REMARKS ON THE PAPER OF V. GUILLEMIN AND K. OKIKIOLU: “SUBPRINCIPAL TERMS IN SZEGÖ ESTIMATES”

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1. Introduction

Let $M$ be a smooth compact manifold without boundary, dim $M = d$ and let $A$ and $B$ be pseudodifferential operators (PsDO) acting in the space $L^2(M)$ of half-densities on $M$. We assume that $A$ is a positive elliptic PsDO of order 1 and that $B$ is a PsDO of order 0. Denote by $a(x, \xi)$ and $b(x, \xi)$, $(x, \xi) \in T^*M \backslash 0$, the principal symbols of the operators $A$ and $B$ respectively. The spectrum of $A$ is discrete and therefore its spectral projection $P_\lambda$, $\lambda \geq 0$, is an operator of a finite rank. Let

\[ V_a(x, \xi) = \sum_{j=1}^d \frac{\partial a}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial a}{\partial x_j} \frac{\partial}{\partial \xi_j}, \quad (x, \xi) \in T^*M \backslash 0, \]

be the bicharacteristic vector field on $T^*M \backslash 0$ associated with $a$. A point $(x, \xi) \in T^*M \backslash 0$ is called periodic with a period $t$ if $\exp(tV_a)(x, \xi) = (x, \xi)$.

Guillemin and Okikiolu [GO] have recently announced the following result:

**Theorem 1.** Let $a(x, \xi) = a(x, -\xi)$ and the subprincipal symbol of $A$ is equal to zero. Suppose that for any $t > 0$ the set of $t$-periodic points is of measure zero with respect to the invariant measure $dx \, d\xi$ on the cotangent bundle $T^*M \backslash 0$. Then

\[ \text{Tr}(P_\lambda BP_\lambda)^k = \text{Tr} P_\lambda B^k P_\lambda - \lambda^{d-1}(2\pi)^{-d} \gamma_k(A, B) + o(\lambda^{d-1}), \quad k \geq 2, \]

where

\[ \gamma_k(A, B) = \frac{d}{8\pi} \sum_{m=1}^{k-1} \frac{k}{m(k-m)} \times \]

\[ \int_{a<1} \int_{-\infty}^{\infty} \frac{(b^m_t(x, \xi) - b^m(x, \xi))(b^{k-m}_t(x, \xi) - b^{k-m}(x, \xi))}{t^2} \, dt \, dx \, d\xi, \]

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and where \( b_t(x, \xi) = (\exp(tV_a))^* b(x, \xi) = b(\exp(tV_a)(x, \xi)) \).

Obviously this result can be reformulated for the trace \( \text{Tr} Q_k(P_\lambda BP_\lambda) \), where \( Q_k \) is an arbitrary polynomial of degree \( k \) and such that \( Q_k(0) = 0 \). Moreover, under certain conditions on the pseudodifferential operator \( B \) the paper [GO] also contains a corresponding asymptotic formula for \( \text{Tr} \log(P_\lambda BP_\lambda) \).

The purpose of this paper is to extend Theorem 1 to the case where instead of \( Q_k \) (or \( \log \)) one deals with an arbitrary function \( \psi \in C^2(\mathbb{R}^1) \).

2. The main result

Let

\[
K := \bigcup_{0 \leq t \leq 1} t \sigma(B) \subset \mathbb{R}^1,
\]

where \( \sigma(B) \) is the spectrum of the operator \( B \). Clearly, \( K \) is a closed bounded interval. In order to formulate our main result we introduce the transformation

\[
W\psi(t, s) = \int_s^t \int_t^s \frac{\psi'(u) - \psi'(v)}{u - v} du \, dv, \quad \psi \in C^2(K), \quad t, s \in K.
\]

One can easily see that \( W \) is a linear continuous map from \( C^2(K) \) into \( C^1(K \times K) \) such that \( |W\psi(t, s)| \leq \|\psi''\|_{C(K)} |t - s|^2 \). The kernel of the map \( W \) consists of the first degree polynomials.

**Theorem 2.** Let \( \psi \in C^2(K) \) and \( B \) be a selfadjoint PsDO of order 0. Then under the conditions of Theorem 1

\[
\text{Tr} P_\lambda \psi(P_\lambda BP_\lambda)P_\lambda = \text{Tr} P_\lambda \psi(B)P_\lambda - \lambda^{d-1}(2\pi)^{-d} \gamma_\psi(A, B) + o(\lambda^{d-1}),
\]

where

\[
\gamma_\psi(A, B) = \frac{d}{8\pi} \int_{a < 1} \int_{-\infty}^\infty \frac{W\psi(b_t(x, \xi), b(x, \xi))}{t^2} dt \, dx \, d\xi.
\]

From the properties of the map \( W \) it follows that

\[
\gamma_{\psi_0}(A, B) \min_{u \in K} \psi''(u) \leq \gamma_\psi(A, B) \leq \gamma_{\psi_0}(A, B) \max_{u \in K} \psi''(u),
\]

where \( \psi_0(u) = u^2/2 \) and

\[
\gamma_{\psi_0}(A, B) = \frac{d}{8\pi} \int_{a < 1} \int_{-\infty}^\infty \left( \frac{b_t(x, \xi) - b(x, \xi)}{t} \right)^2 dt \, dx \, d\xi.
\]

This implies that \( \gamma_\psi(A, B) \) is a linear continuous functional on the space \( C^2(K) \).

If \( \psi \in C^\infty(K) \), then \( \psi(B) \) is a PsDO of order 0. Its principal symbol coincides with \( \psi(b(x, \xi)) \), and subprincipal symbol is given by

\[
\text{sub} \psi(B)(x, \xi) = \psi'(b(x, \xi)) \text{ sub} B(x, \xi).
\]
By the methods of [DG] one can prove that under the conditions of Theorem 1
(7) \[ \text{Tr} P_\lambda \psi(B) P_\lambda = (2\pi)^{-d} \int_{|\lambda| < 1} \left( \lambda^d \psi(b(x,\xi)) + \lambda^{d-1} \text{sub} \psi(B)(x,\xi) \right) dx \, d\xi + o(\lambda^{d-1}). \]

This result can be deduced from (4.2.6) in [SV] in the same way as the two-term asymptotic formula for the counting function \( N(\lambda) \). It also follows from Proposition 29.1.2 in [H] (Hörmander’s formula contains an extra term which is, as was pointed out by D. Vassiliev, actually equal to zero).

Combining Theorem 2 with (7) we obtain

**Corollary 3.** Let \( \psi \in C^\infty(\mathbb{R}^1) \). Then under the conditions of Theorem 2

(8) \[ \text{Tr} P_\lambda \psi(P_\lambda B P_\lambda) P_\lambda = (2\pi)^{-d} \left[ \lambda^d \int_{|\lambda| < 1} \psi(b(x,\xi)) \, dx \, d\xi \right. \]

\[ - \lambda^{d-1} \left( \gamma_\psi(A, B) - \int_{|\lambda| < 1} \text{sub} \psi(B)(x,\xi) \, dx \, d\xi \right) \] \[ + o(\lambda^{d-1}). \]

### 3. Auxiliary statements

The proof of Theorem 2 is based on a version of an abstract result obtained in [LS1] (see also [LS2]). Let \( B \) be a bounded selfadjoint operator, \( P \) be an orthogonal projection in a Hilbert space \( \mathcal{H} \), and \( K \) be the compact set defined by (4). Denote by \( \mathcal{S}_1 \) and \( \mathcal{S}_2 \) respectively the trace class and the Hilbert-Schmidt class of operators in \( \mathcal{H} \).

**Proposition 4.** Let \( PB \in \mathcal{S}_2 \). Then for any function \( \psi \) whose second derivative lies in \( L^\infty(K) \) we have

\[ P\psi(B)P - P\psi(PBP)P \in \mathcal{S}_1 \]

and

(9) \[ \left| \text{Tr}\left(P\psi(B)P - P\psi(PBP)P\right) \right| \leq \frac{1}{2} \|\psi''\|_{L^\infty(K)} \|PB(I - P)\|_{\mathcal{S}_2}^2. \]

The next statement concerns the map \( W \) defined in (5).

**Proposition 5.** For an arbitrary polynomial

\[ \mathcal{Q}_k(x) = \sum_{m=0}^k a_m x^m, \]

we have

(10) \[ W\mathcal{Q}_k(t, s) = \sum_{m=2}^k a_m \sum_{n=1}^{m-1} \frac{m}{n(m - n)} (t^n - s^n)(t^{m-n} - s^{m-n}). \]
Proof. It is sufficient to check (10) for \( Q_k(x) = x^k, \ k \geq 2 \). In this case
\[
WQ_k(t,s) = k \int_s^t \int_s^u \frac{u^{k-1} - v^{k-1}}{u-v} \, du \, dv \\
= k \int_s^t \int_s^u \sum_{n=1}^{k-1} u^{n-1}v^{k-1-n} \, du \, dv = k \sum_{n=1}^{k-1} \frac{1}{n(k-n)} (t^n - s^n)(t^{k-n} - s^{k-n}).
\]
The proof is complete.

4. The proof of Theorem 2

Let \( \{Q_j\}_{j=1}^{\infty} \) be a sequence of polynomials approximating \( \psi \) in \( C^2(K) \). Given \( \varepsilon > 0 \) we choose \( k_0 \) such that that \( \|\psi - Q_k\|_{C^2(K)} \leq \varepsilon \) for all \( k \geq k_0 \). Obviously
\[
\text{Tr} \left( P_\lambda \psi(B)P_\lambda - P_\lambda \psi(P_\lambda BP_\lambda)P_\lambda \right) = T_1 + T_2,
\]
where
\[
T_1(\lambda, A, B) := \text{Tr} \left( P_\lambda Q_k(B)P_\lambda - P_\lambda Q_k(P_\lambda BP_\lambda)P_\lambda \right)
\]
and
\[
T_2(\lambda, A, B) := \text{Tr} \left( P_\lambda (\psi - Q_k)(B)P_\lambda - P_\lambda (\psi - Q_k)(P_\lambda BP_\lambda)P_\lambda \right).
\]
From Proposition 4 we obtain
\[
|T_2(\lambda, A, B)| \leq \frac{1}{2} \|\psi - Q_k\|_{C^2(K)} \|P_\lambda B(I - P_\lambda)\|_{\mathcal{S}_2}^2 \leq \frac{\varepsilon}{2} \|P_\lambda B(I - P_\lambda)\|_{\mathcal{S}_2}^2.
\]
The well known asymptotic properties of the spectrum of the operator \( A \) (see for example [LS2, Section 2]) imply the estimate
\[
\|P_\lambda B(I - P_\lambda)\|_{\mathcal{S}_2}^2 = O(\lambda^{d-1})
\]
and, therefore, there exists a constant \( C \) independent of \( \varepsilon \), such that
\[
\limsup_{\lambda \to \infty} \lambda^{1-d} |T_2(\lambda, A, B)| \leq \varepsilon C.
\]
Applying Theorem 1 to the trace (11) and taking into account Proposition 5 we obtain
\[
\lim_{\lambda \to \infty} \lambda^{1-d} T_1(\lambda, A, B) = (2\pi)^{-d} \gamma_{Q_k}(A, B)
\]
and thus
\[
\limsup_{\lambda \to \infty} |\lambda^{1-d} \text{Tr} \left( P_\lambda \psi(B)P_\lambda - P_\lambda \psi(P_\lambda BP_\lambda)P_\lambda \right) - (2\pi)^{-d} \gamma_{\psi}(A, B)| \leq \varepsilon C.
\]
Since \( \gamma_{\psi}(A, B) \) is a continuous linear functional on \( C^2(K) \), (12) implies that
\[
\limsup_{\lambda \to \infty} |\lambda^{1-d} \text{Tr} \left( P_\lambda \psi(B)P_\lambda - P_\lambda \psi(P_\lambda BP_\lambda)P_\lambda \right) - (2\pi)^{-d} \gamma_{\psi}(A, B)| \leq 2\varepsilon C
\]
for sufficiently large \( k \). Since \( \varepsilon \) can be chosen arbitrarily small, this completes the proof of Theorem 2.
References


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