

PRINCIPAL G -BUNDLES OVER ELLIPTIC CURVES

ROBERT FRIEDMAN, JOHN W. MORGAN, AND EDWARD WITTEN

1. Introduction

Let E be an elliptic curve with origin p_0 , and let G be a complex simple algebraic group. For simplicity, we shall only consider the case where G is simply connected, although all of the methods discussed below can be extended to the case of a general group G . The goal of this note is to announce some results concerning the moduli of principal holomorphic G -bundles over E . Detailed proofs, as well as a more thorough discussion of the case where E is allowed to be singular or to vary in families and of the connections with del Pezzo surfaces, elliptic $K3$ surfaces, and Calabi-Yau manifolds which are elliptic or $K3$ fibrations, will appear elsewhere.

Grothendieck [21] considered principal holomorphic G -bundles over \mathbb{P}^1 , and showed that it was always possible to reduce the structure group to a Cartan subgroup, i.e. to a maximal (algebraic) torus in G . Atiyah [1] classified all holomorphic vector bundles over an elliptic curve (in other words, the cases $G = SL(n, \mathbb{C})$ or $G = PGL(n, \mathbb{C})$), without however considering the problem of trying to construct a moduli space or find a universal bundle. In [16], [17], and [18], this problem is studied in the rank two case with a view toward constructing relative moduli spaces in families. This approach has been generalized to arbitrary rank in [20]. A great deal of work has been done on the moduli spaces and stacks of G -bundles over a curve of genus at least two, partly motivated by the study of conformal blocks and the Verlinde formulas, by very many authors, e.g. [5], [15]. A basic method here is to relate the moduli stack to an appropriate loop group. Related constructions in the case of genus one have been carried out by Baranovsky-Ginsburg [4], based on unpublished work of Looijenga (see for example [13]). They relate semistable G -bundles to conjugacy classes in a corresponding affine Kac-Moody group. Recently Brückert [9] has constructed a Steinberg-type cross-section for the adjoint quotient of the affine Kac-Moody group whose image lies in the set of regular elements, and this construction leads to a moduli space for semistable G -bundles which is equivalent to the one we

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construct in Section 4 below. (We are indebted to Slodowy for calling our attention to the work of Brückert and sketching an argument for the equivalence of the approach described above with the one we give in this paper.) Finally, many of the results in this note, along with applications to physics, are discussed in [19].

The contents of this note are as follows. We will be concerned with the classification of semistable G -bundles. As is typical in invariant theory or moduli problems, the classification will be up to a coarser equivalence than isomorphism, which is usually called S-equivalence and will be defined more precisely in Section 2. In Section 2, we describe the moduli space of semistable G -bundles over E via flat connections for the maximal compact subgroup K of G , or equivalently via conjugacy classes of representations $\rho: \pi_1(E) \rightarrow K$. Such bundles, which for a simply connected group G are exactly the bundles whose structure group reduces to a Cartan subgroup, have an automorphism group which is as large as possible in a certain sense within a fixed S-equivalence class. The main result here is a theorem due to Looijenga and Bernshtein-Shvartsman which describes this moduli space as a weighted projective space. At the end of the section, we connect this description, in the case where $G = E_6, E_7, E_8$, with the moduli space of del Pezzo surfaces of degree 3, 2, 1 respectively and with the deformation theory of simple elliptic singularities. In Section 3, we describe regular G -bundles, which by contrast with flat bundles have automorphism groups whose dimensions are as small as possible within a fixed S-equivalence class. The generic G -bundle is both flat and regular. However at special points of the moduli space we can choose either a unique flat representative or a unique regular representative, and it is the regular representatives which fit together to give holomorphic families. In Section 4, we show how special unstable bundles over certain maximal parabolic subgroups can be used to give another description of the moduli space in terms of regular bundles and obtain a new proof of the theorem of Looijenga and Bernshtein-Shvartsman. Finally, in the last section we discuss the existence of universal bundles and give a brief description of how our construction can be twisted with the help of a certain spectral cover.

2. Split semistable bundles

We fix notation for the rest of this paper. As before, E denotes an elliptic curve with origin p_0 . Let G be a simple and simply connected complex Lie group of rank r , and let $\xi \rightarrow E$ be a holomorphic principal G -bundle over E . The following definition differs from that given in Ramanathan [32], but is equivalent to it.

Definition 2.1. The principal bundle $\xi \rightarrow E$ is *semistable* if the associated vector bundle $\text{ad } \xi$ is a semistable vector bundle. The principal bundle $\xi \rightarrow E$ is *unstable* if it is not semistable.

Note that, if ξ is stable in the sense of [32], it is still possible for the vector bundle $\text{ad } \xi$ to be strictly semistable. However, in our case (G simply connected),

there are essentially no properly stable bundles over E , and so the above definition will suffice for our purposes.

If ξ is an unstable bundle, the structure group of ξ reduces canonically to a parabolic subgroup P of G , the *Harder-Narasimhan parabolic* associated to ξ (see for example [31] or [2], pp. 589–590). The canonical reduction holds over a general base curve. In the case of a base curve E of genus one, it is easy to see that the structure group further reduces to a Levi factor of P .

Recall the following standard terminology: a *family* of principal G -bundles over E parametrized by a complex space (or scheme) S is a principal G -bundle Ξ over $E \times S$. The family Ξ is a family of *semistable* principal G -bundles over E if $\Xi|_{E \times \{s\}} = \Xi_s$ is semistable for all $s \in S$. Finally, let ξ and ξ' be two semistable bundles over E . We say that ξ and ξ' are *S -equivalent* if there exists a family of semistable bundles Ξ parametrized by an irreducible S and a point $s \in S$ such that, for $t \neq s$, $\Xi|_{E \times \{t\}} \cong \xi$ and $\Xi|_{E \times \{s\}} \cong \xi'$. More generally, we let S -equivalence be the equivalence relation generated by the above relation.

The following holds only under our assumption that G is simply connected.

Proposition 2.2. *Let ξ be a semistable principal G -bundle, and suppose that the rank of G is r . Then $h^0(E; \text{ad } \xi) \geq r$. Equivalently, $\dim \text{Aut}_G \xi \geq r$, where $\dim \text{Aut}_G \xi$ denotes the group of global automorphisms of ξ (as a G -bundle).*

Definition 2.3. Let ξ be a semistable principal G -bundle. We call ξ *regular* if $h^0(E; \text{ad } \xi) = r$, or equivalently if $\dim \text{Aut}_G(\xi) = r$. We call ξ *split* if its structure group reduces to a Cartan subgroup of G , i.e. a maximal (algebraic) torus.

It is easy to check that split bundles have the following closure property: if there exists a family of semistable bundles Ξ parametrized by an irreducible S and a point $s \in S$ such that, for $t \neq s$, the bundles $\Xi|_{E \times \{t\}}$ are split and all isomorphic to each other, then $\Xi|_{E \times \{s\}}$ is isomorphic to $\Xi|_{E \times \{t\}}$, $t \neq s$, and thus it is split as well. In general, however, the condition of being split is neither open nor closed. On the other hand, by the upper semicontinuity theorem, regularity is an open condition: if Ξ is a family of semistable bundles parametrized by S and $\Xi|_{E \times \{s\}}$ is regular, then $\Xi|_{E \times \{t\}}$ is regular for all t in an open neighborhood of s .

To describe the set of split bundles, we introduce flat bundles on the compact group. Let K be a maximal compact subgroup of G . Then K is a compact, simple and simply connected Lie group. If \mathfrak{k} is the Lie algebra of K and \mathfrak{g} is the Lie algebra of G , then \mathfrak{g} is the complexification of \mathfrak{k} . Given a representation $\rho: \pi_1(E) \cong \mathbb{Z} \oplus \mathbb{Z} \rightarrow K$, we can form the associated principal K -bundle $(\tilde{E} \times K)/\pi_1(E) \rightarrow E$, where $\pi_1(E)$ acts on \tilde{E} , the universal cover of E , in the usual way, and on K via ρ . We shall call such a K -bundle a *flat* K -bundle. Using the inclusion $K \subset G$, we can also view a flat K -bundle as a G -bundle, and we shall also incorrectly refer to the induced G -bundle as a flat K -bundle. We will need the following version of the theorem of Narasimhan-Seshadri [29] and Ramanathan [32] (see also Atiyah-Bott [2] and Donaldson [12]):

Theorem 2.4. *Let $\xi \rightarrow E$ be a semistable principal G -bundle. Then there is a flat K -bundle S -equivalent to ξ , and it is unique up to isomorphism of flat K -bundles. More precisely, there is a family of semistable principal G -bundles Ξ over $E \times \mathbb{C}$, such that, for $t \neq 0$, $\xi_t = \Xi|_{E \times \{t\}} \cong \xi$, and such that $\xi_0 = \Xi|_{E \times \{0\}}$ is the G -bundle associated to a flat K -bundle via the inclusion $K \subset G$. Finally, two flat K -bundles are isomorphic as G -bundles if and only if they are isomorphic as K -bundles.*

We note that Theorem 2.4 also holds for a non-simply connected group. The special feature of simply connected groups which we need to describe the moduli space of flat K -bundles is contained in the following result of Borel [7] (see also [22] for the analogous algebraic result, due to Springer and Steinberg):

Theorem 2.5. *Let K be a compact, simple, and simply connected Lie group, and let r_1 and r_2 be two commuting elements of K . Then there exists a maximal torus T in K with $r_1, r_2 \in T$.*

Since $\pi_1(E) \cong \mathbb{Z} \oplus \mathbb{Z}$, to give a representation $\rho: \pi_1(E) \rightarrow K$ is to give two commuting elements $r_1, r_2 \in K$. Thus a flat K -bundle reduces to a T -bundle. In particular, we see that for a simply connected group G , every G -bundle associated to a flat bundle is split, and conversely. On the other hand, if G is not simply connected, every split bundle lifts to the universal cover \tilde{G} of G , so that a G -bundle which does not lift to \tilde{G} cannot be split. Thus the correct notion for unliftable bundles is that of a flat bundle.

Returning to the case of a simply connected group G , let T be a maximal torus in the compact group K . One checks that two homomorphisms from $\pi_1(E)$ to T are conjugate by an element of K if and only if they are conjugate by an element of the normalizer of T in K . Thus we have:

Theorem 2.6. *There is a natural bijection from the set of flat K -bundles up to isomorphism, or equivalently the set of semistable G -bundles up to S -equivalence, to the set $\text{Hom}(\pi_1(E), T)/W$, where W is the Weyl group of K , acting in the usual way on the maximal torus T .*

Fix a maximal torus T in K . If $\Lambda = \pi_1(T)$, then $T \cong U(1) \otimes_{\mathbb{Z}} \Lambda$. Moreover, since K is simply connected, if $\mathfrak{t}_{\mathbb{R}}$ denotes the real Lie algebra of T , then $\Lambda \subset \mathfrak{t}_{\mathbb{R}}$ is the lattice generated by the coroots α^{\vee} , where $\alpha \in \mathfrak{t}_{\mathbb{R}}^*$ is a root. Now given a homomorphism $\rho: \pi_1(E) \cong \mathbb{Z} \oplus \mathbb{Z} \rightarrow K$, the image of ρ is generated by two commuting elements of K and so, after conjugation, lies in T . The set of flat T -bundles is naturally

$$\text{Hom}(\pi_1(E), T) = \text{Hom}(\pi_1(E), U(1) \otimes_{\mathbb{Z}} \Lambda) \cong \text{Hom}(\pi_1(E), U(1)) \otimes_{\mathbb{Z}} \Lambda.$$

Now $\text{Hom}(\pi_1(E), U(1))$ is the set of flat line bundles on E , and is naturally identified with $\text{Pic}^0 E$. Since we have fixed a base point $p_0 \in E$, we can further identify $\text{Pic}^0 E$ with E . Thus the space of flat T -bundles is naturally $E \otimes_{\mathbb{Z}} \Lambda$. On the other hand, as we are classifying not flat T -bundles but flat K -bundles,

we must take the quotient of $E \otimes_{\mathbb{Z}} \Lambda$ by the action of the Weyl group W of G acting on $E \otimes_{\mathbb{Z}} \Lambda$ via the natural action of W on Λ . We have thus described the coarse moduli space of semistable G -bundles over E as $(E \otimes_{\mathbb{Z}} \Lambda)/W$. A different proof of this result has been given by Laszlo [25].

The varieties $(E \otimes_{\mathbb{Z}} \Lambda)/W$ have been studied by Looijenga [27] and Bernshtein-Shvartsman [6], who proved the following theorem:

Theorem 2.7. *Let E be an elliptic curve and let Λ be the coroot lattice of a simple root system R with Weyl group W . Then $(E \otimes_{\mathbb{Z}} \Lambda)/W$ is a weighted projective space $WP(g_0, \dots, g_r)$, where the weights g_i are given as follows: $g_0 = 1$, and the remaining roots g_i are found by choosing a set of simple roots $\alpha_1, \dots, \alpha_r$, and then writing the coroot $\tilde{\alpha}^\vee$ dual to the highest root $\tilde{\alpha}$ as a linear combination $\sum_i g_i \alpha_i^\vee$ of the coroots dual to the simple roots. In case R is simply laced, we can identify the dual coroot $\alpha^\vee \in R^\vee$ to α with α , and consequently the g_i are the coefficients of $\tilde{\alpha}$ in terms of the basis $\alpha_1, \dots, \alpha_r$.*

The proof of [27] and [6] makes use of formal theta functions for a complexified affine Weyl group. We shall outline a different proof of (2.7) below.

Since it will be important to motivate the construction of Section 4, let us give Looijenga's reason for studying the space $(E \otimes_{\mathbb{Z}} \Lambda)/W$. Let (X, x_0) be the germ of a simple elliptic singularity whose minimal resolution has a single exceptional component which is a smooth elliptic curve E with self-intersection $-3, -2$, or -1 . These are exactly the simple elliptic singularities which can be realized as hypersurface singularities in $(\mathbb{C}^3, 0)$, and we shall refer to them as being of type $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$ respectively. These singularities are weighted cones over E corresponding to a line bundle L on E of degree 3, 2, or 1, and thus have a \mathbb{C}^* -action. Moreover \mathbb{C}^* also acts on the tangent space to the deformations of (X, x_0) . The zero weight directions (in other words those directions fixed by the \mathbb{C}^* -action) correspond to deforming (X, x_0) in an equisingular family by deforming E . The remaining weights are negative, and deformations in the neagative weight space correspond to deforming (X, x_0) to a rational double point (RDP) singularity or smoothing it. The local action of \mathbb{C}^* on the negative weight deformations may be globalized, and the quotient corresponding to the singularity \tilde{E}_r is a weighted projective space $WP(g_0, \dots, g_r)$, where the weights g_i are those defined above for the root system E_r . On the other hand, by the general theory of negative weight deformations of singularities with \mathbb{C}^* -actions, and in particular by work of Pinkham [30], Looijenga [26], and later Mérindol [28], the points of this weighted projective space parametrize triples (\bar{S}, D, φ) , where \bar{S} is a generalized del Pezzo surface of degree $9 - r$ (i.e., \bar{S} has at worst rational double point singularities and the inverse of the dualizing sheaf $K_{\bar{S}}$ is ample on \bar{S} , with $K_{\bar{S}}^2 = 9 - r$), $D \in |-K_{\bar{S}}|$ is a smooth divisor, not passing through the singularities of \bar{S} , and φ is an isomorphism from D to the fixed elliptic curve E such that $\varphi^* L = N_{D/\bar{S}}$. The moduli of such triples (\bar{S}, D, φ) can be described directly in terms of the defining equations for \bar{S} and can also be checked directly to be a weighted projective space with the correct weights.

(Similar but slightly more involved arguments also handle the case of degree 4 and 5, in which case the singularity is a codimension two complete intersection, in the case of degree 4, and the corresponding root system is D_5 , or a Pfaffian singularity in case the degree is 5, and the root system is A_4 .)

Now an elementary Torelli-type theorem shows that the pair (\bar{S}, D) (ignoring the extra structure of φ) is determined by the homomorphism $\psi_0: H_0^2(S; \mathbb{Z}) \rightarrow D$, where S is the minimal resolution of \bar{S} and $H_0^2(S; \mathbb{Z})$ is the orthogonal complement of $[K_S]$ in $H^2(S; \mathbb{Z})$, given as follows: represent a class $\lambda \in H_0^2(S; \mathbb{Z})$ by a holomorphic line bundle L on S such that $\deg(L|D) = 0$, and define $\psi_0(\lambda)$ to be the element $L|D \in \text{Pic}^0 D \cong D$. But $H_0^2(S; \mathbb{Z})$ is isomorphic to the root lattice for the corresponding root system E_r , and this isomorphism is well-defined up to the action of the Weyl group. The choice of the isomorphism φ enables one to extend the map ψ_0 to a map $\psi: H^2(S; \mathbb{Z})/\mathbb{Z}[D] \rightarrow E$, essentially because on the fixed curve E we can choose a $(9-r)^{\text{th}}$ root of the line bundle L , and conversely the choice of such a root fixes an isomorphism from D to E which lines up L with $N_{D/\bar{S}}$. Now $H^2(S; \mathbb{Z})/\mathbb{Z}[D]$ is dual to the coroot lattice Λ of the root system E_r , and ψ defines an element of $E \otimes_{\mathbb{Z}} \Lambda$, well-defined modulo the action of W . In this way, we have identified $WP(g_0, \dots, g_r)$ with $(E \otimes_{\mathbb{Z}} \Lambda)/W$. Let \bar{S} be the result of contracting all of the curves on S not meeting D . Thus \bar{S} has certain rational double point (RDP) singularities. Under the identification of the moduli space of pairs (S, D) with the set of $\psi: \Lambda^{\vee} \rightarrow E$, it is not difficult to show that the RDP singularities on S correspond to homomorphisms ψ such that there is a sub-root lattice $\Lambda' \subseteq \text{Ker } \psi$. In fact, the maximal such lattice Λ' describes the type of the RDP singularities on \bar{S} . Here the main point is to show, by a Riemann-Roch argument, that if $\gamma \in \text{Ker } \psi$ with $\gamma^2 = -2$, then $\pm\gamma$ is represented by an effective curve on S disjoint from D , and thus gives a singular point on the surface obtained by contracting all such curves. In this way, there is a link between subgroups of E_r , $r = 6, 7, 8$, and singularities of the corresponding del Pezzo surfaces.

3. Regular bundles

Recall that, for a simply connected group G , the bundle ξ is *regular* if $h^0(E; \text{ad } \xi)$ is equal to the rank of G . We begin by giving a detailed description of the set of regular bundles in case G is one of the classical groups. At the end of the section we shall outline the general structure of regular bundles. Let us give a preliminary definition:

Definition 3.1. Let I_n be the vector bundle of rank n and trivial determinant on E defined inductively as follows: $I_1 = \mathcal{O}_E$, and I_n is the unique nonsplit extension of I_{n-1} by \mathcal{O}_E . More generally, if λ is a line bundle on E of degree zero, we define $I_n(\lambda) = I_n \otimes \lambda$.

An easy argument shows that the algebra $\text{Hom}(I_n, I_n)$ is isomorphic to $\mathbb{C}[t]/(t^n)$, and in particular it is a commutative unipotent \mathbb{C} -algebra of dimension n .

If V is an arbitrary semistable vector bundle of degree zero over E and λ is a line bundle of degree zero over E , let $V_\lambda \subseteq V$ be the sum of all of the subbundles of V which are filtered by a sequence of subbundles whose successive quotients are isomorphic to λ . An easy argument shows that V_λ itself is the maximal such subbundle with this property and that $V = \bigoplus_\lambda V_\lambda$. A straightforward induction classifies the possible V_λ as a direct sum $\bigoplus_j I_{k_j}(\lambda)$. From this, it is easy to check:

Proposition 3.2. *Let V be a semistable vector bundle over E with trivial determinant, i.e. V is a principal $SL(n)$ -bundle over E . If $V \cong \bigoplus_{i=1}^r I_{d_i}(\lambda_i)$, where the λ_i are line bundles on E of degree zero, such that $\lambda_1^{d_1} \otimes \cdots \otimes \lambda_r^{d_r} = \mathcal{O}_E$ and $\sum_i d_i = n$, then V is regular if and only if $\lambda_i \neq \lambda_j$ for all $i \neq j$.*

To deal with the case of the symplectic or orthogonal group, the main point is to decide when a bundle V carries a nondegenerate alternating or symmetric form. The crucial case is that of I_n . In this case, we have the following:

Proposition 3.3. *There exists a nondegenerate alternating pairing on I_n if and only if n is even. There exists a nondegenerate symmetric pairing on I_n if and only if n is odd. In both cases, every two such nondegenerate pairings on I_n are conjugate under the action of $\text{Aut } I_n$.*

With this said, we can describe the regular symplectic bundles. It is simplest to describe them via the standard representation:

Proposition 3.4. *Let V be a vector bundle of rank $2n$ over E with a nondegenerate alternating form, and suppose that the dimension of the group of symplectic automorphisms of V is n . Then there exist positive integers d_i and nonnegative integers a_j , $0 \leq j \leq 3$, with $\sum_i d_i + \sum_j a_j = n$, such that V is isomorphic to*

$$\bigoplus_i (I_{d_i}(\lambda_i) \oplus I_{d_i}(\lambda_i^{-1})) \oplus I_{2a_0} \oplus I_{2a_1}(\eta_1) \oplus I_{2a_2}(\eta_2) \oplus I_{2a_3}(\eta_3),$$

where the λ_i are line bundles of degree zero, not of order two, such that, for all $i \neq j$, $\lambda_i \neq \lambda_j^{\pm 1}$, and η_1, η_2, η_3 are the three distinct line bundles of order two on E . Conversely, suppose that V is a vector bundle as given above. Then V has a nondegenerate alternating form, all such forms have a group of symplectic automorphisms of dimension exactly n , and every two nondegenerate alternating forms on V are equivalent under the action of $\text{Aut } V$.

In particular, we see that a regular symplectic bundle is always a regular bundle in the sense of $SL(2n)$ -bundles.

For $SO(2n)$ and $SO(2n+1)$, the situation is a little more complicated for two reasons. First, we shall only consider those bundles which can be lifted to $Spin(2n)$ or $Spin(2n+1)$, but shall not describe here the actual choice of a lifting. Secondly, because of (3.3), it turns out that a regular $SO(n)$ -bundle does not always give a regular $SL(n)$ -bundle.

Proposition 3.5. *Let V be a vector bundle of rank $2n$ over E with a nondegenerate symmetric form, and suppose that the dimension of the group of orthogonal automorphisms of V is n . Finally suppose that V can be lifted to a principal $Spin(2n)$ -bundle. Then V is isomorphic to*

$$\bigoplus_i (I_{d_i}(\lambda_i) \oplus I_{d_i}(\lambda_i^{-1})) \oplus \bigoplus_j (I_{2a_j+1}(\eta_j) \oplus \eta_j)$$

where the λ_i are line bundles of degree zero, not of order two, such that, for all $i \neq j$, $\lambda_i \neq \lambda_j^{\pm 1}$, $\eta_0 = \mathcal{O}_E$, η_1, η_2, η_3 are the four distinct line bundles of order two on E , and the second sum is over some subset (possibly empty) of $\{0, 1, 2, 3\}$. Conversely, every such vector bundle V has a nondegenerate symmetric form, all such forms have a group of orthogonal automorphisms of dimension exactly n , and every two nondegenerate symmetric forms on V are equivalent under the action of $\text{Aut } V$.

Here the symmetric form on $I_{2a_0+1} \oplus \mathcal{O}_E$ consists of the orthogonal direct sum of the nondegenerate form on the factor I_{2a_0+1} given by (3.3), together with the obvious form on \mathcal{O}_E , and similarly for the summands $I_{2a_i+1}(\eta_i) \oplus \eta_i$. Moreover, not all of the summands $I_{2a_j+1}(\eta_j) \oplus \eta_j$ need be present in V . We remark that, if a vector bundle $\bigoplus_j I_{d_j}(\lambda_j)$ is isomorphic to its dual, and the sum of all the factors where $\lambda_j = \eta_i$ for some i has odd rank, then the same must be true for all of the η_i . Thus, if the automorphism group of V is to be as small as possible, then either V is as described in (3.5) or V is of the form

$$\bigoplus_i (I_{d_i}(\lambda_i) \oplus I_{d_i}(\lambda_i^{-1})) \oplus I_{2a_0+1} \oplus I_{2a_1+1}(\eta_1) \oplus I_{2a_2+1}(\eta_2) \oplus I_{2a_3+1}(\eta_3).$$

But in this last case V does not lift to a $Spin(2n)$ -bundle.

The case of $SO(2n+1)$, which we shall not state explicitly, is completely analogous, except that the summand $I_{2a_0+1} \oplus \mathcal{O}_E$ is replaced by the odd rank summand I_{2a_0+1} , which must always be present.

We return now to the study of regular bundles over a general group G .

Proposition 3.6. *Let ξ be a semistable principal G -bundle over E . Then the structure group of ξ reduces to an abelian subgroup of G . If furthermore ξ is regular, the structure group of ξ reduces to an abelian subgroup of $\text{Aut}_G \xi$, which naturally sits inside G up to conjugation.*

In fact, one can take the structure group of ξ to be of the following form. Let ξ_0 be the split bundle S -equivalent to ξ , corresponding to the representation $\rho: \pi_1(E) \rightarrow T \subset K$. Let T_0 be the image of ρ . Then there exists a subgroup U of G commuting with T_0 , which is either trivial or a 1-parameter commutative unipotent subgroup, such that the structure group of ξ reduces to $T_0 U$.

We now describe the set of bundles which are simultaneously regular and split. If ξ is split, then ξ corresponds to a point of $(E \otimes_{\mathbb{Z}} \Lambda)/W$. After lifting

this point to an element μ of $E \otimes_{\mathbb{Z}} \Lambda$, we see that we can describe $\text{ad } \xi$ as follows. A root α defines a homomorphism $\Lambda \rightarrow \mathbb{Z}$, and thus a homomorphism $E \otimes_{\mathbb{Z}} \Lambda \rightarrow E \cong \text{Pic}^0 E$. Denote the image of μ in E by $\alpha(\mu)$ and the corresponding line bundle by $\lambda_{\alpha(\mu)}$. Then, as vector bundles,

$$\text{ad } \xi \cong \mathcal{O}_E^r \oplus \bigoplus_{\alpha} \lambda_{\alpha(\mu)}.$$

Hence ξ is regular if and only if, for every root α , $\alpha(\mu) \neq 0$.

In particular, there is a nonempty Zariski open subset of $(E \otimes_{\mathbb{Z}} \Lambda)/W$ such that all of the corresponding split bundles are regular. In fact, on this open subset, S-equivalence is the same as isomorphism.

At the other extreme, we can consider bundles which are S-equivalent to the trivial bundle. The split representative for the S-equivalence class corresponds to the image of $0 \in E \otimes_{\mathbb{Z}} \Lambda$ in $(E \otimes_{\mathbb{Z}} \Lambda)/W$, which has the unique preimage $0 \in E \otimes_{\mathbb{Z}} \Lambda$. To describe the regular representative, or more precisely its adjoint bundle, we first recall the definition of the *Casimir weights* d_1, \dots, d_r of a root system R . These can be defined to be the numbers $m_i + 1$, where the m_i are the exponents of R (cf. [8], V (6.2)), and they are also the degrees of a set of homogeneous generators for the invariants of the symmetric algebra of the vector space corresponding to the root system R under the action of the Weyl group. To describe a regular bundle S-equivalent to the trivial bundle, we shall describe its adjoint bundle. (Here an $(\text{ad } G)$ -bundle has in general finitely many liftings to a G -bundle, but exactly one of these will turn out to be S-equivalent to the trivial bundle.)

Proposition 3.7. *There is a unique regular G -bundle ξ S-equivalent to the trivial bundle. As vector bundles over E ,*

$$\text{ad } \xi \cong \bigoplus_i I_{2d_i-1},$$

where the d_i are the Casimir weights of the root system of G .

The bundle $\text{ad } \xi$ can be seen to be an $(\text{ad } G)$ -bundle as follows: start with the bundle $I_3 = \text{Sym}^2 I_2$. It is an $SL(2)$ bundle which descends to an $SO(3)$ -bundle. Now there is a “maximal” embedding of $SO(3)$ in G , unique up to conjugation. Thus there is a representation ρ of $SO(3)$ on the Lie algebra \mathfrak{g} . Under this representation \mathfrak{g} decomposes as a direct sum

$$\mathfrak{g} = \bigoplus_i \text{Sym}^{2d_i-2}(\mathbb{C}^2),$$

where we view \mathbb{C}^2 as the standard representation of $SL(2)$, and thus its odd symmetric powers give representations of $SO(3)$. In particular, the G -bundle induced by ρ gives rise to the $(\text{ad } G)$ -bundle described above.

We can generalize the above picture for the trivial bundle to an arbitrary bundle. Let ξ be an arbitrary semistable G -bundle and let ξ_0 be the unique split bundle S -equivalent to ξ . Then $\text{Aut } \xi_0$ is up to isogeny a product of N factors G_i , where each factor G_i is either simple or isomorphic to \mathbb{C}^* . Let $\mu \in E \otimes_{\mathbb{Z}} \Lambda$ be a representative for the class of ξ_0 . The Lie algebra $H^0(E; \text{ad } \xi_0)$ of $\text{Aut } \xi_0$ is identified with

$$\mathfrak{h} \oplus \bigoplus_{\alpha(\mu)=0} \mathfrak{g}^{\alpha},$$

where \mathfrak{g}^{α} is the root space corresponding to the root α (and thus in particular the rank of this reductive Lie algebra is r). We then have:

Proposition 3.8. *With notation as above, let ξ_{reg} be a regular semistable bundle S -equivalent to ξ_0 . Let r_i be the rank of G_i , where by definition $r_i = 1$ if $G_i \cong \mathbb{C}^*$, and let $d_{ij}, 1 \leq j \leq r_i$ be the Casimir weights of G_i , where we set $d_{i1} = 1$ if $G_i \cong \mathbb{C}^*$. Then the maximal subbundle of $\text{ad } \xi_{\text{reg}}$ which is filtered by subbundles whose successive quotients are \mathcal{O}_E is*

$$(\text{ad } \xi_{\text{reg}})_{\mathcal{O}_E} = \bigoplus_{i=0}^N \bigoplus_{j=1}^{r_i} I_{2d_{ij}-1}.$$

From this, it is possible in principle to give a complete description of $\text{ad } \xi_{\text{reg}}$. As a consequence of Proposition 3.6, one can show:

Proposition 3.9. *Let ξ be a semistable principal G -bundle. Then ξ is S -equivalent to a unique regular semistable bundle and to a unique split bundle.*

There are thus two canonical representatives for every S -equivalence class, depending on whether we choose the regular or the split representative. For an open dense subset of bundles, these two representatives will in fact coincide. As should be clear from Section 2, the split representatives arise most naturally from the point of view of flat connections. However, if we try to find a universal holomorphic G -bundle, then we must work instead with regular bundles. In fact, even working locally, it is not possible to fit the split bundles together into a universal bundle, even for $SL(n)$.

Finally, we make some comments about the automorphism group of a regular bundle.

Proposition 3.10. *Let ξ be a regular semistable G -bundle. If $\text{Aut}_G(\xi)$ is the automorphism group of ξ and $(\text{Aut}_G(\xi))^0$ is the component of $\text{Aut}_G(\xi)$ containing the identity, then $(\text{Aut}_G(\xi))^0$ is abelian. Moreover, $\text{Aut}_G(\xi)$ is itself abelian if and only if ξ corresponds to a smooth point of the moduli space of S -equivalence classes of semistable G -bundles.*

In fact, a careful analysis of the root systems involved shows that the singular locus of the moduli space corresponding to $\mathbb{Z}/d\mathbb{Z}$ -isotropy is smooth and irreducible, of dimension equal to the number of i such that $d|g_i$, in the notation of Theorem 2.7. Of course, this statement also follows directly from Theorem 2.7.

4. The parabolic construction

In this section, we describe a method of constructing families of regular semi-stable G -bundles. The motivation is as follows: we seek to find an analogue for bundles of the singularities picture outlined above in Section 2. That is, we seek to find a mildly “singular” (in other words, unstable) G -bundle ξ_0 together with a \mathbb{C}^* -action on its deformation space, such that the weighted projective space corresponding to the quotient of the negative weight deformations of ξ_0 by \mathbb{C}^* is both the weighted projective space $WP(g_0, \dots, g_r)$ and is the coarse moduli space of semistable G -bundles modulo S -equivalence. (Actually, with our conventions the action of \mathbb{C}^* will be by positive weights.) It will also turn out that the points of the weighted projective space parametrize regular G -bundles, as opposed to split bundles, and will thus enable us to find locally a universal G -bundle away from the orbits where \mathbb{C}^* does not act freely. In fact, in many cases we can use this construction to produce a global universal G -bundle.

To pursue this idea further, we have seen that unstable G -bundles over E reduce to a parabolic subgroup of E , and further to a Levi factor L . Conversely, fix a maximal parabolic subgroup P of G and a Levi decomposition $P = LU$, where U is the unipotent radical of P and L is the reductive or Levi factor. Then U is normal, all Levi factors are conjugate in P , and the quotient homomorphism $P \rightarrow L$ is well-defined. The group L is never semisimple; in fact, since P is a maximal parabolic, the connected component of the center of L is \mathbb{C}^* . The maximal parabolic subgroup P has a canonical character $\chi: P \rightarrow \mathbb{C}^*$ (the unique primitive dominant character), which is induced from a character $L \rightarrow \mathbb{C}$. Using this character, we can define the determinant line bundle of a principal L -bundle over E . Fix an L -bundle η , such that $\det \eta$ has negative degree. The induced G -bundle ξ_0 is unstable, because ξ_0 also reduces to the opposite parabolic to P , and the determinant line bundle for the primitive dominant character of the opposite parabolic has positive degree. Consider the set of all P -bundles ξ such that the associated L -bundle (via the homomorphism $P \rightarrow L$) is η . It is straightforward to classify all such bundles: the group L acts by conjugation on U , and the L -bundle η and the action of L on U define a sheaf of unipotent groups $\underline{U}(\eta)$ on E , which is in general nonabelian. The set of all isomorphism classes of P -bundles ξ which reduce to η may then be identified with the cohomology set $H^1(E; \underline{U}(\eta))$. The \mathbb{C}^* in the center of L then acts on $H^1(E; \underline{U}(\eta))$. Cohomology sets similar to $H^1(E; \underline{U}(\eta))$, arising from the H^1 of a sheaf of unipotent groups over a base curve, have been studied in a different context by Babbitt and Varadarajan [3], following ideas of Deligne, as well as by Faltings [14]. Using similar ideas, one can show that $H^1(E; \underline{U}(\eta))$ has a (non-canonical) linear structure and that \mathbb{C}^* acts linearly in this structure with positive weights (following certain standard conventions), so that the quotient is isomorphic to a weighted projective space.

In the case of $SL(n)$, it is easy to make these ideas explicit. The maximal parabolic subgroups of $SL(n)$ correspond to filtrations $\{0\} \subset \mathbb{C}^d \subset \mathbb{C}^n$, where $0 < d < n$. For each such d , there is a unique stable bundle W_d over E of rank d such that $\det W_d = \mathcal{O}_E(p_0)$. The unstable bundle which we consider

is then $W_d^* \oplus W_{n-d}$, and it has a nontrivial \mathbb{C}^* -action which acts trivially on $\det(W_d^* \oplus W_{n-d})$. In this case, a straightforward argument shows:

Theorem 4.1. *Let V be a regular semistable vector bundle of rank n . Then there is an exact sequence*

$$0 \rightarrow W_d^* \rightarrow V \rightarrow W_{n-d} \rightarrow 0.$$

Moreover, the automorphism group of V acts transitively on the set of subbundles of V isomorphic to W_d^ whose quotients are isomorphic to W_{n-d} . Finally, if V is a nonsplit extension of W_{n-d} by W_d^* , then V is in fact a regular semistable vector bundle.*

We note that in this case the parabolic subgroup in question is

$$P = \left\{ \begin{pmatrix} A & B \\ O & D \end{pmatrix} : A \in GL(d), D \in GL(n-d), \det A \cdot \det D = 1 \right\},$$

the Levi factor of P is given by

$$L = \left\{ \begin{pmatrix} A & O \\ O & D \end{pmatrix} : A \in GL(d), D \in GL(n-d), \det A \cdot \det D = 1 \right\},$$

and the unipotent radical U of P , which in this case is abelian, is given by

$$U = \left\{ \begin{pmatrix} I & B \\ O & I \end{pmatrix} : B \text{ is a } d \times (n-d) \text{ matrix} \right\}.$$

It is easy then to identify $H^1(E; \underline{U}(\eta))$ with the usual sheaf cohomology group $H^1(E; W_{n-d}^* \otimes W_d^*)$ and the \mathbb{C}^* -action with the usual one, up to a factor. In this way, the moduli space of regular semistable vector bundles over E of rank n and trivial determinant is identified with \mathbb{P}^{n-1} , a fact which could also be established by spectral cover methods [20]. The full tangent space to the deformations of the unstable bundle $W_d^* \oplus W_{n-d}$ keeping the determinant trivial is $H^1(E; \text{ad}(W_d^* \oplus W_{n-d}))$. This group contains the subgroup $H^1(E; W_{n-d}^* \otimes W_d^*)$ which is tangent to the set of extensions described above. The one remaining direction has \mathbb{C}^* -weight zero, which corresponds to moving the point p_0 on E and which should be viewed as a one parameter family of locally trivial deformations.

In the case of $SL(n)$, or equivalently the root system A_{n-1} , every maximal parabolic subgroup has an abelian unipotent radical and there is an appropriate construction from any such subgroup giving the moduli space of regular semistable G -bundles. In all other cases, we have the following:

Theorem 4.2. *Let G be a complex, simple, and simply connected group, not of type A_n . Then there exists a unique maximal parabolic subgroup P of G , up to*

conjugation, such that, if L is the Levi factor of P , then there exists an L -bundle η with the following properties:

- (i) The connected component of the automorphism group of η as an L -bundle is \mathbb{C}^* .
- (ii) The line bundle $\det \eta$ has negative degree, and so the G -bundle ξ_0 induced by η is unstable.
- (iii) If U is the unipotent radical of P , then the nonabelian cohomology set $H^1(E; \underline{U}(\eta))$ has the structure of affine $(r+1)$ -dimensional space.
- (iv) There exists a linear structure on $H^1(E; \underline{U}(\eta))$ for which the natural copy of $\mathbb{C}^* \subseteq \text{Aut}_G \xi_0$ acts linearly, fixing the trivial element, and with positive weights. The stabilizer of every nontrivial element of $H^1(E; \underline{U}(\eta))$ is finite, and the quotient $(H^1(E; \underline{U}(\eta)) - \{0\})/\mathbb{C}^*$ is a weighted projective space $WP(g_0, \dots, g_r)$.
- (v) If ξ is a P -bundle over E corresponding to an element of $H^1(E; \underline{U}(\eta)) - 0$, then ξ is a regular semistable bundle.

In all cases, the bundle η with the above properties is uniquely specified by requiring that $\det \eta = \mathcal{O}_E(-p_0)$.

In fact, (iv) and (v) are a consequence of the other properties. If we do not specify that $\det \eta = \mathcal{O}_E(-p_0)$, then it is still the case that $\det \eta$ must have degree -1 , and so η is specified up to translation on E .

We note that all of the weights are equal, in other words the weighted projective space is an ordinary projective space, exactly in the cases A_n and C_n , in other words for the groups $SL(n+1)$ and $Sp(2n)$. In all other cases, for a simply connected group G , the weighted projective space will in fact have singularities.

To describe the maximal parabolic subgroups which arise in Theorem 4.2, note first that maximal parabolic subgroups of G , up to conjugation, are in one-to-one correspondence with the vertices of the Dynkin diagram of the corresponding root system. In case G is D_n or E_6, E_7, E_8 , the maximal parabolic subgroup in Theorem 4.2 corresponds to the unique trivalent vertex of the Dynkin diagram. In the remaining cases, the vertex in question is the unique vertex meeting the multiple edge which is the long root. (Such vertices will be trivalent in an appropriate sense except for the case C_n .)

Let us describe the construction explicitly for the remaining classical groups. The simplest case after A_n is the case of $Sp(2n)$, in other words C_n . In this case the parabolic in question corresponds to those elements of $Sp(2n)$ which preserve a totally isotropic subspace of dimension n . Thus

$$P = \left\{ \begin{pmatrix} T & B \\ O & {}^t T^{-1} \end{pmatrix} : T \in GL(n), T^{-1}B = {}^t(T^{-1}B) \right\},$$

the Levi factor of P is given by

$$L = \left\{ \begin{pmatrix} T & O \\ O & {}^t T^{-1} \end{pmatrix} : T \in GL(n) \right\},$$

and the unipotent radical U of P , which in this case is also abelian, is given by

$$U = \left\{ \begin{pmatrix} I & B \\ O & I \end{pmatrix} : {}^t B = B \right\}.$$

The unstable symplectic bundle corresponding to η is the bundle $W_n^* \oplus W_n$, with the first factor embedded as a totally isotropic subbundle and the second as its dual. It is easy then to identify $H^1(E; \underline{U}(\eta))$ with the usual sheaf cohomology $H^1(E; \text{Sym}^2 W_n^*)$. Here \mathbb{C}^* acts with constant weight, so that the quotient is an ordinary (smooth) \mathbb{P}^{n-1} .

Next we consider $Spin(2n)$, although here it will be more convenient to work in $SO(2n)$. The natural analogue of the construction for the symplectic group would lead to the unstable bundle $W_n^* \oplus W_n$, together with the symmetric non-degenerate form for which W_n^* is isotropic and which identifies the dual of W_n^* with the complementary W_n . Such orthogonal bundles do not lift to $Spin(2n)$, although this construction does identify all of the regular semistable $SO(2n)$ -bundles with $w_2 \neq 0$ with the projective space on $H^1(E; \bigwedge^2 W_n^*)$, which is a \mathbb{P}^{n-2} . For liftable $SO(2n)$ -bundles, we use the parabolic subgroup corresponding to the trivalent vertex, which is the subgroup of $g \in SO(2n)$ preserving an isotropic subspace of rank $n-2$. In this case the unipotent radical is nonabelian. The bundle η , viewed as an unstable $SO(2n)$ -bundle ξ_0 , is the bundle

$$\xi_0 = W_{n-2}^* \oplus Q_4 \oplus W_{n-2},$$

where W_{n-2}^* is an isotropic subspace, Q_4 is the $SO(4)$ -bundle $\mathcal{O}_E \oplus \eta_1 \oplus \eta_2 \oplus \eta_3$, in the notation of Section 3, with a diagonal nondegenerate symmetric pairing, and Q_4 is orthogonal to the direct sum $W_{n-2}^* \oplus W_{n-2}$. More invariantly $Q_4 = \text{Hom}(W_2, W_2)$ with the quadratic form given by the trace. Note that neither Q_4 nor $W_{n-2}^* \oplus W_{n-2}$ lifts to a $Spin$ -bundle, and hence the direct sum is liftable. In this case, the nonabelian cohomology set $H^1(E; \underline{U}(\eta))$ (for $Spin(2n)$) has a weight 1 piece given by $H^1(Q_4 \otimes W_{n-2}^*)$, of rank 4, and a weight 2 piece given by $H^1(E; \bigwedge^2 W_{n-2}^*)$, which has rank $n-3$. Similar results hold for $Spin(2n+1)$, by replacing Q_4 by

$$Q_3 = \eta_1 \oplus \eta_2 \oplus \eta_3 = \text{ad } W_2.$$

Returning to the general case, let us show that the weighted projective space $WP(g_0, \dots, g_r)$ arising from the parabolic construction can be naturally identified with $(E \otimes_{\mathbb{Z}} \Lambda)/W$, thus giving a new proof of Looijenga's theorem. One first shows that there exists a universal G -bundle over $E \times H^1(E; \underline{U}(\eta))$ in the appropriate sense. By general properties, there is a \mathbb{C}^* -equivariant map from $H^1(E; \underline{U}(\eta)) - 0$ to the moduli space of semistable G -bundles, in other words to $(E \otimes_{\mathbb{Z}} \Lambda)/W$.

Theorem 4.3. *The induced map $WP(g_0, \dots, g_r) \rightarrow (E \otimes_{\mathbb{Z}} \Lambda)/W$ is an isomorphism.*

The essential point of the proof is to compare the determinant line bundles on the two sides, and then to use the elementary fact that a degree one morphism

from a weighted projective space to a normal variety is an isomorphism. On the weighted projective side, the determinant line bundle is always Cartier, and in fact it is the line bundle $K_{WP^r}^2$. On the other side, it is easy to calculate the preimage of the determinant line bundle in $E \otimes_{\mathbb{Z}} \Lambda$. At least in the case of a simply laced root system R , the fact that the degree of the morphism in question is one then follows from the fact that the order of the Weyl group is $r!(g_1 \cdots g_r) \det R$ [8].

The parabolic construction also leads to a proof of the existence of universal bundles in certain cases. For a fixed G , we denote by $\mathcal{M}_E = \mathcal{M}_E(G)$ the moduli space of regular semistable G -bundles over E and by \mathcal{M}_E^0 the smooth locus of \mathcal{M}_E .

Theorem 4.4. *If $G = SL(n)$, let P_d be the maximal parabolic subgroup of $SL(n)$ stabilizing the flag $\{0\} \subset \mathbb{C}^d \subset \mathbb{C}^n$, and if $G \neq SL(n)$, let P be the maximal parabolic subgroup of G described in Theorem 4.2. Let n_P be the positive integer defined as follows:*

- (i) *If $G = SL(n)$ and $P = P_d$, then $n_{P_d} = n / \gcd(d, n)$.*
- (ii) *If G is of type C_n , B_n with n even, or D_n with n odd, then $n_P = 2$.*
- (iii) *In all other cases, $n_P = 1$.*

Let \bar{G} be the quotient of G by the unique subgroup of the center of G of order n_P . Then the universal G -bundle over $E \times H^1(E; \underline{U}(\eta))$ descends to a universal \bar{G} -bundle $\bar{\Xi}$ on $E \times \mathcal{M}_E^0$.

Let us mention the analogous results for families of elliptic curves over a base B . Let $\pi: Z \rightarrow B$ be a flat family, all of whose fibers are smooth elliptic curves or more generally irreducible curves of arithmetic genus one (i.e. smooth, nodal, or cuspidal curves). Let σ be a section of π meeting each fiber in a smooth point. Associated to Z is the line bundle L on B defined by $L^{-1} = R^1\pi_*\mathcal{O}_Z$, which can be identified with $\mathcal{O}_Z(\sigma)|_{\sigma}$ under the isomorphism $\sigma \rightarrow B$ induced by π . We want to describe the parabolic construction along the family Z . To do so, recall that we have the weights g_i of (2.8), which we assume ordered so that $g_i \leq g_{i+1}$. Recall also that we have defined the Casimir weights d_1, \dots, d_r of a root system R in Section 3. We order the d_i by increasing size, except in the case of D_n , where we order the d_i by: $2, 4, n, 6, 8, \dots, 2n - 2$.

Our result in families can then be somewhat loosely stated as follows:

Theorem 4.5. *Suppose that $G \neq E_8$. The parabolic construction then globalizes over Z to give a bundle of nonabelian cohomology groups over B . This bundle is a bundle of affine spaces with a \mathbb{C}^* -action which is isomorphic to the vector bundle*

$$\mathcal{O}_B \oplus L^{-d_1} \oplus \cdots \oplus L^{-d_r}.$$

Via this isomorphism \mathbb{C}^ acts diagonally on the line bundles in the direct sum, by the weight g_i on the factor L^{-d_i} (and with weight $g_0 = 1$ on the factor \mathcal{O}_B). The associated bundle of weighted projective spaces is then a universal relative moduli space for G -bundles which are regular and semistable on every fiber.*

A result closely related to Theorem 4.5 was established by Wirthmüller [33], who also noted the exceptional status of E_8 . We note that, from our point of view, in the case of E_8 there is a family of weighted projective spaces over the open subset B' of B over which the fibers of π are either smooth or nodal. However, this family is not the quotient of a vector bundle minus its zero section by \mathbb{C}^* acting diagonally. Furthermore, the construction degenerates in an essential way at the cuspidal curves. A similar phenomenon appears if we try to classify generalized del Pezzo surfaces of degree one with an appropriate hyperplane section.

5. Automorphism sheaves and spectral covers

In this section, we fix G and denote by $\mathcal{M}_E(G) = \mathcal{M}_E$ the moduli space of regular semistable G -bundles over E . Likewise, given an elliptic fibration with a section $\pi: Z \rightarrow B$ whose fibers are smooth elliptic curves or nodal or cuspidal cubics (except in the case $G = E_8$ where we will not allow cuspidal fibers), we have a relative moduli space $\mathcal{M}_{Z/B} = \mathcal{M}_{Z/B}(G)$. Thus in all cases $\mathcal{M}_{Z/B}$ is a bundle of weighted projective spaces.

Because of fixed points for the \mathbb{C}^* action, the universal G -bundle over $E \times H^1(E; \underline{U}(\eta))$ does not descend to a universal G -bundle over $E \times \mathcal{M}_E$, even locally, near the singular points of \mathcal{M}_E , and a similar statement holds in families. However, let \mathcal{M}_E^0 denote the smooth locus of \mathcal{M}_E , and similarly for $\mathcal{M}_{Z/B}^0$. Then locally in either the classical or étale topology there exists a universal bundle Ξ over $E \times \mathcal{M}_E^0$, and similarly for $Z \times_B \mathcal{M}_{Z/B}^0$. As we have seen in Theorem 4.4, there also exists a \bar{G} -bundle $\bar{\Xi}$ over $E \times \mathcal{M}_E^0$, where \bar{G} is a quotient of G by a subgroup of the center of order at most two. In particular, a universal adjoint bundle always exists. In this section, we describe the issues of the existence and uniqueness of a global universal bundle over $E \times \mathcal{M}_E^0$ or $Z \times_B \mathcal{M}_{Z/B}^0$.

There are other questions closely related to these. Given a family $\pi: Z \rightarrow B$ as above, suppose that Ξ is a G -bundle over Z such that $\Xi|_{\pi^{-1}(b)}$ is a semistable bundle for all b for which $\pi^{-1}(b)$ is smooth. Then Ξ defines a section of $\mathcal{M}_{Z/B}$ over the open subset of B consisting of such b . At the singular points of $\mathcal{M}_{Z/B}$, the section is locally liftable to the affine bundle of cohomology groups over B . Conversely, a locally liftable section defines local G -bundles over $\pi^{-1}(U)$ for all sufficiently small open sets U of B (in the classical or étale topology). Note that the parabolic construction extends over the singular fibers of π (except for cuspidal fibers in case $G = E_8$), dictating the correct definition of regular semistable G -bundles for a singular fiber. When does a locally liftable section of $\mathcal{M}_{Z/B}$ actually determine a G -bundle over Z ? More generally, how can we describe the set (possibly empty) of all bundles corresponding to a given section? For simplicity, we shall assume that the section does not pass through the singular points of $\mathcal{M}_{Z/B}$. Thus, if there existed a relative universal bundle over $Z \times_B \mathcal{M}_{Z/B}^0$, we could simply pull this bundle back by the section to obtain a bundle over B . While a relative universal bundle does not usually exist, there

are many cases where a section does indeed determine a G -bundle. However, our answers are complete only in the cases $G = SL(n), Sp(2n)$.

Working for the moment with a single curve E , over an open subset of \mathcal{M}_E^0 where there exists a local universal bundle Ξ , there is an associated group scheme $\underline{\text{Aut}}(\Xi)$. Because the associated automorphism groups are abelian on \mathcal{M}_E^0 , as follows from (3.10), these local group schemes piece together to give an abelian group scheme over \mathcal{M}_E^0 , whose associated sheaf of sections will be denoted \mathcal{A} . In the usual way, the obstruction to finding a global universal principal G -bundle over $E \times \mathcal{M}_E^0$ lies in $H^2(\mathcal{M}_E^0; \mathcal{A})$, and if this obstruction is zero, then the set of all such principal bundles is a principal homogeneous space over $H^1(\mathcal{M}_E^0; \mathcal{A})$. More generally, given an elliptic fibration $\pi: Z \rightarrow B$ as above, we can fit together the automorphism group schemes of local universal bundles to find an abelian group scheme over $\mathcal{M}_{Z/B}^0$ whose fiber over every point $b \in B$ is the group scheme constructed above. Let \mathcal{A}_B denote the sheaf of sections of this group scheme. Given a section s of $\mathcal{M}_{Z/B}^0 \rightarrow B$, we can pull back the the above group scheme to obtain a group scheme over B , whose sheaf of sections we denote by $\mathcal{A}_B(s)$. Just as in the case of a single smooth elliptic curve, the obstruction to finding a G -bundle over Z corresponding to the section s lies in $H^2(B; \mathcal{A}_B(s))$, and if this obstruction is zero, then the set of all such bundles is a principal homogeneous space over $H^1(B; \mathcal{A}_B(s))$.

Let us describe the sheaf \mathcal{A} in the case of $SL(n)$ and a fixed elliptic curve E in more detail. For each integer d , $1 \leq d \leq n-1$, one can construct a universal extension \mathcal{E}_d over $E \times \mathbb{P}^{n-1}$, viewing \mathbb{P}^{n-1} as $\text{Ext}^1(W_{n-d}, W_d^*)$, which fits into an exact sequence

$$0 \rightarrow \pi_1^* W_d^* \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^{n-1}}(1) \rightarrow \mathcal{E}_d \rightarrow \pi_1^* W_{n-d} \rightarrow 0.$$

Clearly, $\det \mathcal{E}_d$ has trivial restriction to each slice $E \times \{s\}$ but is not in fact trivial. On the other hand, since the restriction of \mathcal{E}_d to every fiber is regular and semistable, $\pi_{2*} \text{Hom}(\mathcal{E}_d, \mathcal{E}_d)$ is a sheaf of locally free commutative \mathbb{C} -algebras over \mathbb{P}^{n-1} of rank n , and thus corresponds to a finite morphism $\nu: T \rightarrow \mathbb{P}^{n-1}$ of degree n , which we shall call the *spectral cover* of \mathbb{P}^{n-1} . It is straightforward to identify the base \mathbb{P}^{n-1} with the complete linear system $|np_0|$ and the cover T with the incidence correspondence in $\mathbb{P}^{n-1} \times E$, in other words

$$T = \left\{ \left(\sum_{i=1}^n e_i, e \right) : \sum_{i=1}^n e_i \in |np_0|, e = e_i \text{ for some } i \right\}.$$

Thus T is smooth, and it has the structure of a \mathbb{P}^{n-2} -bundle over E such that the \mathbb{P}^{n-2} fibers are mapped to hyperplanes in \mathbb{P}^{n-1} under ν . Another way to describe T is as follows: let $\Lambda \cong \mathbb{Z}^{n-1}$ as the sublattice of \mathbb{Z}^n of vectors whose sum is zero, acted on by the Weyl group \mathfrak{S}_n , so that $\mathbb{P}^{n-1} = |np_0| = (E \otimes \Lambda) / \mathfrak{S}_n$. Let $W_0 = \mathfrak{S}_{n-1} \subset \mathfrak{S}_n$ be the stabilizer of the vector $e_n \in \mathbb{Z}^n$. Then $T = (E \otimes \Lambda) / W_0$.

A standard argument shows that, if \mathcal{V} is a vector bundle over $E \times \mathbb{P}^{n-1}$ whose restriction to every slice is isomorphic to the corresponding restriction of \mathcal{E}_d ,

then $\pi_{2*} \operatorname{Hom}(\mathcal{V}, \mathcal{E}_d)$ is locally free of rank one over $\pi_{2*} \operatorname{Hom}(\mathcal{E}_d, \mathcal{E}_d) = \nu_* \mathcal{O}_T$, and thus corresponds to a line bundle on T , and conversely every line bundle on T defines a vector bundle \mathcal{V} with the above property. It is helpful to compare this situation with the one usually encountered in algebraic geometry, where we try to make a moduli space of simple vector bundles and then the only choice is to twist by the pullback of a line bundle from the moduli space factor.

From this it follows that, in the case of $SL(n)$, the automorphism sheaf \mathcal{A} is given by the kernel of the norm homomorphism $\nu_* \mathcal{O}_T^* \rightarrow \mathcal{O}_{\mathbb{P}^{n-1}}^*$. Hence there is an exact sequence

$$0 \rightarrow H^1(\mathbb{P}^{n-1}; \mathcal{A}) \rightarrow \operatorname{Pic} T \rightarrow \operatorname{Pic} \mathbb{P}^{n-1} \rightarrow H^2(\mathbb{P}^{n-1}; \mathcal{A}) \rightarrow H^2(\mathcal{O}_T^*) \rightarrow 0.$$

Thus, $H^1(\mathbb{P}^{n-1}; \mathcal{A}) \cong \mathbb{Z} \times E$ for $n > 2$ and $H^1(\mathbb{P}^1; \mathcal{A}) \cong E$, and $H^2(\mathbb{P}^{n-1}; \mathcal{A}) \cong H^3(T; \mathbb{Z})$. There is also an analogue of the above exact sequence where we take étale cohomology. In this case, $H^1(\mathbb{P}^{n-1}; \mathcal{A})$ is unchanged and $H^2(\mathbb{P}^{n-1}; \mathcal{A}) \cong H_{\text{ét}}^2(T; \mathbb{G}_m)$, which is a torsion group. The obstruction to gluing together local families (in either the classical or étale topology) of $SL(n)$ -bundles to make a global $SL(n)$ -bundle over $E \times \mathbb{P}^{n-1}$ lives in $H^2(\mathbb{P}^{n-1}; \mathcal{A})$, and in case the obstruction is zero the set of all such bundles is then a principal homogeneous space over $H^1(\mathbb{P}^{n-1}; \mathcal{A})$. In our case, a direct construction using the pushforward of appropriate line bundles on $E \times T$ shows that the obstruction in $H^2(\mathbb{P}^{n-1}; \mathcal{A})$ vanishes. Thus the family of universal $SL(n)$ -bundles \mathcal{V} over $E \times \mathbb{P}^{n-1}$ is parametrized by $\mathbb{Z} \times E$ for $n > 2$ and by E if $n = 2$. In case we consider the corresponding situation in families $Z \rightarrow B$, then there exist mod 2 obstructions to finding a principal $SL(n)$ -bundle over the entire family, and these obstructions are not in general zero. On the other hand, there always exists a universal $GL(n)$ -bundle V such that $V|_{\pi^{-1}(b)}$ has trivial determinant for all b , so that $\det V$ is pulled back from B . See [20] for more detail in the case of vector bundles.

Similar explicit constructions can be carried out for the symplectic group. Let $\Lambda = \mathbb{Z}^n$, and let the Weyl group $W = \mathfrak{S}_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$ act on Λ , where the symmetric group acts by permuting the basis elements and $(\mathbb{Z}/2\mathbb{Z})^n$ acts by sign changes. Then $(E \otimes \Lambda)/W = \operatorname{Sym}^n \mathbb{P}^1 = \mathbb{P}^n$. Let $W_0 = \mathfrak{S}_{n-1} \ltimes (\mathbb{Z}/2\mathbb{Z})^{n-1}$ be the subgroup of W fixing the last basis vector, and set $T^{\text{sp}} = (E \otimes \Lambda)/W_0 = \mathbb{P}^{n-1} \times T$. The group W_0 is a subgroup of index two in the larger group $W_1 = \mathfrak{S}_{n-1} \ltimes (\mathbb{Z}/2\mathbb{Z})^n$, and there is an induced involution ι on T^{sp} , with quotient $T^{\text{sp}}/\iota = S = \mathbb{P}^{n-1} \times \mathbb{P}^1$. We then have:

Proposition 5.1. *For the symplectic group $Sp(2n)$, the automorphism sheaf \mathcal{A}^{sp} over $\mathcal{M}_E(Sp(2n)) \cong \mathbb{P}^n$ is given by*

$$\mathcal{A}^{\text{sp}} = \{ f \in \nu_* \mathcal{O}_{T^{\text{sp}}}^* : \iota^* f = f^{-1} \}.$$

Using (5.1) one can show that there is a universal bundle over $E \times \mathcal{M}_E(Sp(2n))$ as well, and that the set of all universal bundles is parametrized by E . Thus we have constructed universal bundles over $E \times \mathcal{M}_E$ in the two cases where the

moduli space is smooth. It then follows from Theorem 4.4 that a universal bundle exists over $\mathcal{M}_E^0(G)$ in all cases, with the possible exception of $G = Spin(4n+1)$ and $G = Spin(4n+2)$.

We return to the case of a general G and analyze the structure of the sheaf \mathcal{A} over \mathcal{M}_E^0 . Since \mathcal{A} is the sheaf of sections of an abelian algebraic group scheme, there is the exponential map \exp from the corresponding sheaf of Lie algebras $\text{Lie } \mathcal{A}$ to \mathcal{A} . The kernel of \exp is a constructible sheaf, which we denote by $\underline{\Lambda}$, and the image of \exp is the sheaf \mathcal{A}^0 which, locally, consists of all sections of $\underline{\text{Aut}}(\Xi)$ passing through the identity component of every fiber. First we note that $\mathcal{A} = \mathcal{A}^0$ on the Zariski open subset U of \mathcal{M}^0 consisting of split bundles, where the fiber over $x \in U$ of the group scheme corresponding to \mathcal{A} is $(\mathbb{C}^*)^r$ and is connected. If the root system for G is simply laced, we can say more:

Proposition 5.2. *Suppose that G is simply laced. If $G \neq SL(2)$, then the set*

$$\{ \xi \in \mathcal{M}^0 : \mathcal{A}_\xi \neq \mathcal{A}_\xi^0 \}$$

has codimension at least two in \mathcal{M}^0 .

As a consequence, in the relative setting, for G simply laced, if $\dim B = 1$ and $G \neq SL(2)$, then for a generic section s of $\mathcal{M}_{Z/B}^0$, we can always assume that $\mathcal{A}_B(s) = \mathcal{A}_B^0(s)$. The above proposition does not hold if G is not simply laced; for example, it fails for $Sp(2n)$.

Next we turn to $\mathcal{A}^0 = \text{Lie } \mathcal{A} / \underline{\Lambda}$. Note that, in case there is a universal bundle Ξ over $E \times \mathcal{M}^0$, then $\text{Lie } \mathcal{A} = R^0 p_{2*}(\text{ad } \Xi)$ is dual to $R^1 p_{2*}(\text{ad } \Xi)$, which is the tangent bundle to \mathcal{M}^0 . Thus $\text{Lie } \mathcal{A} = \Omega_{\mathcal{M}^0}^1$ is the cotangent bundle. In fact, this statement always holds, since a universal bundle exists locally and the automorphism sheaf is abelian. Another way to describe the cotangent bundle is as follows: let \mathfrak{h} be the Lie algebra of a Cartan subgroup of G . Then the Weyl group acts on $E \otimes \Lambda$ and on the trivial vector bundle $\mathcal{O}_{E \otimes \Lambda} \otimes_{\mathbb{C}} \mathfrak{h}$, and the sheaf of W -invariant sections is a coherent sheaf over $(E \otimes \Lambda)/W = \mathcal{M}$ whose restriction to \mathcal{M}^0 is locally free, and in fact is $\Omega_{\mathcal{M}^0}^1$. The constructible sheaf $\underline{\Lambda}$ can be described as follows. Let U be the open subset of \mathcal{M}^0 over which the map $E \otimes \Lambda \rightarrow \mathcal{M}$ is unramified, and let $i: U \rightarrow \mathcal{M}^0$ be the inclusion. Then the action of W on Λ gives a locally constant sheaf $\underline{\Lambda}_0$ on U , and $\underline{\Lambda} = i_* \underline{\Lambda}_0$. The map $\Lambda \rightarrow \mathfrak{h}$ induces an inclusion $\underline{\Lambda} \rightarrow (\mathcal{O}_{E \otimes \Lambda} \otimes_{\mathbb{C}} \mathfrak{h})^W$, and this is the same as the inclusion $\underline{\Lambda} \rightarrow \text{Lie } \mathcal{A}$.

This picture is related to the general theory of spectral covers of [24] and [10] (as has also been noted by Donagi in [11]). Suppose that ϖ is an element of \mathfrak{h} such that $W \cdot \varpi$ spans \mathfrak{h} over \mathbb{C} . In the typical application, ϖ is (the dual of) a minuscule weight, if such exist. Let W_0 be the stabilizer of ϖ . If we set $T = (E \otimes \Lambda)/W_0$, then there is a surjection $\nu: T \rightarrow \mathcal{M}$. By pure algebra,

$$\nu_* \mathcal{O}_T = (\mathcal{O}_{E \otimes \Lambda} \otimes_{\mathbb{C}} \mathbb{C}[W/W_0])^W.$$

On the other hand, there is a surjection $\mathbb{C}[W/W_0] \rightarrow \mathfrak{h}$ whose kernel consists of the relations in the orbit $W \cdot \varpi$. Correspondingly, there is a surjection

$$(\mathcal{O}_{E \otimes \Lambda} \otimes_{\mathbb{C}} \mathbb{C}[W/W_0])^W \rightarrow (\mathcal{O}_{E \otimes \Lambda} \otimes_{\mathbb{C}} \mathfrak{h})^W.$$

In particular $H^1(\mathcal{M}; \text{Lie } \mathcal{A})$ is a quotient of $H^1(\mathcal{M}; \nu_* \mathcal{O}_T)$.

Now suppose that we are in the relative case of an elliptic fibration $\pi: Z \rightarrow B$. There is then a relative universal moduli space $\mathcal{M}_{Z/B}$ (with the usual care in the case of E_8). The covers $T \rightarrow \mathcal{M}$ defined over every smooth fiber extend to a finite morphism $\mathcal{T}_{Z/B} \rightarrow \mathcal{M}_{Z/B}$. A section s of the map $\mathcal{M}_{Z/B}^0 \rightarrow B$ defines a finite cover C_s of B , which we will call the *spectral cover* in this case. Of course, C_s need not be smooth or even reduced. In case $\dim B = 1$ and s is generic, the above discussion identifies the connected components of $H^1(B; \mathcal{A}_B(s))$ with an abelian variety which is a quotient of the Jacobian $J(C_s)$, and which is called the *Prym-Tyurin variety* of the spectral cover.

A straightforward dimension argument shows:

Proposition 5.3. *Suppose that $\dim B = 1$ and that $\mathcal{A}_B(s)_b = \mathcal{A}_B^0(s)_b$ for at least one point $b \in B$. Then $H^2(B; \mathcal{A}_B(s)) = 0$. In other words, there exists a universal G -bundle over B corresponding to the section s .*

If however $\mathcal{A}_B(s)_b \neq \mathcal{A}_B^0(s)_b$ for all $b \in B$, then it is possible for there not to exist a universal G -bundle over B corresponding to s , even when $G = SL(2)$. For $\dim B$ arbitrary, the possible obstructions in the case of $SL(n)$ are analyzed in detail in [20].

Let us work out the twisting group $H^1(B; \mathcal{A}_B(s))$ explicitly in the simplest cases $G = SL(n), Sp(2n)$, with $\dim B$ arbitrary:

Proposition 5.4. *Suppose that $G = SL(n)$. Let $C_s \rightarrow B$ be the spectral cover defined above. Then*

$$H^1(B; \mathcal{A}_B(s)) = \text{Ker}\{\text{Norm}: \text{Pic}(C_s) \rightarrow \text{Pic } B\}.$$

If $G = Sp(2n)$, let C_s be the corresponding degree $2n$ cover of B , let $\iota: C_s \rightarrow C_s$ be the induced involution, and let $f: C_s \rightarrow D_s$ be the degree two quotient of C_s by ι . Then

$$H^1(B; \mathcal{A}_B(s)) = \text{Ker}\{\text{Norm}: \text{Pic}(C_s) \rightarrow \text{Pic } D_s\}.$$

Thus, in case B is a curve, $H^1(B; \mathcal{A}_B(s))$ is the generalized Prym variety of the cover $C_s \rightarrow D_s$. Similar results hold for the remaining classical groups $Spin(2n)$ and $Spin(2n+1)$.

On the other hand, suppose that $G = E_6, E_7, E_8$, that $\dim B = 1$, and that the section s is generic. In this case, there is an associated fibration of del Pezzo surfaces $p: Y \rightarrow B$, where Y is a smooth threefold. Moreover Z is included as a smooth divisor on Y so that $p|_Z = \pi: Z \rightarrow B$. Let $J^3(Y)$ denote the

intermediate Jacobian of Y . There is an induced morphism $J^3(Y) \rightarrow J(B)$, where $J(B)$ is the ordinary Jacobian of B coming from the homomorphism $H^*(Y) \rightarrow H^*(Z) \rightarrow H^{*-2}(B)$. Denote the kernel of the morphism $J^3(Y) \rightarrow J(B)$ by $J^3(Y/B)$. Finally set

$$H_0^{2,2}(Y; \mathbb{Z}) = \text{Ker}\{H^4(Y; \mathbb{Z}) \rightarrow H^2(B; \mathbb{Z})\} / \mathbb{Z} \cdot [Y_t],$$

where Y_t is a general fiber of p . In general $H_0^{2,2}(Y; \mathbb{Z})$ is a finite group. We then obtain the following theorem, first proved by Kanev [24] in the case $B = \mathbb{P}^1$ via the Abel-Jacobi homomorphism:

Proposition 5.5. *In the above situation, there is an exact sequence*

$$0 \rightarrow J^3(Y/B) \rightarrow H^1(\mathcal{A}_B(s)) \rightarrow H_0^{2,2}(Y; \mathbb{Z}) \rightarrow 0.$$

We note that we can interpret $H^1(\mathcal{A}_B(s))$ as a relative Deligne cohomology group.

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DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY, NEW YORK, NY 10027
E-mail address: rf@math.columbia.edu, jm@math.columbia.edu

SCHOOL OF NATURAL SCIENCES, INSTITUTE FOR ADVANCED STUDY, PRINCETON NJ 08540
E-mail address: witten@ias.edu