each other.

2. We proceed by induction on $\dim \Delta$. The cone $\delta$ spanned by $\tau_i$ and $\tau_j$ must lie in some cone of $\Delta$, say $\delta^*$, which we may take of minimal dimension. We follow the second construction of the barycentric subdivision described in the preliminaries. Either $\dim \delta^* \leq m - 1$, so $\delta$ is in the barycentric subdivision of the $m - 1$-skeleton of $\Delta$, in which case the statement follows by the inductive assumption, or $\dim \delta^* = m$, in which case only one of $\tau_1$ and $\tau_2$ can be its barycenter, and the other is again a barycenter of a cone in the $m - 1$ skeleton.

3. From lemma 2.2, since the decomposition $B(\Delta)$ of $\Delta$ is equivariant, $G$ acts toroidally on $X_B(\Delta)$. Let $E_1, E_2 = g(E_1) \subset X_B(\Delta) \setminus U$ be divisors corresponding to edges $e_1, e_2$ in $B(\Delta)$. If $E_1 \cap E_2 \neq \emptyset$, there is a cone in $B(\Delta)$ containing $e_1, e_2$ as edges. From part (2), $\dim \delta_{e_1} \neq \dim \delta_{e_2}$, so $g(e_1)$ can not equal to $e_2$. This contradicts the fact that the morphism is equivariant and $g(E_1) = E_2$.

Proposition 2.4. There is a positive $G$-equivariant order function on $B(\Delta)$ such that the associated ideal $I$ induces a blowing up $B_I X_B(\Delta)$, which is a nonsingular $G$-strict toroidal embedding, on which $G$ acts toroidally.

Proof. By the previous proposition, we know that $G$ acts toroidally and strictly on $X_B(\Delta)$. It follows from Lemma 2.1 that the quotient $B(\Delta)/G$ is a conical polyhedral complex, since no cone has two distinct edges in $B(\Delta)$ which are identified in the quotient. We can use the argument of [KKMS], I §2, lemmas 1-3, to get an order function $\text{ord} : B(\Delta)/G \to \mathbb{R}$ which induces a simplicial subdivision with every cell of index 1. Denote by $\pi : B(\Delta) \to B(\Delta)/G$ the quotient map. Then $\text{ord} \circ \pi$ is an order function subdividing $B(\Delta)$ into simplicial cones of index 1. Let $I$ be the corresponding ideal sheaf. The blow up of $X_B(\Delta)$ along $I$ is a nonsingular strict toroidal embedding $U \subset B_I X_B(\Delta)$. By lemma 2.2, $G$ acts on $B_I X_B(\Delta)$ strictly and toroidally.
Proof of Theorem 0.2. Let $G \subset \text{Aut}(U \subset X)$ be as in the theorem. The morphism $X_{B(\Delta)} \to X$ is projective, and by the last two propositions there is a projective, toroidal $G$-equivariant morphism $Y \to X$ where $Y$ is nonsingular and such that $G$ acts strictly and toroidally on $Y$.

Remark. With a little more work we can obtain a canonical choice of a toroidal equivariant resolution of singularities. One observes that the cones in the barycentric subdivision have canonically ordered coordinates, which agree on intersecting cones: for a cone $\delta$ choose the unit coordinate vectors $e_i$ to be primitive lattice vectors generating the edges $\tau$, where $i = \dim \delta$, the dimension of the cone of which $\tau$ is a barycenter. Recall that in order to resolve singularities, one successively takes the subdivisions centered at lattice points $w_j$ which are not integrally generated by the vectors $e_i$. These $w_j$ are partially ordered according to the lexicographic ordering of their canonical coordinates, in such a way that if $w_j \neq w_k$ have the same coordinates (e.g. if $g(w_1) = w_2$), they do not lie in a the same cone, and therefore we can take the centered subdivision simultaneously.

We conclude this section with a simple proposition about quotients:

**Proposition 2.5.** Let $U \subset X$ be a strict toroidal embedding, and let $G \subset \text{Aut}(U \subset X)$ be a finite group acting strictly and toroidally. Then $(X/G, U/G)$ is a strict toroidal embedding.

**Proof.** Since the quotient of a toric variety by a finite subgroup of the torus is toric, we conclude that $X/G$ is still a toroidal embedding, by the definition of toroidal embedding. We need to show that it is strict. Let $q : X \to X/G$ be the quotient map. Let $Z \subset X \setminus U$ be an irreducible component. Then $q(Z) = \cup_g g(Z)$. Since the action is strict, we have $q(\cup_g g(Z)) \simeq Z/\text{Stab}(Z)$, which is normal.

3. Proof of theorem 0.1

Given $Z, X$ with $G$ action, $G$ finite, we may blow up $Z$ and therefore we might as well assume that $Z$ is a divisor. Let $Y = X/G$, $Z/G$ be the quotient, $B$ the branch locus. Define $W = B \cup Z/G$. Let $(Y', W') \to (Y, W)$ be a resolution of singularities of $Y$ with $W'$ a strict divisor of normal crossings. Let $X'$ be the normalization of $Y'$ in $K(X)$, and $Z'$ the inverse image of $W'$. Let $U = X' \setminus Z'$. By Abhyankar’s lemma, clearly $U \subset X'$ is a strict toroidal embedding, on which $G$ acts toroidally (moreover, it is $G$-strict). Applying theorem 0.2 we obtain a nonsingular strict toroidal embedding $U \subset X_1 \to X'$ as required.

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