ON CLASSIFICATION OF DYNAMICAL $r$-MATRICES

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ABSTRACT. Using the gauge transformations of the Classical Dynamical Yang-Baxter Equation introduced by P. Etingof and A. Varchenko in [EV], we reduce the classification of dynamical $r$-matrices $r$ on a commutative subalgebra $\mathfrak{i}$ of a Lie algebra $\mathfrak{g}$ to a purely algebraic problem, under some assumption on the symmetric part of $r$. We then describe, for a simple complex Lie algebra $\mathfrak{g}$, all non skew-symmetric dynamical $r$-matrices on a commutative subalgebra $\mathfrak{i} \subset \mathfrak{g}$ which contains a regular semisimple element. This interpolates results of P. Etingof and A. Varchenko ([EV], when $\mathfrak{i}$ is a Cartan subalgebra) and results of A. Belavin and V. Drinfeld for constant $r$-matrices ([BD]). This classification is similar, and in some sense simpler than the Belavin-Drinfeld classification.

1. The classical Yang-Baxter equation

Let $\mathfrak{g}$ be a Lie algebra. The CYBE is the following algebraic equation for an element $r \in \mathfrak{g} \otimes \mathfrak{g}$:

\[
[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0.
\]

(1)

Solutions of this equation are called $r$-matrices. In the theory of quantum groups, one is mainly interested in $r$-matrices satisfying

\[
r + r^{21} \in (S^2 \mathfrak{g})^\mathfrak{g}.
\]

(2)

See [CP] for the links with the theory of quantum groups, and [Che] for links with Conformal Field Theory and the Wess-Zumino-Witten model on $\mathbb{P}^1$. The geometric interpretation of the CYBE was given by Drinfeld in terms of Poisson-Lie groups ([Dr1]).

2. The Belavin-Drinfeld classification

Notations. Let $\mathfrak{g}$ be a simple complex Lie algebra with a nondegenerate invariant form $(\ , \ )$, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra and $\Delta$ the root system. For $\alpha \in \Delta$, let $\mathfrak{g}_\alpha$ denote the root subspace associated to $\alpha$. Let $W$ be the Weyl group and $s_\alpha$, $\alpha \in \Delta$ the reflection with respect to $\alpha^\perp$. Finally, let $\Omega \in S^2 \mathfrak{g}$ and $\Omega_\mathfrak{h} \in S^2 \mathfrak{h}$ be the inverse elements to the form $(\ , \ )$. Notice that $(S^2 \mathfrak{g})^\mathfrak{g} = C\Omega$.

For any polarization $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, we denote by $\Pi$ or $\Pi(\mathfrak{n}_\pm)$ the corresponding set of simple positive roots, by $\Delta_+$ the set of positive roots and by $\mathfrak{b}_+ = \mathfrak{n}_+ \oplus \mathfrak{h}$ the Borel subalgebras. For $\Gamma \subset \Pi$, set $(\Gamma) = \mathbb{Z}\Gamma \cap \Delta$, and let $\mathfrak{g}_\Gamma$ be the subalgebra generated by $\mathfrak{g}_\alpha$, $\alpha \in (\Gamma)$. We will write $\mathfrak{g}_\Gamma = \mathfrak{n}_+(\Gamma) \oplus \mathfrak{h}(\Gamma) \oplus \mathfrak{n}_-(\Gamma)$.
for the induced polarization and $W(\Gamma)$ for the subgroup of $W$ generated by $s_{a}$, $a \in \Gamma$. Let us fix a polarization of $\mathfrak{g}$.

**Definition.** A Belavin-Drinfeld triple is a triple $(\Gamma_1, \Gamma_2, \tau)$ where $\Gamma_1, \Gamma_2 \subset \Pi$ and $\tau : \Gamma_1 \rightarrow \Gamma_2$ is a norm-preserving bijection satisfying the following “nilpotency” condition:

“For any $\gamma_1 \in \Gamma_1$, there exists $n > 0$ such that $\tau^n(\gamma_1) \in \Gamma_2 \setminus \Gamma_1$.”

Let $(\Gamma_1, \Gamma_2, \tau)$ be a Belavin-Drinfeld triple. For each choice of Chevalley generators $(e_{\alpha}, f_{\alpha}, h_{\alpha})_{\alpha \in \Gamma_i}$, $i = 1, 2$, the isomorphism $\tau$ induces a Lie algebra isomorphism $\mathfrak{g}_{\Gamma_1} \rightarrow \mathfrak{g}_{\Gamma_2}$ (by $e_{\alpha} \mapsto e_{\tau(\alpha)}$, $f_{\alpha} \mapsto f_{\tau(\alpha)}$, $h_{\alpha} \mapsto h_{\tau(\alpha)}$). Define a partial order on $\Delta_+$ by setting $\alpha < \beta$ if there exists $n > 0$ such that $\tau^n(\alpha) = \beta$ (in particular, $\alpha \in \Gamma_1$ and $\beta \in \Gamma_2$).

**Definition.** A basis $(x_{\alpha})_{\alpha \in \Delta}$ of $n_+ \oplus n_-$ is called admissible if $(x_{\alpha}, x_{-\alpha}) = 1$ and $\tau(x_{\alpha}) = x_{\tau(\alpha)}$ for $\alpha \in \langle \Gamma_1 \rangle$.

**Theorem 1 (Belavin-Drinfeld).** Let $\mathfrak{g}$ be a simple complex Lie algebra.

1. Let $(\Gamma_1, \Gamma_2, \tau)$ be a Belavin-Drinfeld triple, $(x_{\alpha})$ an admissible basis, and let $r_0 \in \mathfrak{h} \otimes \mathfrak{h}$ be such that

\[ r_0 + r_0^{21} = \Omega_{\mathfrak{h}}, \]

\[ (\tau(\alpha) \otimes 1)r + (1 \otimes \alpha)r = 0 \quad \text{for} \quad \alpha \in \Gamma_1. \]

Then

\[ r = r_0 + \sum_{\alpha \in \Delta_+} x_{-\alpha} \otimes x_{\alpha} + \sum_{\alpha, \beta \in \Delta_+, \alpha < \beta} x_{-\alpha} \wedge x_{\beta} \]

is an r-matrix satisfying $r + r^{21} = \Omega$.

2. Any r-matrix satisfying $r + r^{21} = \Omega$ is of the above type for a suitable polarization of $\mathfrak{g}$.

This theorem is proved in [BD]. For instance, the standard r-matrix for a fixed polarization $r = \frac{\Omega_{\mathfrak{h}}}{2} + \sum_{\alpha \in \Delta_+} x_{-\alpha} \otimes x_{\alpha}$ corresponds to $\Gamma_1 = \Gamma_2 = \emptyset$.

**Remark.** Skew-symmetric r-matrices admit a well known interpretation in terms of nondegenerate 2-cocycles on Lie subalgebras of $\mathfrak{g}$ ([Dr1]), but their classification is unavailable since it requires a classification of Lie subalgebras in $\mathfrak{g}$.

**3. The dynamical Yang-Baxter equation**

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$ and $\mathfrak{l} \subset \mathfrak{g}$ a subalgebra. An element $x \in \mathfrak{g} \otimes \mathfrak{g}$ will be called $\mathfrak{l}$-invariant if

\[ [k \otimes 1 + 1 \otimes k, x] = 0 \quad (\forall k \in \mathfrak{l}). \]
For \( x \in g^\otimes 3 \), we let \( \text{Alt}(x) = x^{123} + x^{231} + x^{312} \). Let \( D \subset \mathfrak{l}^* \) be any open region.

The CDYBE is the following differential equation for a holomorphic \( \mathfrak{l} \)-invariant function \( r : D \to g \otimes g \):

\[
\text{Alt}(dr) + [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0,
\]

(7)

where the differential of \( r \) is considered as a holomorphic function

\[
dr : D \to g \otimes g \otimes g, \quad \lambda \mapsto \sum_i x_i \otimes \frac{\partial r^{23}}{\partial x_i}(\lambda), \quad (\lambda \in \mathfrak{l}^*),
\]

for any basis \((x_i)\) of \( \mathfrak{l} \). In this case,

\[
\text{Alt}(dr) = \sum_i x_i^{(1)} \frac{\partial r^{23}}{\partial x_i} + \sum_i x_i^{(2)} \frac{\partial r^{31}}{\partial x_i} + \sum_i x_i^{(3)} \frac{\partial r^{12}}{\partial x_i}.
\]

The solutions to this equation are called dynamical \( r \)-matrices. Dynamical \( r \)-matrices which are relevant to the theory of quantum groups are those satisfying the following condition, analogous to (2):

\[
\text{Generalized unitarity: } r(\lambda) + r^{21}(\lambda) \in (S^2 g)^g.
\]

(8)

**Remark.** The CDYBE was first written down by G. Felder and C. Wiezcerkowski in connection with the Wess-Zumino-Witten model on elliptic curves ([FW]). The relation with elliptic quantum groups is explained in [Fe]. A geometric interpretation of the CDYBE analogous to the theory of Poisson-Lie groups for the CYBE is given in [EV].

### 4. Gauge transformations

We recall some results from [EV]. We suppose here that \( \mathfrak{l} \) is commutative and we let \( D \) be the formal polydisc centered at the origin. Let \( G \) be a complex Lie group such that \( \text{Lie}(G) = g \), and let \( L \) be the connected subgroup of \( G \) such that \( \text{Lie}(L) = \mathfrak{l} \). Let \( G^L \) be the centralizer of \( L \) in \( G \) and \( g^L \) its Lie algebra. We will denote by \((g \otimes g)^L\) the space of all \( \mathfrak{l} \)-invariant elements in \( g \otimes g \).

Let \( g : D \to G^L \) be any holomorphic function; the 1-form \( \eta = g^{-1}dg \) gives rise to a function \( \overline{\eta} : D \to \mathfrak{l} \otimes g^L \). If \( r : D \to (g \otimes g)^L \) is an \( \mathfrak{l} \)-invariant function satisfying (8), we set

\[
r^g = (g \otimes g)(r - \overline{\eta} + \overline{\eta}^{21})(g^{-1} \otimes g^{-1}).
\]

**Proposition 1.** The function \( r \) is a dynamical \( r \)-matrix if and only if the function \( r^g \) is.

Thus the group \( \text{Map}(D, G^L) \) is a gauge transformation group for the CDYBE. Notice that this group is not commutative if \( G^L \) isn’t.

**Theorem 2.** Let \( \rho, r : D \to g^\otimes 2 \) be two dynamical \( r \)-matrices satisfying (8) such that \( r(0) = \rho(0) \). There exists \( g \in \text{Map}(D, G^L) \) such that \( \rho = r^g \).
This shows that the space of dynamical r-matrices is, up to gauge equivalence, finite dimensional. Proofs of the above results can be found in [EV].

We will now prove a converse of Theorem 2 which reduces the CDYBE to a purely algebraic equation under some assumption on the symmetric part $\Omega$ of $r$: let $\Omega \in (S^2 g)^g$, let $g\Omega$ be the ideal in $g$ generated by the components of $\Omega$ and denote by $g\Omega = \bigoplus_{\lambda} g\Omega(\lambda)$ the generalized weight space decomposition of $g\Omega$ with respect to the adjoint action of $l$. The condition we will need is the following:

\[ (*) \quad g^l \text{ acts semisimply on } g\Omega(0). \]

Suppose that (*) is fulfilled and let $z(g^l)$ denote the center of $g^l$. Then we have a decomposition $g\Omega(0) = z_0(g^l) \oplus V$ where $z_0(g^l) = z(g^l) \cap g\Omega(0)$ and $V$ is the sum of all non-trivial irreducible $g^l$-modules in $g\Omega(0)$. It is clear that $l \cap V = \{0\}$. We will say that a complement $l'$ of $l$ in $g$ is admissible if $V \subset l'$, and write $\pi : g \to l$ for the projection along $l'$. Notice that by $g^l$-invariance of $\Omega$,

\[ (9) \quad \Omega \in z^2_0(g^l) \oplus S^2 V \oplus \bigoplus_{\lambda \neq 0} g\Omega(\lambda) \otimes g\Omega(-\lambda). \]

We will denote by $CYB : g^2 \to g^3$ the map:

\[ r \mapsto [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]. \]

It is more convenient to work with the skew-symmetric part of $r$. If $r(\lambda) + r^{21}(\lambda) = \Omega \in (S^2(g))^g$, we set $s(\lambda) = r(\lambda) - \frac{\Omega}{2}$. It is easy to see that the CDYBE for $r$ is equivalent to the following equation for $s$:

\[ (10) \quad \text{Alt}(ds) + CYB(s) + \frac{1}{4} CYB(\Omega) = 0. \]

Recall that as $\Omega$ is symmetric and invariant, $CYB(\Omega) = [\Omega_{13}, \Omega_{23}]$.

**Theorem 3.** Let $G$ be a complex Lie group and $L \subset G$ a connected commutative subgroup. Let $g, l, g^l$ denote the Lie algebras of $G, L$ and $G^L$. Let $\Omega \in (S^2 g)^g$. Then

1. Let $l'$ be any complement of $l$ in $g$. Any dynamical r-matrix $r(\lambda)$ on $l$ such that $r(\lambda) + r^{21}(\lambda) = \Omega$ is gauge equivalent to a dynamical r-matrix $\tilde{r}(\lambda)$ such that $\tilde{r}(0) \in \Omega_g + (\Lambda^2(l'))^l$.

2. Suppose that condition (*) is true and let $l'$ be any admissible complement of $l$ in $g$. Let $r_0 \in \frac{\Omega}{2} + (\Lambda^2(l'))^l$ satisfy
(11) \[ CYB(r_0) \in \text{Alt}(l \otimes g \otimes g), \]
such that \( s_0 = r_0 - \Omega/2 \) is a regular point of the algebraic manifold
\[ M_\Omega = \{ s \in (\Lambda^2(l'))^1 | CYB(s + \Omega/2) \in \text{Alt}(l \otimes g \otimes g) \}. \]

Then there exists a dynamical r-matrix \( r(\lambda) : D \to \Omega + \Lambda^2(l')^1 \) such that \( r(0) = r_0 \).

The condition (*) is satisfied in the following two interesting special cases: when \( \Omega = 0 \) (triangular case) or when \( g_l \) acts semisimply on \( g \) (for instance, \( G \) is reductive and \( L \) is contained in a maximal torus of \( G \) or more generally, if \( G_L \) is reductive).

The proof of this theorem will occupy the rest of this section. Let us first prove part 1:

**Lemma 1.** Any dynamical r-matrix such that \( r(\lambda) + r^{21}(\lambda) = \Omega \) is gauge-equivalent to a dynamical r-matrix \( \tilde{r}(\lambda) \) such that \( \tilde{r}(0) \in \Omega + \Lambda^2(l')^1 \).

**Proof.** Let \( \eta \in l \otimes g_l^1 \) be such that \( r(0) - \eta + \eta^{21} \in \Omega + \Lambda^2(l')^1 \). There exists a function \( g : D \to G_L^1 \) such that \( g^{-1}dg(0) = \eta \) (see [EV], Lemma 1.3). It is easy to see that \( \tilde{r} = r^g \) satisfies the desired conditions. \( \square \)

Let us now prove part 2. We will interpret the CDYBE (10) as a consistent system of differential equations defined on \( M_\Omega \).

For \( s \in M_\Omega \), (10) is equivalent to

\[ (\pi \otimes 1 \otimes 1) \text{Alt}(ds) = -(\pi \otimes 1 \otimes 1)(CYB(s) + \frac{1}{4}CYB(\Omega)). \]

This reduces to

(12) \[ ds = -(\pi \otimes 1 \otimes 1)([s^{12}, s^{13}] + \frac{1}{4}CYB(\Omega)), \]
or, in coordinates \( (x_i) \), where \( (x_i) \) is a basis of \( l \),

\[ \frac{\partial s}{\partial x_i} = -(x_i \otimes 1 \otimes 1)([s^{12}, s^{13}] + \frac{1}{4}CYB(\Omega)). \]

**Lemma 2.** The system (12) is consistent.
Lemma 3.

Proof. Set \( X : M_\Omega \to I \otimes g \otimes g \), \( s \mapsto (\pi \otimes 1 \otimes 1)([s^{12}, s^{13}] + \frac{1}{4} CYB(\Omega)) \). By definition, the curvature of (12) is given by

\[
\sum_{i,j} x_i \otimes x_j \otimes \left( \frac{\partial^2 s}{\partial x_i \partial x_j} - \frac{\partial^2 s}{\partial x_j \partial x_i} \right)
= (\pi \otimes \pi \otimes 1 \otimes 1) \left( \left\{ [s^{23}, [s^{12}, s^{14}]] + [s^{23}, \frac{1}{4} CYB(\Omega)^{124}] + [[s^{12}, s^{13}], s^{24}] + \left[ \frac{1}{4} CYB(\Omega)^{123}, s^{24} \right] } \right.
- \left\{ [s^{13}, [s^{21}, s^{24}]] + [s^{13}, \frac{1}{4} CYB(\Omega)^{214}] + [[s^{21}, s^{23}], s^{14}] + \left[ \frac{1}{4} CYB(\Omega)^{213}, s^{14} \right] \right\}
= (\pi \otimes \pi \otimes 1 \otimes 1) \left( \left\{ [s^{23}, [s^{12}, s^{14}]] + [[s^{12}, s^{13}], s^{24}] - [s^{13}, [s^{21}, s^{24}]] - [[s^{21}, s^{23}], s^{14}] \right\}
+ \frac{1}{4} \left\{ [s^{13} + s^{23}, CYB(\Omega)^{124}] - [s^{14} + s^{24}, CYB(\Omega)^{123}] \right\} \right).
\]

By the Jacobi identity,

\[
[s^{23}, [s^{12}, s^{14}]] = [[s^{21}, s^{23}], s^{14}], \quad [[s^{12}, s^{13}], s^{24}] = [s^{13}, [s^{21}, s^{24}]].
\]

By \( g \)-invariance of \( CYB(\Omega) \), we have

\[
[s^{13} + s^{23}, CYB(\Omega)^{124}] = [s^{34}, CYB(\Omega)^{124}],
[s^{14} + s^{24}, CYB(\Omega)^{123}] = -[s^{34}, CYB(\Omega)^{123}].
\]

Overall, we have the following expression for the curvature of (12):

\[
\frac{1}{4} (\pi \otimes \pi \otimes 1 \otimes 1)([CYB(\Omega)^{123} + CYB(\Omega)^{124}, s^{24}] - \frac{1}{4} ([\pi \otimes \pi \otimes 1) CYB(\Omega), s])
\]

But (9) and the fact that \( l' \) is admissible imply that \( (\pi \otimes \pi \otimes 1) CYB(\Omega) = 0 \). Thus, (12) is consistent.

Lemma 3. The system (12) is defined on \( M_\Omega \), i.e the vector fields defined by (12) are tangent to \( M_\Omega \).

Proof. Let \( x^* \in \mathfrak{g}^* \), and set \( h = (x^* \otimes 1 \otimes 1) ([s^{12}, s^{13}] + \frac{1}{4} CYB(\Omega)) \). Since \( s \in \Lambda^2(\mathfrak{l}') \) we have \( (x^* \otimes 1 \otimes 1) [s^{12}, s^{13}] \in \Lambda^2(\mathfrak{l}') \). Moreover, the admissibility of \( \mathfrak{l}' \) and (9) together imply that \( (x^* \otimes 1 \otimes 1)(CYB(\Omega)) \in (\Lambda^2 \mathfrak{l}')^l \) since \( [l \otimes 1, S^2 z_0(\mathfrak{g}')] = 0 \). Thus \( h \in \Lambda^2 \mathfrak{l}' \).

To conclude the proof of Lemma 3 and Theorem 3, we now show that

\[
[s^{12}, h^{13}] + [s^{12}, h^{23}] + [s^{13}, h^{23}]
+ [h^{12}, s^{13}] + [h^{12}, s^{23}] + [h^{13}, s^{23}] \in \text{Alt}(I \otimes g \otimes g).
\]
To make the presentation more clear, we will use the pictorial technique to represent expressions and make computations: we associate to each morphism from a \( n \)-tensor to a \( m \)-tensor a diagram in the following way: the operation of taking the commutator is represented by

\[
\begin{array}{c}
\text{a} \\
\Downarrow
\end{array}
\begin{array}{c}
\text{b} \\
\text{[a,b]}
\end{array}
\]

Applying a linear form \( x^* \) will be denoted by

\[
\begin{array}{c}
\text{a} \\
\Downarrow
\end{array}
\begin{array}{c}
\text{x*} \quad x^*(a)
\end{array}
\]

Finally, we will represent \( s \) and \( \Omega/2 \), which can be thought of as maps from a 0-tensor to a 2-tensor, by

\[
\begin{array}{c}
\Omega/2 \\
\Downarrow
\end{array}
\]

For instance,

\[
\text{CYB}(s) = + + +
\]

**Lemma 4.** We have \( x^{*(3)}[CYB(s + \Omega/2)^{123}, s^{34}] \in \text{Alt}(l \otimes g \otimes g) \) or, in pictures (modulo \( \text{Alt}(l \otimes g \otimes g) \))

\[
\begin{array}{c}
\Downarrow
\end{array}
\begin{array}{c}
\text{CYB}(s) \quad = \quad + \quad + \quad +
\end{array}
\]

**Proof.** Recall that \( CYB(s + \Omega/2) \in \text{Alt}(l \otimes g \otimes g) \). Thus the only part of the above expression which can lie outside of \( \text{Alt}(l \otimes g \otimes g) \) is obtained from the \( g \otimes g \otimes l \)-part of \( CYB(s) \). But if \( y \in l \),

\[
(x^* \otimes 1)[y \otimes 1, s] = -(x^* \otimes 1)[1 \otimes y, s]
\]
by $t$-invariance of $s$. This last expression is zero since $s \in (\Lambda^2(l'))^t$. Lemma 4 is proved.

It is clear how to generalize Lemma 4 to other expressions of the form

$$x^{s(k)}[CYB(s + \Omega/2)^{123}, s^{k4}].$$

Now, (13) can be drawn as

![Diagram of expressions](attachment:image.png)
but by Lemma (4) we have, modulo $\text{Alt}(l \otimes g \otimes g)$,

$$
\begin{align*}
&\quad + \quad = \quad + \\
&\quad + \quad = \quad + \\
&\quad + \quad = - \quad - 
\end{align*}
$$

It is easy to check that the sum of the terms of type $[\text{CYB}(s), s]$ in this last expression is zero by the Jacobi identity. Moreover, by $g$-invariance of $\Omega$, we have

$$
\begin{align*}
&\quad = - \\
&\quad = \\
&\quad = 
\end{align*}
$$
Thus, modulo $\text{Alt}(\mathfrak{l} \otimes \mathfrak{g} \otimes \mathfrak{g})$, (13) reduces to

\[
\begin{array}{ccc}
\begin{array}{c}
\end{array} & + & \begin{array}{c}
\end{array} & + & \begin{array}{c}
\end{array} \\
\begin{array}{c}
\end{array} & + & \begin{array}{c}
\end{array} & + & \begin{array}{c}
\end{array} \\
\begin{array}{c}
\end{array} & + & \begin{array}{c}
\end{array} & - & \begin{array}{c}
\end{array} \\
\end{array}
\]

The sums of terms in each column is zero by Jacobi Identity. This concludes the proof of Theorem 3.

\[\Box\]

5. Classification of dynamical $r$-matrices

Let $\mathfrak{g}$ be a simple Lie algebra and let $\Omega \in (S^2\mathfrak{g})^\mathfrak{g}$ be the Casimir element. In that case, (8) becomes

\[(14) \quad r(\lambda) + r^{21}(\lambda) = \epsilon \Omega.\]

We will classify all solutions of equations (6,7,14) when $\epsilon \neq 0$ and when $\mathfrak{l}$ contains a semisimple regular element. In particular, in this case, the centralizer $\mathfrak{h}$ of $\mathfrak{l}$ is the unique Cartan subalgebra containing $\mathfrak{l}$. Notice that we can assume that $\epsilon = 1$ (since the assignment $r(\lambda) \rightarrow \epsilon r(\epsilon \lambda)$ is a gauge transformation of (7)). We can also assume that the restriction of $(\ ,\ )$ to $\mathfrak{l}$ is nondegenerate. Indeed, for any dynamical $r$-matrix, we can replace $\mathfrak{l}$ by the largest subspace of $\mathfrak{h}$ for which $r$ is invariant, and such a subspace is real. Let $\mathfrak{h}_0$ be the orthogonal complement of $\mathfrak{l}$ in $\mathfrak{h}$ and let $i : \mathfrak{l} \hookrightarrow \mathfrak{h}$ be the inclusion map. We will also write $(\ ,\ )$ for the induced bracket on $\mathfrak{l}^*$. Let $\Omega_{\mathfrak{h}_0}$ denote the Casimir element of the restriction of $(\ ,\ )$ to $\mathfrak{h}_0$. 
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5.1. Statement of the theorem. Let $g = n_+ \oplus h \oplus n_-$ be a polarization of $g$.

Definition. A generalized Belavin-Drinfeld triple is a triple $(\Gamma_1, \Gamma_2, \tau)$ where $\Gamma_1, \Gamma_2 \subset \Pi$, and $\tau : \Gamma_1 \rightarrow \Gamma_2$ is a norm-preserving bijection.

In other terms, in a generalized Belavin-Drinfeld triple, we drop the nilpotency condition. We will say that a generalized Belavin-Drinfeld triple is $l$-graded if $\tau$ preserves the decomposition of $g$ in $l$-weight spaces. If $(\Gamma_1, \Gamma_2, \tau)$ is a generalized Belavin-Drinfeld triple, we will denote by $\tilde{\Gamma}_1$ the largest subset of $\Gamma_1 \cap \Gamma_2$ which is stable under $\tau$, and $\tilde{\Gamma}_2 = \Gamma_2 \setminus \Gamma_3$. It is clear that $(\tilde{\Gamma}_1, \tilde{\Gamma}_2, \tau)$ is a Belavin-Drinfeld triple. As before, for each choice of Chevalley generators $(e_\alpha, f_\alpha, h_\alpha)_{\alpha \in \Gamma_i}$, the map $\tau$ induces isomorphisms $g_{\tilde{\Gamma}_1} \rightarrow g_{\tilde{\Gamma}_2}$ and $\tau : g_{\Gamma_3} \rightarrow g_{\Gamma_3}$.

For $\lambda \in \mathfrak{l}^*$, consider the map:

$$ K(\lambda) : n_+(\Gamma_1) \rightarrow n_+(\Gamma_2) $$

$$ e_\alpha \mapsto \frac{1}{2} e_\alpha + e^{-(\alpha, \lambda)}_{\tau} \frac{\tau(e_\alpha)}{1 - e^{-(\alpha, \lambda)}_{\tau}}. $$

Notice that we have

$$ K(\lambda)(e_\alpha) = \frac{1}{2} e_\alpha + \sum_{n>0} e^{-n(\alpha, \lambda)}_{\tau} e_\alpha. $$

This sum is finite for $\alpha \notin \langle \Gamma_3 \rangle$.

Theorem 4. Let $g$ be a simple Lie algebra with nondegenerate invariant bilinear form $(.,.)$, $l \subset g$ a commutative subalgebra containing a regular semisimple element on which $(.,.)$ is nondegenerate, $h$ the Cartan subalgebra containing $l$ and $h_0$ the orthogonal complement of $l$ in $h$. Then

1. Any dynamical $r$-matrix is gauge-equivalent to a dynamical $r$-matrix $\tilde{r}$ such that

$$ \tilde{r}(\lambda) - \tilde{r}(\lambda)^{21} \in (l^1)^{\otimes 2} = (\bigoplus_{\alpha \neq 0} g_\alpha \oplus h_0)^{\otimes 2}. $$

2. Let $(\Gamma_1, \Gamma_2, \tau)$ be an $l$-graded generalized Belavin-Drinfeld triple and let $(e_\alpha, f_\alpha, h_\alpha)_{\Gamma_i}$ be a choice of Chevalley generators. Let $r_{h_0, h_0}$ be in $h_0 \otimes h_0$ satisfy the equation

$$ (\tau(\alpha) \otimes 1)r_{h_0, h_0} + (1 \otimes \alpha)r_{h_0, h_0} = \frac{1}{2}((\alpha + \tau(\alpha)) \otimes 1)\Omega_{h_0}. $$

Then

$$ r(\lambda) = \frac{1}{2} \Omega + r_{h_0, h_0} + \sum_{\alpha \in (\Gamma_1) \cap \Delta_+} K(\lambda)(e_\alpha) \wedge e_{-\alpha} + \sum_{\alpha \in \Delta_+, \alpha \notin \langle \Gamma_1 \rangle} \frac{1}{2} e_\alpha \wedge e_{-\alpha} $$

is a solution the CDYBE satisfying (15).

3. Any solution of the CDYBE satisfying (15) is of the above type for a suitable polarization of $g$.
The proof of this theorem will occupy the rest of this section. Our methods are greatly inspired by the paper [BD]. Notice that 1. follows from Theorem 3, but we will describe the gauge transformations explicitly in this case.

**Notations.** Let \( \Delta \subset \mathfrak{h}^* \) be the root system of \( \mathfrak{g} \) with respect to \( \mathfrak{h} \) and set \( \Delta_1 = i^*(\Delta) \subset \mathfrak{l}^* \). We will denote by \( \mathfrak{g}_\alpha \) the weight subspace associated to \( \tilde{\alpha} = i^*(\alpha) \in \Delta_1 \), and we set \( \mathfrak{g}\pi = \mathfrak{h}_0 \). It is clear that

\[
\mathfrak{g}\pi = \bigoplus_{\beta \in \Delta, \; i^*(\beta) = \pi} \mathfrak{g}\beta
\]

In particular, \( (\ , \ ) \) is a pairing \( \mathfrak{g}\pi \times \mathfrak{g}_{-\pi} \rightarrow \mathbb{C} \).

A vector space \( V \subset \mathfrak{g} \) will be called \( \mathfrak{h} \)-graded (resp. \( l \)-graded) if it is an \( \mathfrak{h} \)-submodule (resp. \( \mathfrak{l} \)-submodule) of \( \mathfrak{g} \). Finally, let \( \Omega' \in (\mathfrak{l}^*)^2 \) denote the Casimir (inverse element) of the restriction of \( (\ , \ ) \) to \( \mathfrak{l}^2 = \mathfrak{h}_0 \bigoplus \mathfrak{g}_{\pi} \).

Now let \( r : \mathfrak{l}^{\ast} \supset D \rightarrow (\mathfrak{g} \otimes \mathfrak{g})^l \) be a formal power series satisfying (14) (with \( \epsilon = 1 \)). By (6), we can write

\[
r(\lambda) = \frac{1}{2} \Omega + r_{l,l}(\lambda) + r_{l,0}(\lambda) + r_{0,l}(\lambda) + (\varphi(\lambda) \otimes 1)\Omega',
\]

where \( r_{l,l}(\lambda) \in l \otimes l \), \( r_{l,0}(\lambda) \in l \otimes \mathfrak{h}_0 \), \( r_{0,l}(\lambda) \in \mathfrak{h}_0 \otimes l \) and where \( \varphi(\lambda) \in \text{End}(\mathfrak{h}_0 \bigoplus \mathfrak{g}_{\pi}) \) is a sum of maps \( \varphi_{\pi}(\lambda) \in \text{End}(\mathfrak{g}_{\pi}) \). By the unitarity condition, \( r_{l,l}(\lambda) \in \Lambda^2 l \), \( r_{l,0}(\lambda) = -r_{0,l}(\lambda) \) and \( \varphi_{-\pi}(\lambda) = -\varphi^*(\lambda) \).

With these notations, the CDYBE splits into 4 components: the \( l \otimes l \otimes l \)-part, the \( l \otimes l \otimes \mathfrak{h}_0 \)-part, the \( l \otimes \mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha} \)-part and the \( \mathfrak{g}_\alpha \otimes \mathfrak{g}_\beta \otimes \mathfrak{g}_\gamma \)-part where \( \tilde{\alpha} + \tilde{\beta} + \tilde{\gamma} = 0 \).

- The \( l \otimes l \otimes l \)-part: let us set \( r_{l,l} = \sum_{i,j} C_{i,j}(\lambda)x_i \otimes x_j \). This part of the CDYBE can then be written:

\[
\frac{\partial C_{i,k}}{\partial x_i} + \frac{\partial C_{k,i}}{\partial x_j} + \frac{\partial C_{i,j}}{\partial x_k} = 0 \quad \forall i, j, k
\]

and says that \( \sum_{i,j} C_{i,j}dx_i \wedge dx_j \) is a closed 2-form.

- The \( l \otimes l \otimes \mathfrak{h}_0 \)-part: let us set \( r_{l,0} = \sum_{i,j} D_{i,j}(\lambda)x_i \otimes y_j \) for some basis \( (y_j) \) of \( \mathfrak{h}_0 \). This part of the CDYBE is

\[
\frac{\partial D_{i,j}}{\partial x_k} = \frac{\partial D_{k,j}}{\partial x_i} \quad \forall i, k, j
\]

and says that for any \( j \), \( \sum_i D_{i,j}(\lambda)dx_i \) is a closed 1-form.

Since \( r \) is defined on a polydisc, the above forms are exact. Let \( f : D \rightarrow \mathfrak{h}_0 \) be such that \( df(\lambda) = \sum_i D_{i,j}(\lambda)dx_i \otimes y_j \) and let \( \xi \) be a 1-form on \( D \) such that \( d\xi = \sum_{i,j} C_{i,j}dx_i \wedge dx_j \). Then \( \xi \) defines a function \( \xi : D \rightarrow \mathfrak{l} \). The gauge transformation which should be applied to \( r \) to make it satisfy (15) is easily seen to be the following: \( r(\lambda) \mapsto r(\lambda)^g = \frac{1}{2} \Omega + (e^{-ad f(\lambda)}\varphi(\lambda)e^{ad f(\lambda)} \otimes 1)\Omega' \) where \( g(\lambda) = e^{f(\lambda)}e^{-\xi(\lambda)} \).
Thus, we can assume that \( r_{I,1} = r_{I,0} = 0 \), in which case the remaining components of the CDYBE can be written in the following way:

- The \( l \otimes g_{\alpha} \otimes g_{-\alpha} \)-part:

\[
(20) \quad d\varphi_{\alpha} + (\varphi_{\alpha}^2 - \frac{1}{4})dh_{\alpha} = 0.
\]

In particular, \( r_{b_0, b_0} \in \Lambda^2 h_0 \) is constant.

- The \( g_{\alpha} \otimes g_{\beta} \otimes g_{\gamma} \)-part where \( \alpha + \beta + \gamma = 0 \):

\[
(21) \quad \Lambda(\varphi_{\alpha} \otimes \varphi_{\beta} \otimes 1 + \varphi_{\alpha} \otimes 1 \otimes \varphi_{\gamma} + 1 \otimes \varphi_{\beta} \otimes \varphi_{\gamma} + \frac{1}{4}Id) = 0
\]

where \( \Lambda : g_{\alpha} \otimes g_{\beta} \otimes g_{\gamma} \to \mathbb{C}, x \otimes y \otimes z \mapsto ([x, y], z) \).

This set of equations is sufficient by skew-symmetry of the CDYBE.

### 5.2. The Cayley transform

Let us set \( A_{\pm} = \text{Im}(\varphi(\lambda) \pm \frac{1}{2}) \), \( I_{\pm} = \text{Ker}(\varphi(\lambda) \mp \frac{1}{2}) \). Notice that, by (20), \( A_{\pm} \) and \( I_{\pm} \) are indeed independent of \( \lambda \). Furthermore, \( A_{\pm}, I_{\pm} \) are l-graded by the weight-zero condition, \( I_{\pm} \subset A_{\pm} \) and \( A_{\pm} = I_{\pm}^l \) by the unitarity condition. Notice also that \( A_{+} + A_{-} \oplus l = g \). Now consider

\[
\psi(\lambda) = \frac{\varphi - \frac{1}{2}}{\varphi + \frac{1}{2}} : A_{+}/I_{+} \to A_{-}/I_{-}.
\]

Extend \( \psi(\lambda) \) to \( \tilde{\psi}(\lambda) : l \oplus A_{+}/I_{+} \to l \oplus A_{-}/I_{-} \) by setting \( \tilde{\psi}|_l = Id \). It is clear that \( \psi \) is a well-defined linear isomorphism. The following proposition is crucial:

**Proposition 2.** The maps \( \varphi_{\alpha} \) satisfy (20, 21) if and only if the following hold:

1. \( A_{\pm} \oplus l \) is a subalgebra of \( g \) and \( I_{\pm} \oplus l \) is an ideal of \( A_{\pm} \oplus l \).
2. There exists a (constant) map \( \psi_0 : l \oplus A_{+}/I_{+} \to l \oplus A_{-}/I_{-} \) such that \( \psi(\lambda)|_{g_{\alpha}} = e^{-\langle \alpha, \lambda \rangle}\psi_0|_{g_{\alpha}} \).
3. The map \( \psi_0 \) is a Lie algebra map:

\[
(22) \quad [\psi_0(x), \psi_0(y)] = \psi_0[x, y].
\]

**Proof.** Assume that \( \varphi \) satisfies (20, 21) and let \( a \in g_{\alpha}, b \in g_{\beta}, c \in g_{\gamma} \) with \( \alpha + \beta + \gamma = 0 \). From (21), we have

\[
([\varphi_{\alpha} + \frac{1}{2}]a, [\varphi_{\beta} + \frac{1}{2}]b, c) + ([a, [\varphi_{\beta} + \frac{1}{2}]b], [\varphi_{\gamma} - \frac{1}{2}]c) + ([\varphi_{\alpha} - \frac{1}{2}]a, b, [\varphi_{\gamma} - \frac{1}{2}]c) = 0.
\]

Since \( \varphi_{\gamma} = -\varphi_{-\gamma}^* \), and \( (, ) \) is a nondegenerate pairing \( g_{\gamma} \otimes g_{-\gamma} \to \mathbb{C} \), this implies that \( A_{+} \oplus l \) is a Lie subalgebra of \( g \). Note that the term in \( l \) is necessary here.
since \([\mathfrak g_{\alpha}, \mathfrak g_{-\alpha}] \not\subset \mathfrak g_{\Gamma} = \mathfrak h_0\), but is not consequential as \(A_+\) is \(t\)-graded. The second claim of (i) follows from the relation

\[
\left(\left(\varphi_{\alpha} - \frac{1}{2}\right)a, (\varphi_{\beta} - \frac{1}{2})b\right) + \left(\left(\varphi_{\alpha} + \frac{1}{2}\right)b\right), (\varphi_{\gamma} + \frac{1}{2})c) + \left(\left(\varphi_{\alpha} - \frac{1}{2}\right)a, (\varphi_{\gamma} + \frac{1}{2})c\right) = 0.
\]

The proof is the same for \(A_-\) and \(I_-\). The equivalence of (ii) and (20) follows from the equality

\[
d\psi|_{\mathfrak g_{\alpha}} = \frac{d\varphi_{\alpha}(\varphi_{\alpha} + \frac{1}{2}) - (\varphi_{\alpha} - \frac{1}{2})d\varphi_{\alpha}}{(\varphi_{\alpha} + \frac{1}{2})^2}
= -\frac{(\varphi_{\alpha}^2 - \frac{1}{4})}{(\varphi_{\alpha} + \frac{1}{2})^2}dh_{\alpha}
= -\langle\alpha, \lambda\rangle \psi|_{\mathfrak g_{\alpha}}.
\]

where we used (20). Finally it follows from (21) that

\[
(\varphi_{\alpha+\beta} - \frac{1}{2})\left(\left(\varphi_{\alpha} + \frac{1}{2}\right)a, (\varphi_{\beta} + \frac{1}{2})b\right) = (\varphi_{\alpha+\beta} + \frac{1}{2})\left(\left(\varphi_{\alpha} - \frac{1}{2}\right)a, (\varphi_{\beta} - \frac{1}{2})b\right).
\]

This implies (iii).

Conversely, if (i-iii) are satisfied then for any \(x \in \mathfrak g_{\alpha}, y \in \mathfrak g_{\beta} (\bar{\alpha} + \bar{\beta} \neq 0)\) there exist \(z \in \mathfrak g_{\alpha+\beta}\) such that

\[
\left((\varphi_{\alpha} - \frac{1}{2})x, (\varphi_{\beta} - \frac{1}{2})y\right) = (\varphi_{\alpha+\beta} - \frac{1}{2})z.
\]

Since \(\psi\) is a Lie algebra map, \([((\varphi_{\alpha} + \frac{1}{2})x, (\varphi_{\beta} + \frac{1}{2})y] - (\varphi_{\alpha+\beta} + \frac{1}{2})z \in \ker (\varphi_{\alpha+\beta} - \frac{1}{2})\). Subtracting, we obtain \([((\varphi_{\alpha} + \frac{1}{2})x, y] + [x, (\varphi_{\beta} + \frac{1}{2})y] - [x, y] - z = \ker (\varphi_{\alpha+\beta} - \frac{1}{2})\).

Applying \((\varphi - \frac{1}{2})\) and dropping the indices, we have

\[
(\varphi - \frac{1}{2})\left(\left((\varphi + \frac{1}{2})x, y\right) + [x, (\varphi + \frac{1}{2})y] - [x, y]\right) = [(\varphi - \frac{1}{2})x, (\varphi - \frac{1}{2})y].
\]

Thus,

\[
\left((\varphi + \frac{1}{2})x, (\varphi + \frac{1}{2})y\right) - (\varphi + \frac{1}{2})\left(\left((\varphi - \frac{1}{2})x, y\right) + [x, (\varphi - \frac{1}{2})y]\right) = 0.
\]

which is equivalent to (21).

We will call the triple \((A_+, A_-, \psi_0)\) the Cayley transform of \(\varphi\). We are now reduced to the classification of all triples satisfying (i-iii) and which arise as a Cayley transform (Cayley triples).

### 5.3. Classification of Cayley triples

Let \((A_+, A_-, \psi_0)\) be a Cayley triple. If \(\mathfrak g = \mathfrak n_+ \oplus \mathfrak h \oplus \mathfrak n_-\) is a polarization of \(\mathfrak g\) and \(\Gamma \subset \Pi(\mathfrak n_+)\) we will denote by \(\mathfrak q_{\Gamma}^+\) (resp. \(\mathfrak q_{\Gamma}^-\)) the subalgebra generated by \(\mathfrak n_+\) and \(\mathfrak g_{-\alpha}, \alpha \in \Gamma\) (resp. generated by \(\mathfrak n_-\) and \(\mathfrak g_{\alpha}, \alpha \in \Gamma\)). We denote by \(\mathfrak p_{\Gamma}^\pm = \mathfrak h + \mathfrak q_{\Gamma}^\pm\) the parabolic subalgebras associated to \(\Gamma\).
Proposition 3. There exists a polarization \( g = n_+^1 \oplus h \oplus n_-^1 \), two subsets \( \Gamma_+, \Gamma_- \subset \Pi(n_+^1) \) and two vector spaces \( V_+, V_- \subset h \) with \( V_+^1 \subset V_\pm \) such that
\[
I \oplus A_+ = q_{\Gamma_+}^+ \oplus V_+, \quad I \oplus A_- = q_{\Gamma_-}^- \oplus V_-.
\]

Proof. Notice that \( (I \oplus A_+)^\perp = I_+ \subset I \oplus A_+ \). It is known, (c.f [Bou, chap.VIII, §10, Thm. 1] or [BD]), that this implies that \( I \oplus A_+ = q_{\Gamma_+}^+ \oplus V_+ \) for some polarization \( g = n_+^1 \oplus h' \oplus n_-^1 \). Similarly, \( I \oplus A_- = q_{\Gamma_-}^- \oplus V_- \) for some polarization \( g = n_+^1 \oplus h'' \oplus n_-^1 \). Moreover, \( I \) acts semisimply on \( A_\pm \) so \( I \subset h' \), \( I \subset h'' \). But \( I \) contains a regular element, thus \( I = h' = h'' \). Proposition 3 is now an easy consequence of the following lemma:

Lemma 5. Let \( g \) be a simple Lie algebra and \( h \) a Cartan subalgebra. Let \( a_1 \) and \( a_2 \) be two parabolic subalgebras containing \( h \) such that \( a_1 + a_2 = g \). Then there exists a polarization \( g = n_+ \oplus h \oplus n_- \) and \( \Gamma_+, \Gamma_- \subset \Pi \) such that \( a_1 = p_{\Gamma_+}^1 \) and \( a_2 = p_{\Gamma_-}^1 \).

Proof. Let \( n_+ \oplus h \oplus n_- \) be a polarization of \( g \) such that \( b_+ \subset a_1 \) and for which \( \dim (n_+ \cap a_2) \) is minimal. We claim that \( b_- \subset a_2 \). Suppose on the contrary that there exists a simple root \( \alpha \in \Pi \) such that \( g_{-\alpha} \not\subset a_2 \). Then \( g_{-\alpha} \subset a_1 \) since \( a_1 + a_2 = g \) and \( g_{-\alpha} \subset a_2 \) since \( a_2 \) is parabolic. But then \( s_\alpha n_+ \oplus h \oplus s_\alpha n_- \) is a polarization of \( g \) for which \( s_\alpha b_+ \subset a_1 \) and \( \dim (s_\alpha n_+ \cap a_2) < \dim (n_+ \cap a_2) \). Contradiction. \( \square \)

In particular, \( A_\pm \), \( I_\pm \) are all \( h \)-graded and
\[
I_+ = (q_{\Gamma_+}^+ \oplus V_+)^\perp = \bigoplus_{\alpha \in \Delta_+ \setminus (\Gamma_+)} g_\alpha \oplus (V_+^\perp \cap \mathfrak{h}_0),
I_- = (q_{\Gamma_-}^- \oplus V_-)^\perp = \bigoplus_{\alpha \in \Delta_- \setminus (\Gamma_-)} g_\alpha \oplus (V_-^\perp \cap \mathfrak{h}_0).
\]

Thus \( A_+/I_+ = g_{\Gamma_+} \oplus V_1 \) and \( A_-/I_- = g_{\Gamma_-} \oplus V_2 \) for some suitable \( V_1, V_2 \subset \mathfrak{h}_0 \).

Let \( L_{\pm \frac{1}{2}}(\lambda) \) be the generalized eigenspace of \( \varphi(\lambda) \) associated to \( \pm \frac{1}{2} \). Since \( \varphi \) is a solution of an ordinary differential equation with stationary points at \( \pm \frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \), \( L_{\pm \frac{1}{2}}(\lambda) \) is independent of \( \lambda \) and we will simply denote it by \( L_{\pm \frac{1}{2}} \). Similarly, let \( L' \) be the sum of all other generalized eigenspaces so that \( g = I \oplus L_\frac{1}{2} \oplus L' \oplus L_{-\frac{1}{2}} \).

Proposition 4. There exists a polarization \( g = \bar{\Pi}_+ \oplus h \oplus \bar{\Pi}_- \) and a subset \( \Gamma_3 \subset \Pi(\bar{\Pi}_+) \) such that \( L_{\pm \frac{1}{2}} \subset \bar{\mathfrak{b}}_\pm \), \( L' \subset g_{\Gamma_3} + h \) and \( \varphi(\bar{\Pi}_+) \subset \bar{\Pi}_+ \).

Proof. We will construct a polarization satisfying the above conditions in several steps.

Lemma 6. We have:

(i) \( I \oplus L_{\pm \frac{1}{2}} \) is an \( h \)-graded solvable subalgebra,
(ii) \( I \oplus L' \) is an \( h \)-graded subalgebra,
(iii) we have \( [L_{\pm \frac{1}{2}}, L'] \subset I \oplus L_{\pm \frac{1}{2}} \).
Proof. This follows from the proofs of Lemma 12.3 and Theorem 12.6 in [BD].

Notice that $L_{\pm} \not\subset b_{\pm}^1$ in general. We first construct a polarization $g = n_+^2 \oplus h \oplus n_-^2$ such that $L_{\pm} \subset b_{\pm}^2$. We have $I_{\pm} \subset L_{\pm}$. Notice that $L_{\pm} \cap n_1^1 \subset \g_{r_+} \cap \g_{r_-} = \g_{r_+} \cap \g_{r_-}$ since $n_1^1 \subset (\g_{r_+} \cap I_+)$ and $L_{\pm}$ is solvable. Similarly, $L_{\pm} \cap n_1^1 \subset \g_{r_+} \cap \g_{r_-}$. Moreover, by Lemma 6, $i \oplus (L_{\pm} \cap \g_{r_+} \cap \g_{r_-})$ and $i \oplus (L_{\pm} \cap \g_{r_+} \cap \g_{r_-})$ are disjoint, solvable, $h$-graded subalgebras. By lemma 5 it follows that there exists an element $s$ of the group $W_{\g_{r_+} \cap \g_{r_-}}$ such that $i \oplus (L_{\pm} \cap \g_{r_+} \cap \g_{r_-}) \subset s b_{\pm}^1$. Notice that $s$ permutes elements of $\Delta^+ \setminus (\Gamma_+ \cap \Gamma_-)$, leaving it globally unchanged. Thus, $i \oplus L_{\pm} \subset s b_{\pm}^1$. Set $n_2^1 = s n_1^1$.

Now we construct a polarization of $g$ satisfying the other conditions of proposition 4. Recall that $i \oplus L \subset \g_{r_+} \cap \g_{r_-} \cap (V_1 \cap V_2)$. Thus $(L' \cap n_2^1) \oplus (L_{\pm} \cap n_2^1 (\Gamma_+ \cap \Gamma_-)) = n_2^1 (\Gamma_+ \cap \Gamma_-)$.

Since $[L', L_{\pm}] \subset i \oplus L_{\pm}$ by Lemma 6.(iii), $L_{\pm} \cap n_2^1 (\Gamma_+ \cap \Gamma_-)$ is an ideal of $n_2^1 (\Gamma_+ \cap \Gamma_-)$. But $L' \cap n_2^1$ is a subalgebra. It is easy to see that this implies that $L' \cap n_2^1$ is generated by a set of simple root subspaces of $n_2^1 (\Gamma_+ \cap \Gamma_-)$, i.e $L' \cap n_2^1 = n_2^1 (\Gamma_+ \cap \Gamma_-)$.

Moreover, the restriction of $(\cdot, \cdot)$ to $L'$ is nondegenerate, hence $L' \cap n_2^1 = n_2^1 (\Gamma_+ \cap \Gamma_-)$. Thus $i \oplus \g_{r_+} \subset i \oplus L' \subset i \oplus \g_{r_-} \cap (V_1 \cap V_2)$.

Since $\varphi(\lambda) + \frac{1}{2} i$ is invertible in $L'$, $\psi(\lambda)$ is a well-defined operator $L' \to L'$, satisfying (22), and $\psi(\lambda)(\eta_0 \cap L') \subset \eta_0 \cap L'$. Now, $i$ contains a regular element. Thus there exists a polarization of $g$ compatible with the $l$-weight decomposition. This induces a polarization of $\g_{r_+}$, compatible with the $l$-weight decomposition of $\g_{r_-}$. Hence, there exists $s' \in W_{\Gamma_1} \subset W$ such that $\psi|_{\g_{r_+}}$ is compatible with the polarization $s' n_2^1 \oplus h \oplus n_-^2$. Since $s'$ leaves $s \setminus (\Gamma_1 \cap \Gamma_2)$ globally unchanged, the polarization $g = \pi_+ \oplus h \oplus \pi_-$ with $\pi_+ = s' n_2^1$ and $\Gamma_3 = s' \setminus \Gamma$ satisfies the requirements of proposition 4.

To sum up, we have shown that there exists a polarization $g = \pi_+ \oplus h \oplus \pi_-$, compatible with $\varphi$, subsets $\Gamma_1 = s' s \Gamma_+$, $\Gamma_2 = s' s \Gamma_-$ and $\Gamma_3 \subset \Pi(\pi_+)$ such that $(A_+ / I_+) \cap n_+ = \pi_+(\Gamma_1)$, $A_- \cap n_+ = \pi_+(\Gamma_2)$ and $L' \cap n_+ = \pi_+(\Gamma_3)$.

The map $\psi_0$ now restricts to a Lie algebra isomorphism $\pi_+(\Gamma_1) \to \pi_+(\Gamma_2)$. This isomorphism maps weight spaces to weight spaces as $\psi_0$ preserves $\eta_0$ and $\varphi$ is $l$-invariant. Define $\tau : \Gamma_1 \to \Gamma_2$ by $\psi_0(\g_{r_+}) = \g_{r_+(\alpha)}$. It is a norm-preserving bijection. Thus $(\Gamma_1, \Gamma_2, \Gamma_3)$ is a generalized Belavin-Drinfeld triple. It is clear that $\Gamma_3$ is the largest subset of $\Gamma_1 \cap \Gamma_2$ stable under $\tau$, and that $\psi_0 : \pi_+(\Gamma_3) \to \pi_+(\Gamma_3)$ is a Lie algebra isomorphism. Finally, it is easy to see that the map $\varphi$ is obtained from this data by formulas

$$\varphi(\lambda)(e_\alpha) = \frac{1}{2} e_\alpha \quad (\alpha \notin (\Gamma_1))$$

$$\varphi(\lambda)(e_\alpha) = \frac{1}{2} e_\alpha + \frac{\psi_0}{1 - e(\alpha, \lambda) \psi_0}(e_\alpha) \quad (\alpha \in (\Gamma_1))$$

Conversely, it is clear how to construct from a generalized Belavin-Drinfeld triple $(\Gamma_1, \Gamma_2, \tau)$ the subalgebras $n_+(\Gamma_1), n_+(\Gamma_2), n_+(\Gamma_3)$ and, for each choice of
Chevalley generators, a Lie algebra isomorphism \( \psi_0 : n_+ (\Gamma_1) \to n_+ (\Gamma_2) \), and the map \( \varphi (\lambda) \). Condition (16) on the \( h_0 \otimes h_0 \)-part comes from (21)-see [BD].

6. Examples

6.1. Constant r-matrices. Our results imply the following:

**Corollary 1.** A dynamical r-matrix associated to a generalized Belavin-Drinfeld triple \((\Gamma_1, \Gamma_2, \tau)\) is gauge equivalent to a constant r-matrix if and only if \( \Gamma_3 = \emptyset \).

6.2. \( h \)-invariant dynamical r-matrices. When \( l = h \), our classification coincides with that given in [EV]: the only \( h \)-graded generalized Belavin-Drinfeld triple is of the form \((\Gamma, \Gamma, \tau = Id)\). The dynamical r-matrices obtained are then (up to gauge transformations and choice of Chevalley generators):

\[
\begin{align*}
    r (\lambda) &= \frac{\Omega}{2} + \frac{1}{2} \sum_{\alpha \in \Delta_+, \alpha \notin (\Gamma) \cap \Delta_+} e_\alpha \wedge e_{-\alpha} + \frac{1}{2} \sum_{\alpha \in (\Gamma) \cap \Delta_+} \coth{\left( \frac{1}{2} (\alpha, \lambda) e_\alpha \wedge e_{-\alpha} \right)}.
\end{align*}
\]

6.3. Example for \( sl_3 \) and \( sl_n \). The first nontrivial example is for \( g = sl_3 \): fix a polarization \( \mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\gamma \in \Delta} \mathfrak{g}_{\gamma} \), where \( \Delta^+ = \{ \alpha, \beta, \alpha + \beta \} \) and set \( l = \mathbb{C} h_0 \).

Consider the generalized Belavin-Drinfeld triple with \( \Gamma_1 = \Gamma_2 = \{ \alpha, \beta \} \) and \( \tau : \alpha \mapsto \beta, \beta \mapsto \alpha. \) In this case, we can choose the map \( \psi_0 \) to be the following

\[
\begin{align*}
    e_\alpha &\mapsto e_\beta, & h_\alpha &\mapsto h_\beta, & e_{-\alpha} &\mapsto e_{-\beta} \\
    e_\beta &\mapsto e_\alpha, & h_\beta &\mapsto h_\alpha, & e_{-\beta} &\mapsto e_{-\alpha} \\
    e_{\alpha + \beta} &\mapsto -e_{\alpha + \beta}, & e_{-\alpha - \beta} &\mapsto -e_{-\alpha - \beta}.
\end{align*}
\]

The corresponding dynamical r-matrix is given by:

\[
\begin{align*}
    r (\lambda) &= \frac{\Omega}{2} + r_{h_0, h_0} + \frac{1}{2} \coth(\alpha, \lambda) e_\alpha \wedge e_{-\alpha} + \frac{1}{2} \coth(\beta, \lambda) e_\beta \wedge e_{-\beta} + \frac{1}{2} \coth(\alpha, \lambda) e_\alpha \wedge e_{-\alpha} \\
    &\quad + \frac{1}{2} \coth(\alpha, \lambda) e_\alpha \wedge e_{-\alpha} + \frac{1}{2} \coth(\beta, \lambda) e_\beta \wedge e_{-\beta} \\
    &\quad + \frac{1}{2} \coth(\alpha, \lambda) e_\alpha \wedge e_{-\alpha} + \frac{1}{2} \coth(\beta, \lambda) e_\beta \wedge e_{-\beta} + \frac{1}{2} \coth(\alpha, \lambda) e_\alpha \wedge e_{-\alpha} \\
    &\quad + \frac{1}{2} \coth(\beta, \lambda) e_\beta \wedge e_{-\beta}.
\end{align*}
\]

This dynamical r-matrix is gauge-equivalent to the dynamical r-matrix

\[
\begin{align*}
    \tilde{r} (\lambda) &= \frac{\Omega}{2} + r_{h_0, h_0} + r_{l_0, h_0} - r_{l_0, h_0} + \frac{1}{2} \coth(\alpha, \lambda) e_\alpha \wedge e_{-\alpha} + \frac{1}{2} \coth(\beta, \lambda) e_\beta \wedge e_{-\beta} + \frac{1}{2} \coth(\alpha, \lambda) e_\alpha \wedge e_{-\alpha} + \frac{1}{2} \coth(\beta, \lambda) e_\beta \wedge e_{-\beta} + \frac{1}{2} \coth(\alpha, \lambda) e_\alpha \wedge e_{-\alpha} + \frac{1}{2} \coth(\beta, \lambda) e_\beta \wedge e_{-\beta}.
\end{align*}
\]

when

\[
(\alpha \otimes 1 + 1 \otimes \tau (\alpha)) (r_{h_0, h_0} + r_{l_0, h_0} - r_{l_0, h_0}) = \frac{1}{2} (\alpha + \tau (\alpha)) \Omega_h.
\]
In particular, \( \tilde{r}(\lambda) \) interpolates the constant \( r \)-matrix obtained from the Belavin-Drinfeld triple \( (\Gamma_1 = \alpha, \Gamma_2 = \beta, \tau : \alpha \mapsto \beta) \) at \( (\alpha, \lambda) \to \infty \) and the \( r \)-matrix obtained from \( (\Gamma_1 = \beta, \Gamma_2 = \alpha, \tau : \beta \mapsto \alpha) \) at \( (\alpha, \lambda) \to -\infty \).

Remark. The generalization of this example to \( \mathfrak{g} = \mathfrak{sl}_{2n+1} \) is the following. Fix a polarization and let \( I = C h_{\rho} \). Denote by \( \Delta \) the root system and by \( \Pi = (\alpha_1, \ldots, \alpha_{2n}) \) the set of positive simple roots. Let \( i : \alpha_k \mapsto \alpha_{2n+1-k} \) be the involution of the Dynkin diagram. The dynamical \( r \)-matrix obtained from the generalized Belavin-Drinfeld triple \( (\Gamma_1 = \Gamma_2 = \Pi, \tau = i) \) interpolates the constant \( r \)-matrices obtained from the Belavin-Drinfeld triples \( (\Gamma_1 = (\alpha_1, \ldots, \alpha_n), \Gamma_2 = (\alpha_{n+1}, \ldots, \alpha_{2n}), \tau = i) \) and \( (\Gamma_1 = (\alpha_{n+1}, \ldots, \alpha_{2n}), \Gamma_2 = (\alpha_1, \ldots, \alpha_n), \tau = i^{-1}) \).

6.4. Permutation dynamical \( r \)-matrices. Consider \( \mathfrak{g} = \mathfrak{sl}_{2n} \), and let \( \Pi = (\alpha_1, \ldots, \alpha_{2n-1}) \) denote a system of simple roots. For any \( \sigma \in S_n \), we can construct a generalized Belavin-Drinfeld triple by setting \( \Gamma_1 = \Gamma_2 = (\alpha_1, \alpha_3, \ldots, \alpha_{2n-1}) \) and \( \tau : \alpha_{2k-1} \mapsto \alpha_{2\sigma(k)-1} \).

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References


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