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CONTACT DEGREE AND THE INDEX OF FOURIER INTEGRAL OPERATORS

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Abstract. An elliptic Fourier integral operator of order 0, associated to a homogeneous canonical diffeomorphism, on a compact manifold is Fredholm on L^2. The index may be expressed as the sum of a term, which we call the contact degree, associated to the canonical diffeomorphism and a term, computable by the Atiyah-Singer theorem, associated to the symbol. The contact degree is shown to be defined for any oriented-contact diffeomorphism of a contact manifold and is then reduced to the index of a Dirac operator on the mapping torus, also computable by the theorem of Atiyah and Singer. In this case, of an operator on a fixed manifold, these results answer a question of Weinstein in a manner consistent with a more general conjecture of Atiyah.

Introduction

Let X be a compact contact manifold, with oriented contact line bundle $L \subset T^*X$. Let $\mathcal{M}(X)$ be the contact mapping class group of X; that is the group of components of the space of contact diffeomorphisms. We construct a homomorphism which we call the *contact degree*

$$\text{c-deg} : \mathcal{M}(X) \longrightarrow \mathbb{Z}. \tag{1}$$

This construction is based on the notion of a quantization of the contact structure introduced by Boutet de Monvel and Guillemin [5]. In particular if the hyperplane bundle, $W = L^\circ$ on X, is given an almost complex structure which is positive with respect to the conformal symplectic structure and X is given a compatible partially Hermitian metric then in [5] generalized Szegő projections are shown to exist. Although the analysis in [5] is in terms of the Hermite calculus, these projections also lie in the Heisenberg calculus discussed by Beals and Greiner [1] and Taylor [18], and originally described by Dynin [6]. Extending an earlier idea of the second author [8] (for the integrable, that is CR, case) we introduce below the relative index of two such generalized Szegő projections as the index of the Fredholm operator which is their composite acting between their ranges:

$$\operatorname{ind}(S_0, S_1) = \operatorname{ind}(S_1 S_0 : \operatorname{Ran}(S_0) \longrightarrow \operatorname{Ran}(S_1)). \tag{2}$$

We show below in Proposition 2 that this relative index faithfully labels the components of the space of generalized Szegő projections. The action of the

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group of contact diffeomorphisms, by conjugation, on the space of generalized Szegő projections induces the homomorphism (1),

$$\text{c-deg}(\phi) = \operatorname{ind}(S, S_\phi),\ \ S_\phi = (\phi^*)^{-1} S \phi^*. \tag{3}$$

Let Z_ϕ be the *mapping torus* of ϕ, i.e. $X \times [-1, 1]$ with the ends identified by ϕ. The contact structure on X gives Z_ϕ a natural Spin-$\mathbb{C}$ structure. Let $\eth_\phi$ be the associated Dirac operator.

Theorem 1. *For any oriented contact diffeomorphism of a contact manifold the contact degree is given by the index of the Dirac operator associated to the* Spin-$\mathbb{C}$ *structure on the mapping torus, that is*

$$\text{c-deg}(\phi) = \operatorname{ind}(\eth_\phi).$$

This is proved by first exhibiting the contact degree as the spectral flow of a family of Dirac operators on the contact manifold. The space of Dirac operators associated with partial Hermitian structures on the contact manifold is contractible. Since these Dirac operators are elliptic and self-adjoint, the spectral flow along any curve connecting one such Dirac operator to its ϕ-conjugate is well defined.

Theorem 2. *For any contact diffeomorphism,* c-deg(ϕ) *is the spectral flow of the curve of Dirac operators on the contact manifold associated with an isotopy from any one partial Hermitian structure to its ϕ-conjugate.*

Theorem 1 follows from this by a suspension argument.

The contact degree is directly related to a long-open question of Weinstein [20, 19] asking for a geometric formula for the index of elliptic Fourier integral operators. For such operators acting on a fixed manifold the theorem above provides an answer. Namely, Zelditch (see [19], [21] and [22]) had observed that the properties of the integral transformation studied by Guillemin [12] show the equality of the index of the Fourier integral operator and the relative index of the Szegő projection on S^*X and its ϕ conjugate. Combining these results we deduce

Theorem 3. *If Y is a compact manifold and $X = S^*Y$ is its cosphere bundle then for any* (*oriented*) *contact diffeomorphism, ϕ, of X, i.e. a canonical diffeomorphism of $T^*X \setminus 0$,*

$$\operatorname{ind}(F_\phi) = \text{c-deg}(\phi)$$

where F_ϕ is a Fourier integral operator (*see* [13]) *associated to ϕ and with elliptic symbol corresponding to the positive trivialization of the Maslov bundle, hence Fredholm on $L^2(Y)$.*

An explicit cohomological formula for this index, which follows from the Atiyah-Singer index theorem, is discussed in §8 below. We leave open, for the moment, the part of Weinstein's question concerning the index of elliptic Fourier integral operators between different manifolds. It is highly likely that methods closely related to those used here can be applied to the general problem and provide an answer in terms of the conjecture of Atiyah. Note that it is easily seen that our present formula is consistent with that conjecture (see [19]).

The central result here, which is the passage from Proposition 4, essentially a restatement of (3), to Theorem 2, is closely related to the homotopy argument for the Toeplitz index given by Boutet de Monvel in [4]. Indeed his Toeplitz index theorem can be proved in essentially the same way, and so made independent of the Atiyah-Singer index theorem for pseudodifferential operators. This provides an analytic alternative to the usual K-theory path from the theorem for Dirac operators to the general case of pseudodifferential operators.

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1. The analytic construction

For the treatment of (generalized) Szegő projectors and related ideas we refer to the book of Boutet de Monvel and Guillemin [5] and references therein and for the 'Heisenberg algebra' of a contact manifold to that of Beals and Greiner [1] (see also the book of Taylor [18]). The properties of the Heisenberg algebra and the extended Heisenberg algebra will be discussed more fully in a forthcoming paper of G. Mendoza and the present authors, see also [8] and [16].

Let $\Psi^0_{\mathrm{He}}(X)$ be the Heisenberg algebra, of 'parabolic' pseudodifferential operators associated to the contact structure on X. This has a natural ideal, $\mathcal{I}^0_{\mathrm{He}}(X) \subset \Psi^0_{\mathrm{He}}(X)$ consisting of those elements with full symbols which vanish in the lower half of the cotangent bundle. The (non-commutative) principal symbol map for the Heisenberg calculus gives a short exact sequence of algebras

$$0 \longrightarrow \mathcal{I}^{-1}_{\mathrm{He}}(X) \longrightarrow \mathcal{I}^0_{\mathrm{He}}(X) \longrightarrow \mathcal{S}(\tilde{W}) \longrightarrow 0. \tag{4}$$

Here $\tilde{W}$ is a vector bundle isomorphic to T^*X/L. The product on the Schwartz space $\mathcal{S}(\tilde{W})$ is local to each fiber and is given there by the usual 'pseudodifferential' product which can be written in terms of the differential of a contact form

$$a\#b = e^{id\alpha(D)} a \otimes b\big|_{\mathrm{Diag}}.$$

This is isomorphic to the operator product on $\mathcal{S}(\mathbb{R}^{2n})$ acting as kernels of operators on $\mathbb{R}^n$.

The choice of a positive almost complex structure and admissible metric on T^*X/L induces harmonic oscillators on the fibers of $\tilde{W}$. The ground state is a positive Schwartz function on $\mathbb{R}^n$. Let $s \in \mathcal{S}(\tilde{W})$ be the projection onto it. The set of these projections, for different choices and corresponding ground states, is simply connected. Boutet de Monvel and Guillemin show that s can be lifted to a projection in $\mathcal{I}^0_{\mathrm{He}}(X)$, a generalized Szegő projection, or 'quantized contact structure.' We recall and slightly extend their result as

Proposition 1. *For any smooth isotopy of ground-state projections s_t, $t \in I$, arising from an isotopy of partial Hermitian structures, there is a smooth isotopy S_t of generalized Szegő projections with Heisenberg symbols s_t. Two projections with the same symbol differ by an operator of negative order. One can be deformed, through Szegő projections, to differ from the other by a finite rank self-adjoint projection.*

Proof. A global quantization map for the Heisenberg calculus is given in [10, 16] and this shows immediately that the smooth curve s_t can be lifted to a smooth curve $\tilde{S}_t$ in the Heisenberg calculus. These operators satisfy

$$\tilde{S}_t^2 - \tilde{S}_t = E_t(1) \in \mathcal{I}^0_{\mathrm{He}}(X) \subset \Psi^{-1}_{\mathrm{He}}(X).$$

This also shows the uniqueness of the projection up to terms of negative order. The spectrum of $\tilde{S}_t$ is discrete outside 0 and 1. Furthermore $\tilde{S}_t$ can be replaced by its self-adjoint part so the spectrum is real. Using Cauchy's theorem to replace $\tilde{S}_t$ by the projection onto the span of the eigenspaces for eigenvalues larger than $a \in (0, 1)$, for some a not in the spectrum, gives a locally smooth family of generalized Szegő projections near any t. The existence of a global representation easily shows that this integral is in the Heisenberg calculus.

Thus, we may cover the parameter interval, I by subintervals, I_j, $j = 0, \dots, m$ such that only consecutive intervals intersect and on each interval we have a self-adjoint projection, S^j_t, $t \in I_j$, which differs from $\tilde{S}_t$ by a compact operator. From the construction, on each overlap, $I_j \cap I_{j+1}$, either $S^j_t < S^{j+1}_t$ or vice versa, with the differences smoothing operators. As both the nullspaces and ranges of these projectors are infinite dimensional we can modify the projectors, working from left to right, by adding or subtracting finite rank, self-adjoint, smoothing projections so that after finitely many modifications we obtain a smooth family of projections, S_t, $t \in I$. The difference $S'_t - S_t$ is a smooth family of Heisenberg operators of negative order. This proves the first claim.

Consider two generalized Szegő projections S_0, S_1 with the same symbol. They differ by a negative order, hence compact, operator so $S_1 - S_0 = A + B$ where A is a finite rank smoothing operator and B is a self-adjoint operator with norm less than $1/10$ as an operator on $L^2(X)$. It follows that the family $S_1 - tB$, $t \in [0, 1]$, consists of operators with real spectrum never equal to $1/2$. Projecting this family onto the part of the spectrum greater than $1/2$ therefore gives a smooth isotopy of projections from S_1 to S'_0 where S_0 and S'_0 differ by an operator of finite rank. Let $R = \operatorname{Ran} S'_0 \cap \operatorname{Ran} S_0$ and $N = \operatorname{Null} S'_0 \cap \operatorname{Null} S_0$;

both are subspaces of finite codimension in their respective factors, invariant under both projections. In fact

$$S_0\big|_{R\oplus N} = S_0'\big|_{R\oplus N}.$$

Thus $R \oplus N$ is a subspace of finite codimension and its orthocomplement, K is an invariant subspace for both S_0 and S_0'. Suppose that $\dim \operatorname{Ran} S_0\big|_K \geq \dim \operatorname{Ran} S_0'\big|_K$. It is an elementary fact that we can deform $S_0'\big|_K$ through self-adjoint projections to a subprojection of $S_0\big|_K$. This completes the proof of the second claim. □

For any two such projections S, S' (possibly with different choices for symbols s) the composite $S'S$ is Fredholm as a mapping from the range of S to the range of S'. This follows from the fact that, since all ground states are positive, the inner product of the ground states of any two harmonic oscillators on $\mathbb{R}^n$ cannot be zero. Thus the composite symbol $s's$ is an isomorphism from the range of s to that of s', see [22]. We define the relative index as

$$\operatorname{ind}(S, S') = \operatorname{ind}(S'S : \operatorname{Ran} S \longrightarrow \operatorname{Ran} S').$$

From the discussion in the proof of Proposition 1 above it follows that this index equals the difference of the ranks when the second projection is deformed, through projections, to be equal to the first up to finite rank. In particular the relative index assumes all integral values and its vanishing is equivalent to the existence of an isotopy between the two projections. Gathering these conclusions we have shown

Proposition 2. *The set of connected components of the space of generalized Szegő projectors on a compact oriented contact manifold is isomorphic to $\mathbb{Z}$ with the natural $\mathbb{Z}$ action given by the relative index*; *this $\mathbb{Z}$ action induces the homomorphism* c-deg .

It also follows that the relative index satisfies the cocycle condition

$$\operatorname{ind}(S_1, S_2) + \operatorname{ind}(S_2, S_3) = \operatorname{ind}(S_1, S_3). \tag{5}$$

Louis Boutet de Monvel has pointed out that this cocycle condition is also just the additivity of the index. Namely the left side is the index of the composite $S_3S_2 \circ S_2S_1 = S_3S_2S_1$ from $\operatorname{Ran}(S_1)$ to $\operatorname{Ran}(S_3)$. Now S_2 may be deformed to S_1, through a family of Szegő projections up to smoothing errors. Under this deformation $S_3S_t'S_1$ remains Fredholm as a map from $\operatorname{Ran}(S_1)$ to $\operatorname{Ran}(S_3)$, and so has constant index. At the endpoint, $S_3S_1^2 = S_3S_1$, so the index is equal to $\operatorname{ind}(S_1, S_3)$.

Using somewhat different, though related, techniques the first author had earlier defined this relative index for a pair of Szegő projectors induced by a pair of embeddable, strictly pseudoconvex CR–structures with the given underlying contact structure, see [8]. In that paper the opposite sign convention is used in the definition of $\operatorname{ind}(S, S')$.

Now if ϕ is a contact diffeomorphism of X and $S \in \mathcal{I}^0_{\mathrm{He}}(X)$ is any choice of generalized Szegő projection then $S_\phi = (\phi^*)^{-1}S\phi^*$ is another generalized Szegő projection, associated to the image under ϕ of the Hermitian structure. If S' is another choice of generalized Szegő projection then the conjugation invariance of the index shows that $\mathrm{ind}(S'_\phi, S_\phi) = \mathrm{ind}(S', S)$. The cocycle condition then shows that the 'contact degree' defined by

$$\text{c-deg}(\phi) = \mathrm{ind}(S, S_\phi)$$

is independent of the choice of S. The stability of the index shows that it is an homotopy invariant of ϕ, hence defined on $\mathcal{M}(X)$. If ψ is a second contact diffeomorphism then $S_{\phi\circ\psi} = (S_\psi)_\phi$. In view of (5) the contact degree is therefore multiplicative,

$$\begin{aligned}\text{c-deg}(\phi \circ \psi) &= \mathrm{ind}(S, S_{\phi\circ\psi})\\ &= \mathrm{ind}(S, S_\psi) + \mathrm{ind}(S_\psi, (S_\psi)_\phi)\\ &= \text{c-deg}(\psi) + \text{c-deg}(\phi).\end{aligned}$$

Thus we have defined the homomorphism (1).

2. Extended Heisenberg calculus

In the homotopy arguments below we use the 'extended Heisenberg algebra.' We note here some of its properties; for more details see [10]. We use both the algebra, $\Psi^{\mathbb{Z},\mathbb{Z}}_{\mathrm{eH}}(X; E)$ of operators of integral double orders and the more general spaces with real orders $\Psi^{m,m'}_{\mathrm{eH}}(X; E)$, all well defined acting on sections of any vector bundle, E, over the contact manifold. Two closely related algebras, (one non-classical and one not complete) were introduced by Taylor [18]; the use of his algebras would suffice, with some additional complexity in the arguments, for our purposes here. In fact it is likely that the homotopies could be controlled in the 'standard' algebra of pseudodifferential operators of type $\frac{1}{2}$, much as is done by Boutet de Monvel in [4] but this would involve careful examination of commutators.

The main property of $\Psi^{\mathbb{Z},\mathbb{Z}}_{\mathrm{eH}}(X; E)$ is that it contains both the traditional algebra $\Psi^{\mathbb{Z}}(X; E)$, defined without any reference to the contact structure, and the Heisenberg algebra $\Psi^{\mathbb{Z}}_{\mathrm{He}}(X; E)$ which has parabolic homogeneity and for the properties of which we refer to [1], [18]. Of the two indices, the first is 'traditional' order and the second the Heisenberg order:

$$(6) \quad \Psi^k(X; E) \subset \Psi^{k,2k}_{\mathrm{eH}}(X; E),\ k \in \mathbb{R},\ \Psi^m_{\mathrm{He}}(X; E) \subset \Psi^{m,m}_{\mathrm{eH}}(X; E),\ m \in \mathbb{R}.$$

As explained in [10], all three 'full symbol algebras,' the quotients by the ideal of smoothing operators, can be identified with non-commutative products on the spaces of Laurent series 'at infinity' for different compactifications of T^*X. The standard calculus requires the fiber-wise radial compactification of T^*X; the Heisenberg calculus uses the parabolic compactification defined by the contact line bundle. For the extended Heisenberg calculus we begin with the radial compactification and parabolically blow up the submanifold of the boundary defined by the contact line bundle; this is the eH-compactification. We denote the new boundary face (which has two components) by B_L and the lift of the radial boundary by B_S. The inclusions in (6) correspond to the natural maps between the three compactifications. The algebra $\Psi_{\mathrm{eH}}(X;E)$ is a bi-graded algebra of operators on $\mathcal{C}^\infty(X;E)$:

$$\Psi^{k,l}_{\mathrm{eH}}(X;E)\circ\Psi^{k',l'}_{\mathrm{eH}}(X;E)=\Psi^{k+k',l+l'}_{\mathrm{eH}}(X;E),\ k,l,k',l'\in\mathbb{R}.$$

In fact, $\Psi^{\mathbb{Z},\mathbb{Z}}_{\mathrm{eH}}(X;E)$ is a smooth completion of the subalgebra generated by the two subalgebras in (6) and its symbolic properties strongly reflect this. The 'standard symbol' map extends to a homomorphism giving a short exact sequence,

$$0\longrightarrow\Psi^{k-1,l}_{\mathrm{eH}}(X;E)\longrightarrow\Psi^{k,l}_{\mathrm{eH}}(X;E)\xrightarrow{\sigma_S}\mathcal{C}^\infty(B_S;\mathrm{Hom}(E)\otimes G^k_S\otimes G^l_L)\longrightarrow 0.$$

The bundle G_S is the inverse of the conormal bundle to B_S and G_L is the inverse of the conormal bundle to its boundary. It may be identified with the restriction to $B_L\cap B_S$ of the inverse of the conormal bundle to B_L. The composition rule for this symbol map is given by point-wise multiplication of functions.

The 'Heisenberg symbol' is a little more complicated to describe since it is non-commutative. The second boundary hypersurface, B_L, of the eH-compactification of T^*X can be identified with two copies of the radial compactification of the dual bundle W^* to the hyperplane bundle $W\subset TX$ which is the annihilator of the contact line bundle. The interior of each component of B_L is naturally a symplectic vector bundle over X and this means that the 'isotropic' pseudodifferential algebra of operators on $\mathbb{R}^n$, which is a noncommutative product on $\mathcal{C}^\infty(\mathbb{B}^{2n})$ depending only on the symplectic structure of $\mathbb{R}^{2n}\hookrightarrow\mathbb{B}^{2n}$ can be transferred naturally, and smoothly, to the fibers of B_L. In this sense the Heisenberg algebra defines a short exact sequence,

$$0\longrightarrow\Psi^{k,l-1}_{\mathrm{eH}}(X;E)\longrightarrow\Psi^{k,l}_{\mathrm{eH}}(X;E)\xrightarrow{\sigma_{\mathrm{He}}}\mathcal{C}^\infty(B_L;\mathrm{Hom}(E)\otimes G^k_S\otimes G^l_L)\longrightarrow 0.$$

Here, since B_L has two components (one corresponding to the positive direction of L and one to the negative direction) the symbol consists of two operators at each point of X. Thus the Heisenberg symbol defines two smooth families of model operators in the isotropic calculus; the composition rule for this symbol comes from the composition of these operators. The bundle G_L has a trivialization along the fibers of B_L, the constant sections of which commute with the product.

Jointly these two symbol maps capture both orders, and satisfy only the natural compatibility condition of equality at the corner. In particular the elements of $\Psi^{0,0}_{\mathrm{eH}}(X;E)$ are precisely the elements of $\Psi^{\mathbb{Z},\mathbb{Z}}_{\mathrm{eH}}(X;E)$ which are L^2-bounded and the subspace $\Psi^{-1,-1}_{\mathrm{eH}}(X;E)$ is the null space of the joint symbol and consists precisely of the elements of $\Psi^{\mathbb{Z},\mathbb{Z}}_{\mathrm{eH}}(X;E)$ which are compact on $L^2(X;E)$. Note that the ellipticity of an element of $A\in\Psi^{k,l}_{\mathrm{eH}}(X;E)$ means the *invertibility* of this joint symbol; this is equivalent to the ellipiticity in the usual sense for $\sigma_S(A)$ but the model operators defined by the Heisenberg symbol must be invertible on each fiber. The ellipticity of σ_S implies, via the compatibility condition, the microlocal ellipticity of the Heisenberg symbol. Its invertibility on L^2 is therefore equivalent to its invertibility in the isotropic algebra, and its inverse is automatically a model operator.

The usual symbol of an element $A \in \Psi^k(X;E)$ can be interpreted as a section over S^*X, of the bundle $\operatorname{Hom}(E)\otimes G^k$, where G is the inverse conormal bundle to S^*X viewed as the boundary of the radial compactification of T^*X. The bundle G pulls back to the boundary $B_S \cup B_L$ of the extended Heisenberg compactification as $G_S G_L^2$ and any section of G^k pulls back to a section of $G_S^k G_L^{2k}$ which is in the center of the symbol algebra on the Heisenberg face. Thus the usual symbol, $\sigma_k(A)$ determines both the standard symbol and the Heisenberg symbol. If ξ is a vector field transversal to the hyperplane bundle and positive on the contact line then $\sigma_L(\xi)$ is a section of G_L^2 which is positive on the positive side of B_L and in the center of the algebra. Thus $|\sigma_L(\xi)|^{\frac{1}{2}}$ can be used to trivialize G_L.

For a CR manifold with Hermitian structure, $\bar\partial_b$ and $\bar\partial_b^*$ are both operators of order 1 in the Heisenberg, and hence in the extended Heisenberg, algebra. Their symbols are creation and annihilation differentials. Acting on functions the Heisenberg symbol of $\Box_b = \bar\partial_b^*\bar\partial_b$ is the harmonic oscillator shifted by a constant. On the positive side this makes the lowest eigenvalue 0; on the negative side the symbol is strictly positive (as an operator). The Szegő projector is an element of $\Psi_{\mathrm{He}}^{-\infty,0}(X)$ with Heisenberg symbol the projection onto the ground state of this harmonic oscillator on the positive side; since it is in the ideal corresponding to the positive direction of its symbol vanishes identically on the negative side. Boutet de Monvel and Guillemin in [5] construct analogues of these objects for a general contact manifold (with oriented contact line), see Proposition 3.

3. Dirac operators

We use the conventions for Dirac operators from [15]. A discussion of odd and even-dimensional Dirac operators, and the relationship between them as in the Atiyah-Patodi-Singer index theorem, can be found there or in [14].

Suppose an almost complex structure has been chosen on the hyperplane bundle, W of the contact manifold and that this structure is positive with respect to the conformal symplectic structure it inherits. The choice of a contact form therefore defines an Hermitian metric on W; an admissible metric on X is one which restricts to this and gives the contact form length one. The complex structure gives a reduction of the structure bundle of X (of dimension $2n+1$) to $U(n)$. It therefore defines a Spin-$\mathbb{C}$ structure on X. The Dirac operator associated to this structure can be taken to act on the exterior bundle of $W^\dagger$, which is the complex bundle which is the dual to the $(0,1)$–part of $\mathbb{C}\otimes W$. If ∇ is the Levi-Civita connection projected onto W then the Dirac operator is of the following form in terms of the decomposition of $\Lambda^* W^\dagger$ into odd and even parts:

$$\eth = \begin{pmatrix} -\frac{1}{i}\nabla_\xi & \bar\partial_b^* + \bar\partial_b \\ \bar\partial_b^* + \bar\partial_b & \frac{1}{i}\nabla_\xi \end{pmatrix}$$

in the CR case. Similarly in the general case

$$\eth = \begin{pmatrix} -\frac{1}{i}\nabla_\xi & D' \\ D' & \frac{1}{i}\nabla_\xi \end{pmatrix}$$

where D' has the same symbol as $\bar\partial_b + \bar\partial_b^*$ in the integrable case and ξ, as above, is the unit normal to the hyperplane field.

Let Z_ϕ denote the mapping torus defined by a contact diffeomorphism ϕ of X :

$$Z_\phi = X \times [0,1]/(x,0) \simeq (\phi(x),1). \tag{7}$$

Since ϕ is a contact transformation the lift of the hyperplane bundle W to $X \times [0,1]$ projects to a subbundle, W_ϕ, of TZ_ϕ of codimension 2. This has a well defined conformal symplectic structure. The parameter direction determines a trivial line bundle, M, in TZ_ϕ as does the orthocomplement to $W_\phi \oplus M$. Thus $TZ_\phi = W_\phi \oplus \mathbb{R}^2$. On the $\mathbb{R}^2$–factor we use a "constant" almost complex structure. Using the conformal symplectic structure we may choose a positive almost complex structure on W_ϕ. This induces the canonical Spin-$\mathbb{C}$ structure of Z_ϕ and hence defines a Dirac operator $\eth_\phi$.

4. Resolution

If S is a generalized Szegő projection for a pseudo-Hermitian structure on a compact contact manifold then, essentially following Boutet de Monvel and Guillemin [5], we consider the notion of a *resolution* of S. By this we shall mean a Heisenberg pseudodifferential operator, B, of order 1 acting on $\mathcal{C}^\infty(X;\Lambda^* W^\dagger)$ which defines an acyclic graded complex with respect to form degree, has symbol that of the formal $\bar\partial_b$ operator, i.e. the annihilation complex, and is such that

$$\begin{aligned} &\operatorname{Null}(B^{(0)}) = \operatorname{Ran}(S),\ \operatorname{Null}(B^{(j)}) = \operatorname{Ran}(B^{(j-1)}),\ 1 \le j < n, \\ &\qquad\qquad \operatorname{Ran}(B^{(n-1)}) \oplus \operatorname{Ran}(T) = \mathcal{C}^\infty(X;\Lambda^n W^\dagger),\ (\dim X = 2n+1). \end{aligned} \tag{8}$$

Here T is a generalized Szegő projector for the negative of the contact structure. In fact the last condition is superfluous; given the exactness conditions the projector onto the orthocomplement of the range of $B^{(n-1)}$ is automatically a generalized Szegő projector for the canonical line bundle.

In [5] Boutet de Monvel and Guillemin construct a resolution, which is a classical pseudodifferential operator, for their generalized Szegő projections in the Hermite calculus. However, the construction in the appendix to [5] is easily adapted to the Heisenberg calculus. Clearly $B \in \Psi^1_H(X;\Lambda^* W^\dagger)$ is determined, by the requirement that its symbol be the annihilation complex, up to a term of order 0. In the integrable case we may take $B = \bar\partial_b$ and then $B^2 = 0$. In the general case the choice of symbol ensures only that $B^2 \in \Psi^1_H(X; \Lambda^* W^\dagger)$. However, B can be modified by the addition of lower order terms to give a resolution in the sense of (8).

Proposition 3. *Every generalized Szegő projection has a resolution*; *any two such resolutions are smoothly isotopic and any smooth family of Szegő projections has a smooth family of resolutions.*

Proof. The existence of such a resolution follows from the methods of the appendix to [5]. See in particular Theorem 5.9 and the remarks following, where it is noted that the finite-dimensional homology of the resolving complex may be taken to be trivial. To translate the construction there to the framework of the extended Heisenberg calculus consider a differential operator B, on the contact manifold which replaces $\bar{\partial}_b$ in the CR case. The symbol sequence of this operator is exact, in fact its symbolic properties are the same as in the CR case. In particular it is of order 1 in the Heisenberg sense and the Heisenberg symbol of B_0, the action on functions, is the annihilation operator with null space exactly spanned by s, the symbol of the Szegő projector. Similarly the symbol acting on maximal degree forms is the annihilator of the symbol of the 'dual' Szegő projector. In intermediate form degrees the complex defined by B is elliptic in the Heisenberg sense, and hence subelliptic. It follows that there are projections in each form degree, $S_i \in \Psi^0_H(X; \Lambda^* W^\dagger)$, in the Heisenberg calculus with symbols, s_i, which are the orthogonal projections onto the ranges of the symbols in the creation complex.

In fact for a smooth family of structures these projections can be chosen smoothly in all degrees, with the given family in degree 0. Modifying the original choice of B to

$$B'_i(t) = S_{i+1}(t) B_i(t)(\mathrm{Id} - S_i(t)) \tag{9}$$

gives a smooth resolution, in the Heisenberg calculus, up to smoothing errors. Thus the only remaining step is to pass from a complex modulo smoothing errors to an actual complex.

This is done, for a single operator (and in the G-equivariant case) in [5]. Thus we may assume that the smooth family $S(t)$ of generalized Szegő projectors has a smooth family $B'(t)$ of resolutions up to smoothing errors; we may assume that $B'(0)$ is a resolution. If necessary, first replace $B'_0(t)$ by $B''_0(t) = B'_0(t)(\mathrm{Id} - S(t))$, from which it differs by a smoothing family. Then $\mathrm{Ran}(S(t)) \subset \mathrm{Null}(B''(t))$ has finite codimension for each t. Since 0 is isolated in the spectrum of $(B''_0(t))^* B''_0(t)$ we can use the argument in the proof of Proposition 1 to construct a smooth family of orthogonal projections, $S_0(t)$, differing from $S(t)$ by a finite rank smoothing family, such that $\mathrm{Null}(B''_0(t)) \subset \mathrm{Ran}(S_0(t))$ for each t. Now consider $B'''_0(t) = B''_0(t)(I - S_0(t))$ then $\mathrm{Null}(B'''_0(t)) = \mathrm{Ran}(S_0(t))$. Adding a smooth family of finite rank smoothing operators to $B'''_0(t)$ we obtain a smooth family, $B_0(t)$ with $\mathrm{Null}(B_0(t)) = \mathrm{Ran}(S(t))$. Since $B_0(t)(B_0(t))^*$, being isospectral to $(B_0(t))^* B_0(t)$, has 0 isolated in its spectrum, the orthogonal projections onto the ranges of the $B_0(t)$ form a smooth family of projections. This allows the resolution to be extended step by step in terms of form degree, just as in [5].

The existence of a smooth isotopy between any two resolutions of the same projection follows from this argument and the existence of a symbolic isotopy. $\square$

Suppose that S_0 and S_1 are two generalized Szegő projections with $S_0 > S_1$ in the sense that $S_0S_1 = S_1S_0 = S_1$; set $k = \operatorname{ind}(S_0, S_1) \geq 0$. Thus $K_0 = \operatorname{Ran}(S_0) \ominus \operatorname{Ran}(S_1) \subset \mathcal{C}^\infty(X)$ is a vector space of dimension k and $S_0 = S_1 + \pi_0$, with π_0 the orthogonal projection onto K_0. Consider how a resolution for S_0 may be obtained from a resolution for S_1. Namely, the resolution, B_1, for S_1 may be decomposed into a resolution, B_0, for S_0 and a finite dimensional complex. Set

$$B_1^{(0)} = B_0(\operatorname{Id} - \pi_0) = (\operatorname{Id} - \pi_1)B_0$$

where π_1 is the orthogonal projection onto $K_1 = B_0^{(0)}K_0$, which has dimension k. If $n > 1$, choose a subspace $K_2 \subset \mathcal{C}^\infty(X; \Lambda^2 W^\dagger)$ in the complement to the null space of $B^{(2)}$ and set $B_0^{(1)} = B_1^{(1)} + E_1$, where E_1 is a smoothing operator of rank k which is an isomorphism of K_1 onto K_2 and annihilates the range of $B_0^{(0)}$. Proceeding in this manner one constructs a complex, B_0 which is a resolution of S_0. The generalized Szegő projector, T_0, onto the complement of the range of $B_0^{(n-1)}$ for this complex is such that if n is *odd* then $T_0 > T_1$, with $\operatorname{Ran}(T_1) \subset \operatorname{Ran}(T_0)$ a subspace of codimension k and if n is even instead $T_0 < T_1$ with $\operatorname{Ran}(T_0) \subset \operatorname{Ran}(T_1)$ a subspace of codimension k.

With the resolution B_0 constructed from B_1 in this way, consider the two operators

$$D_i = -S_i + B_i + B_i^* + (-1)^n T_i, \ i = 0, 1. \tag{10}$$

These are self-adjoint operators in the Heisenberg calculus for the contact manifold. Both act on $\mathcal{C}^\infty(X; \Lambda^* W^\dagger)$. They are operators of order 1; in fact they differ by a finite rank smoothing operator. Whilst not elliptic they are Fredholm as operators on the same domain, namely the anisotropic Sobolev space which is the completion of $\mathcal{C}^\infty(X; \Lambda^* W^\dagger)$ with respect to the norm

$$\|u\|^2 = \|u\|^2_{L^2} + \|Bu\|^2_{L^2} + \|B^*u\|^2_{L^2}. \tag{11}$$

This Hilbert space is *not* compactly embedded in L^2, since on the image of S it is quasi-isometric to it. Since we shall use it for some time, we shall denote this Hilbert space $\mathcal{H}$. Thus the linear homotopy between these operators, $D_t = (1-t)D_0 + tD_1$, consists of unbounded self-adjoint Fredholm operators with domain $\mathcal{H}$ in L^2, and hence has discrete spectrum near 0. Furthermore, both D_0 and D_1 are invertible, essentially by construction. Thus the spectral flow, at 0, along this family is well defined.

Lemma 1. *The spectral flow of the family D_t on $t \in [0,1]$ at 0 is* $\operatorname{ind}(S_0, S_1)$ (*assumed positive*).

Proof. The spectral flow is that of the finite dimensional family obtained by projection onto

$$K = \bigoplus_j K_j,$$

since this is invariant for the family and outside it the family is constant.

First take n odd. Then, restricted to K the two complexes are

$$K_0 \xrightarrow{0} K_1 \xrightarrow{B_0^{(1)}} K_2 \cdots \xrightarrow{0} K_n,\ S_0 = \operatorname{Id} \text{ on } K_0,\ T_0 = \operatorname{Id} \text{ on } K_n$$
$$K_0 \xrightarrow{B_1^{(0)}} K_1 \xrightarrow{0} K_2 \cdots \xrightarrow{B_1^{(n-1)}} K_n,\ S_1 = 0 \text{ on } K_0,\ T_1 = 0 \text{ on } K_n.$$

Here the $B_0^{(j)}$ are isomorphisms for j odd and the $B_1^{(j)}$ are isomorphisms for j even. It follows that the eigenvalues of $D_0 = -S_0 + B_0 + B_0^* - T_0$ consist of -1 with multiplicity $2k$, $k = \operatorname{ind}(S_0, S_1)$, on $K_0 \oplus K_n$ and on the remaining space of dimension $(n-1)k$ the eigenvalues come in pairs with opposite signs. On the other hand D_1 has all $(n+1)k$ eigenvalues occurring in pairs with opposite signs. Thus the net flow of eigenvalues across 0 for the curve from $t = 0$ to $t = 1$ is k.

For n even, the complexes are again equal off K, on which they take the form

$$K_0 \xrightarrow{0} K_1 \xrightarrow{B_0^{(1)}} K_2 \cdots \xrightarrow{B_0^{(n-1)}} K_n,\ S_0 = \operatorname{Id} \text{ on } K_0,\ T_0 = 0 \text{ on } K_n$$
$$K_0 \xrightarrow{B_1^{(0)}} K_1 \xrightarrow{0} K_2 \cdots \xrightarrow{0} K_n,\ S_1 = 0 \text{ on } K_0,\ T_1 = \operatorname{Id} \text{ on } K_n,$$

with the same invertibility properties. Thus the eigenvalues of $D_0 = -S_0 + B_0 + B_0^*$ on K consist of -1 with multiplicity k on K_0 with the remaining nk eigenvalues occurring in pairs with opposite signs. Similarly $D_1 = B_1 + B_1^* + T_1$ on K has nk eigenvalues occurring in pairs with opposite signs, together with 1 of multiplicity k on K_n. Thus, again the net flow of eigenvalues is k. □

Suppose now that S and S' are any two generalized Szegő projections, for convenience ordered so that $\operatorname{ind}(S, S') \geq 0$. Let D_t, $t \in [-1,1]$ be a curve in the Heisenberg calculus chosen as follows. For $t \in [-1,0]$ let S_t be an isotopy of generalized Szegő projections with $S_{-1} = S$ and such that $S_0 > S_1 = S'$; that is S_0 and S' commute with $S_0 S' = S'$. Take any resolution B_{-1} for $S = S_{-1}$ and deform it as an isotopy of resolutions, B_t for S_t, $t \in [-1,0]$. Let $B = B_1$ be any resolution of $S' = S_1$. Let $B_{\frac12}$ be a resolution of S_1 constructed, as above, from the resolution B_0 of S_0. Let B_t, $t \in [\frac12, 1]$ be an isotopy of resolutions of S_1. Now, define

$$\tilde{D}_t = \begin{cases} S_t + B_t + B_t^* + (-1)^{n-1} T_t, & t \in [-1,0] \cup [\frac12, 1] \\ (1-2t) D_0 + 2t D_{\frac12}, & t \in (0, \frac12). \end{cases} \tag{12}$$

Thus, $\tilde{D}_t$ is a (continuous) curve of self-adjoint Fredholm operators, given by elements of the Heisenberg calculus, it is invertible for $t = -1$ and $t = 1$, so the spectral flow across 0 is well defined. The domains here vary with t, always being the space $\mathcal{H}_t$, given by (11), for the varying resolution; in fact it only depends on the Hermitian structure.

Proposition 4. *The spectral flow, across* 0 *of the curve of self-adjoint operators* $\tilde{D}_t$ *is* $\operatorname{ind}(S, S')$.

Proof. The family is invertible for $t \in [-1, 0]$ and again for $t \in [\frac{1}{2}, 1]$. Thus the spectral flow is simply that of the family, linear in $t \in [0, \frac{1}{2}]$, which is $\operatorname{ind}(S_0, S_1) = \operatorname{ind}(S, S')$ by Lemma 1. □

5. Proof of Theorem 2

Now, suppose that S is any generalized Szegő projector on the compact contact manifold X. Let $S' = S_\phi = (\phi^*)^{-1} S \phi^*$ be the conjugate projector, where ϕ is a contact diffeomorphism. The family $\tilde{D}_t$ considered in Proposition 4 then has spectral flow c-deg(ϕ); for simplicity we shall relabel the parameter so that it runs over $[0, 1]$. Furthermore, if we choose the resolution of S_ϕ to be the ϕ-conjugate of the resolution for S then D_0, associated to S, and D_1, associated to S_ϕ are conjugate operators. The spectral flow on the curve is then well defined independently of the point at which it is measured in $(-1, 1)$, since in this interval the spectrum remains discrete. We now further deform this family; we do this in such a manner that the ends remain conjugate, and hence isospectral, so the spectral flow does not change provided the family remains self-adjoint and Fredholm.

Choose a classical self-adjoint pseudodifferential operator of order 0, M, on X which acts on each of the $\Lambda^j W^\dagger$ and which has symbol that of Id on the positive contact direction and $-\operatorname{Id}$ on the negative. Let M_t be any homotopy, with the same properties at each point, from $M = M_0$ to $M_1 = M_\phi$, its conjugate under ϕ. Written out in terms of the odd-even decomposition of $\Lambda^* W^\dagger$, the operator $\tilde{D}_t$ takes the form

$$\begin{pmatrix} -S'_t & B^*_t + B_t \\ B^*_t + B_t & -T'_t \end{pmatrix} (n \text{ odd}), \begin{pmatrix} -S'_t + T'_t & B^*_t + B_t \\ B^*_t + B_t & 0 \end{pmatrix} (n \text{ even}). \tag{13}$$

Here, S'_t and T'_t are homotopies between S, S_ϕ and T, T_ϕ respectively, which are projections, up to smoothing operators, associated to the ground state of the harmonic oscillator of the same Hermitian structure as B_t. The deformed family

$$\begin{pmatrix} -\epsilon M_t - S'_t & B^*_t + B_t \\ B^*_t + B_t & \epsilon M_t - T'_t \end{pmatrix} (n \text{ odd}), \begin{pmatrix} -\epsilon M_t - S'_t + T'_t & B^*_t + B_t \\ B^*_t + B_t & \epsilon M_t \end{pmatrix} (n \text{ even})$$

remains Fredholm on $\mathcal{H}_t$ and lies in the extended Heisenberg calculus, for $\epsilon \geq 0$. Indeed, the symbol is independent of ϵ in the classical region and in the Heisenberg region, on which M_t acts as the constant given by its symbol, the noncommutative symbol is invertible provided the diagonal term is invertible on the ground state. This is true for all $\epsilon > 0$. Setting $\epsilon = 1$, consider the further deformed family

$$\begin{pmatrix} -M_t - \delta S_t' & B_t^* + B_t \\ B_t^* + B_t & M_t - \delta T_t' \end{pmatrix} (n \text{ odd}), \begin{pmatrix} -M_t - \delta S_t' + \delta T_t' & B_t^* + B_t \\ B_t^* + B_t & M_t \end{pmatrix} (n \text{ even}).$$

The same reasoning shows that this remains Fredholm for $\delta \in [0,1]$.

Up to this point the operators have had domains $\mathcal{H}_t$, given by (11). Consider now a family of self-adjoint elliptic pseudodifferential operators $M_t(s)$ of order s with $M_t(0) = M_t$ and $M_t(1) = \frac{1}{i}\nabla^t_\xi$. Here ξ is the vector normal to the hyperplane bundle which is positive on the positive contact direction and ∇^t is the connection for the Hermitian structure associated to B_t. We further demand that $M_0(s)$ and $M_1(s)$ are always ϕ-conjugate and that the symbol of $M_t(s)$ is equal to that of $\pm|\sigma_1(\xi)|^s$ on the positive and negative contact directions respectively. Clearly these conditions are consistent, and we may even insist that $M_t(s)$ is graded of degree 0. At $\delta = 0$ the form of the previous family no longer depends on the parity of n. Starting from this consider the homotopy of operators

$$\begin{pmatrix} -M_t(s) & B_t^* + B_t \\ B_t^* + B_t & M_t(s) \end{pmatrix}. \tag{14}$$

Since $M_t(s)$ is a classical pseudodifferential operator of order s this remains an element of the extended Heisenberg calculus or order $(1,1)$, until $s = \frac{1}{2}$. For $s \in [0, \frac{1}{2}]$ it is Fredholm on the Sobolev spaces defined similarly to (11), namely as the completion of $\mathcal{C}^\infty(X; \Lambda^* W^\dagger)$ with respect to the norm

$$\|u\|^2 = \|u\|^2_{L^2} + \|M_t(s)u\|^2 + \|B_t u\|^2_{L^2} + \|B_t^* u\|^2_{L^2}. \tag{15}$$

These are closely related to the natural Sobolev spaces associated to elliptic operators of this double order $(1,1)$.

Lemma 2. *For $0 \le s \le \frac{1}{2}$ the Hilbert space obtained from $\mathcal{C}^\infty(X; \Lambda^* W^\dagger)$ by the completion with respect to the norm* (15)*, for any partial Hermitian structure and associated resolution B_t, is the Sobolev space $\mathcal{H}^{1,1}(X; \Lambda^* W^\dagger) \subset L^2(X; \Lambda^* W^\dagger)$ defined by elliptic operators of order $(1,1)$ plus the range of $(1+\Delta)^{s/2}(S+T)$ acting on $L^2(X; \Lambda^* W^\dagger)$ where Δ is the Laplacian of some* (*full*) *metric; for $s = \frac{1}{2}$ this summand can be dropped.*

In the range $s \in [\frac{1}{2}, 1]$ the family is genuinely elliptic as an extended Heisenberg operator of order $(2s, 1)$ and so is Fredholm on the Sobolev space (independent of t) associated with operators of this order. Finally, at $s = 1$, this reduces to the standard Sobolev space $H^1(X; \Lambda^* W^\dagger)$.

At $s = 1$, the family (14) has a classical symbol, which is elliptic in the classical sense, is self-adjoint and the end points are conjugate. The spectral flow of the family, already shown to be c-deg(ϕ), depends only on the principal symbol, provided the end-points are held conjugate. Since the symbol of the family at $s = 1$ is the same as that of the family of Dirac operators associated to the partial Hermitian structure, Theorem 2 is proved.

6. Proof of Theorem 3

Guillemin in [12] considers the push-forward of functions on the coball bundle of a compact manifold, Y, to its base. Grauert in [11] had shown that the ball bundle can always be given a complex structure in which the zero section, which can be identified with the manifold, is totally real; it then looks like a tubular neighborhood of X in T^*X. If the boundary defining function is chosen appropriately, and the tube Ω is taken to be small enough, then it is shown in [12] that the operator of integration over the fibers (with respect to an appropriate fiber volume form) restricted to holomorphic functions

$$F : \left\{u \in \mathcal{C}^\infty(\Omega); \bar{\partial} u = 0\right\} \longrightarrow \mathcal{C}^\infty(Y)$$

is Fredholm. In fact it is shown in [9] to be an isomorphism. This space of holomorphic functions in the tubular domain is canonically isomorphic to the space of smooth CR functions on the boundary, $X = S^*Y$ of Ω. This space is also the range, on $\mathcal{C}^\infty(X)$, of the Szegő projector. Thus F can be replaced by the analogous integral operator G over the spherical fibers of X. Guillemin analyses G by showing that $G \circ G^*$ is an elliptic pseudodifferential operator with positive symbol. The analytic part of his argument, which relies only on the composition formula for Fourier integral operators with complex phase, can be applied almost without change to show that if ϕ is a contact diffeomorphism of X then $G \circ \phi^* \circ G^*$ is an elliptic Fourier integral operator, associated with ϕ and defining the positive trivialization of its Maslov bundle. Since G is Fredholm from the range of S to $\mathcal{C}^\infty(Y)$, this shows that

$$\operatorname{ind}(G\phi^* G^*) = \operatorname{ind}((\phi^*)^{-1} S \phi^* S) : \operatorname{Ran}(S) \longrightarrow \operatorname{Ran}(S_\phi) = \operatorname{ind}(S, S_\phi).$$

This proves Theorem 3.

7. The mapping torus

As already noted in the Introduction, Theorem 1 follows from Theorem 2 by a suspension argument. Closely related results can be found in [2]. For completeness we include an essentially analytic approach to the proof that if $\eth_\phi$ is the Spin-$\mathbb{C}$ Dirac operator on the mapping torus for the contact diffeomorphism ϕ, and $\eth'_t$ is the homotopy, for $t \in [0, 1]$, of Dirac operators for an isotopy of partial Hermitian structures from one on W to its ϕ image, then

$$\operatorname{ind}(\eth_\phi) = \operatorname{SF}(\eth'_t), \tag{16}$$

where SF denotes the spectral flow of the family.

Denote the spectral flow by k as before. Let $D_\theta = -id/d\theta$ be the standard self-adjoint translation-invariant differential operator on the circle of length 2π.

The family $D_\theta(t) = D_\theta - kt$ has spectral flow $-k$ for $t \in [0,1]$. If L_t is the flat complex line bundle over the circle with global sections the functions on $\mathbb{R}$ satisfying the compatibility condition

$$u(t' + 2\pi r) = e^{iktr}u(t'),\ t' \in \mathbb{R},\ r \in \mathbb{Z}$$

then, naturally, $L_1 \equiv L_0$, and $iD_\theta(t)$ may be considered as the twisted exterior differential for L_t, with $d\theta$ used to trivialize 1-forms. Thus $D_\theta(t)$ becomes a family of differential operators on the fibers of the bundle L over the torus $\mathbb{S}_\theta \times \mathbb{S}'_t$ where the second circle has length one. The spectral flow of this family is $-k$.

The mapping torus for the contact diffeomorphism ϕ is a fiber bundle over $\mathbb{S}'_t$ with fiber X. By assumption the Dirac operators $\eth'_t$, acting on the exterior algebra of the hyperplane bundle $W^\dagger$ with varying almost complex structure, has spectral flow k. Thus the direct sum, $A(t)$, of these two families has spectral flow 0 as a family over $\mathbb{S}'_t$.

It follows from the results of [17] that the image of the family in $K^1(\mathbb{S}'_t)$ is 0 and that the combined family has a spectral section. That is, there is a smooth family of projections, $P(t)$ which are of the form

$$P(t) = \begin{pmatrix} P_{00}(t) & P_{01}(t) \\ P_{10}(t) & P_{11}(t) \end{pmatrix}$$

where $P_{00}(t)$ and $P_{11}(t)$ are pseudodifferential operators on the fibers of the two factors and the off-diagonal terms are smoothing operators between the two factors; the crucial property is that there exists a constant R such that if $e_\lambda(t)$ is an eigenfunction of $A(t)$ with eigenvalue λ then

$$\begin{aligned} P(t)e_\lambda = e_\lambda \text{ if } \lambda > R, \\ P(t)e_\lambda = 0 \text{ if } \lambda < -R. \end{aligned}$$

Now consider the positive Dirac operator on the mapping torus. This can be written $G(t)(\partial_t + \eth'_t)$ where $G(t)$ is Clifford multiplication by dt. Thus its index is the same as that of the operator

$$\frac{\partial}{\partial t} + \eth'_t : H^1(Z_\phi; \Lambda^* W^\dagger) \longrightarrow L^2(Z_\phi; \Lambda^* W^\dagger).$$

Consider the direct sum operator $\partial_t + A(t)$. In the decomposition

$$A(t) = P(t)A(t)P(t) + (\mathrm{Id} - P(t))A(t)(\mathrm{Id} - P(t)) + A'(t)$$

the term $A'(t)$ is a smoothing operator. It is therefore compact on $H^1(Z_\phi; \Lambda^* W^\dagger)$, so can be dropped without changing the index, similarly any constant term can be dropped and therefore

$$
\begin{aligned}
\operatorname{ind}(\partial_t + A(t)) = \operatorname{ind}(\partial_t + A_+(t) - A_-(t)),& \\
A_+(t) &= P(t)A(t)P(t) + RP(t), \\
-A_-(t) &= (\mathrm{Id} - P(t))A(t)(\mathrm{Id} - P(t)) + R(\mathrm{Id} - P(t)).
\end{aligned}
$$

Thus, $A_+(t)$, $A_-(t) > 0$ commute. It follows that the index of $\partial_t + A(t)$ is zero. Thus the index of $\eth_\phi$ is $-\operatorname{ind}(\partial_t + D_\theta(t))$. Essentially by construction this latter index is $-k$, so (16) is proved and hence so is Theorem 1.

8. Cohomological formula

The Atiyah–Singer theorem gives the following formula for the index, (see for example [14])

$$
\operatorname{ind}(\eth_\phi) = e^{\frac{c}{2}} \hat{A}(Z_\phi)[Z_\phi]. \tag{17}
$$

Here $c = c_1(T^{1,0}Z_\phi)$ and $\hat{A}$ is the total $\hat{A}$–class. As noted in §3 the tangent bundle splits so

$$
c_1(T^{1,0}Z_\phi) = c_1(W_\phi^{1,0}), \quad \hat{A}(Z_\phi) = \hat{A}(W_\phi).
$$

If X is 3–dimensional then $\hat{A}(W_\phi) = 1$ and therefore we find the simpler formula:

$$
\operatorname{ind}(\eth_\phi) = \frac{1}{4} c_1(W_\phi^{1,0}) \wedge c_1(W_\phi^{1,0})[Z_\phi], \quad \text{if } \dim X = 3. \tag{18}
$$

This formula can be made somewhat more explicit by choosing a Chern–Weil representative for c_1. To do this we choose an Hermitian metric h on the fibers of $W_\phi^{1,0}$. We can lift this metric up to $X \times \mathbb{R}$ and set

$$
W_t^{1,0} = W_\phi^{1,0}\big|_{X \times \{t\}}, \quad h_t = h\big|_{X \times \{t\}}.
$$

Clearly $h_1 = \phi^*(h_0)$, conversely given a hermitian metric, g for $W^{1,0} \to X$ we can define h so that $h_0 = g$. Let $c_1(\Omega)$ be the Chern–Weil representative given by this choice of metric, this form can be decomposed as

$$
c_1(\Omega) = \alpha_t + \beta_t \wedge dt
$$

where $i_{\partial_t}\alpha_t = i_{\partial_t}\beta_t = 0$. By restriction to the fibers of Z_ϕ over the circle with parameter t, α_t must be the Chern–Weil representative of $c_1(W_t^{1,0})$ with respect to the metric h_t. From type considerations it follows that

$$
c_1(W_\phi^{1,0}) \wedge c_1(W_\phi^{1,0})[Z_\phi] = 2 \int_X \int_0^1 \alpha_t \wedge \beta_t \wedge dt. \tag{19}
$$

For a smooth family of forms, γ_t on $X \times \mathbb{R}$ with $i_{\partial_t}\gamma_t = 0$ we set

$$
\dot{\gamma}_t = i_{\partial_t} d\gamma_t \text{ and } d_X\gamma_t = d\gamma_t - dt \wedge \dot{\gamma}_t.
$$

Since $c_1(\Omega)$ is closed and $d_X\alpha_t = 0$ we deduce that $\dot\alpha_t + d_X\beta_t = 0$. Then defining $B_t = \int_0^t \beta_s ds$, it follows that $dB_t = \alpha_0 - \alpha_t$. Using Stokes' formula and these relations we find

$$\begin{aligned}
c_1(W_\phi^{1,0}) \wedge c_1(W_\phi^{1,0})[Z_\phi] &= -2\alpha_1 \wedge B_1[X] - 2\int_X \int_0^1 \dot\alpha_t \wedge \beta_t \wedge dt \\
&= -2\alpha_1 \wedge B_1[X] + 2\int_X \int_0^1 d\beta_t \wedge B_t \wedge dt \\
&= -2\alpha_1 \wedge B_1[X] + 2\int_X \int_0^1 d[\beta_t \wedge B_t \wedge dt] + \beta_t \wedge dB_t \wedge dt \\
&= -2\alpha_1 \wedge B_1[X] + 2\int_X \int_0^1 \beta_t \wedge (\alpha_0 - \alpha_t) \wedge dt.
\end{aligned}$$

Combining this with (19) we deduce that

$$c_1(W_\phi^{1,0}) \wedge c_1(W_\phi^{1,0})[Z_\phi] = \frac{1}{2}[\alpha_0 - \alpha_1] \wedge B_1[X]. \tag{20}$$

Note that as $dB_1 = \alpha_0 - \alpha_1$ we have found an expression for c-deg(ϕ) as a secondary characteristic class on X.

If $W^{1,0}$ is a flat bundle then we may pick h_0 to have zero curvature and therefore $\alpha_1 = \phi^*(\alpha_0) = 0$. Thus we have:

Corollary 1. *If X is a contact 3–manifold such that the contact field is a flat bundle then* c-deg*(ϕ)=0 for all $\phi \in \mathcal{M}(X)$.*

As shown in [7] the hypothesis is satisfied if X is the unit circle bundle in a holomorphic line bundle over a Riemann surface with $W^{1,0}$ the induced S^1–invariant CR–structure. This includes the unit sphere in $\mathbb{C}^2$ with the induced CR–structure or quotients of the three dimensional Heisenberg group by cocompact lattices. The latter case was considered by Zelditch [22]. For application to Weinstein's question we observe that the contact field defined on the boundary of the Grauert tube of a surface is always a trivial plane field. Thus we conclude that c-deg$(\phi) = 0$ for ϕ a contact transformation of the boundary of the Grauert tube over a surface. On a 3–manifold a contact class determines an orientation, namely that of $\theta \wedge d\theta$. If the surface has genus zero then the two contact structures, the one of the Grauert tube and the one on the boundary of a strictly pseudoconvex disk bundle in the canonical bundle, are isotopic. If the genus is greater than 1 then these two contact structures cannot be diffeomorphically equivalent as they belong to opposite orientation classes.

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