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INTERSECTIONAL PAIRS OF n-KNOTS, LOCAL MOVES OF n-KNOTS, AND THEIR ASSOCIATED INVARIANTS OF n-KNOTS

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1. Introduction

Our first purpose is to discuss the following problem. Let S_1^{n+2} and S_2^{n+2} be $(n+2)$-spheres embedded in the $(n+4)$-sphere S^{n+4} $(n \geqq 1)$ which intersect transversely. If we assume $M = S_1^{n+2} \cap S_2^{n+2}$ is PL homeomorphic to the single standard n-sphere we obtain a pair of n-knots, M in S_1^{n+2} and M in S_2^{n+2}. We consider which pairs of n-knots we obtain as above. That is, let (K_1, K_2) be a pair of n-knots. Then we consider whether the pair of n-knots (K_1, K_2) is obtained as above. We give a complete answer to this problem (Theorem 3.1).

In order to get the complete answer, we introduce a local move of n-knots $(n \geq 1)$. Furthermore, we show a relation between the local move and some invariants of n-knots (Theorem 4.1 and Corollary 4.2).

Our second purpose is to discuss the relation between the local move and the invariants of n-knots. In the case of 1-links, there is a great deal known about relations between local moves and knot invariants. (See [V,Wi,Ka2].) Our discussion is a high dimensional version of this theory.

2. Definitions

An (*oriented*) (*ordered*) *m-component n-*(*dimensional*) *link* is a smooth, oriented submanifold $L = \{K_1, ..., K_m\}$ of S^{n+2}, which is the ordered disjoint union of m connected oriented submanifolds, each PL homeomorphic to the standard n-sphere. If $m = 1$, then L is called a *knot*. (This definition is used often. See [CO,L1,L3].)

Let L_1 and L_2 be n-links. L_1 is said to be equivalent to L_2 if there exists an orientation preserving diffeomorphism h of S^{n+2} such that $h|L_1$ is an orientation preserving diffeomorphism from L_1 to L_2. We work in the smooth category.

Definition. (K_1, K_2) is called a *pair of n-knots* if K_1 and K_2 are n-knots. (K_1, K_2, X_1, X_2) is called a *4-tuple of n-knots and* $(n+2)$*-knots* or a *4-tuple of* $(n, n+2)$*-knots* if (K_1, K_2) is a pair of n-knots and X_1 and X_2 are $(n+2)$-knots diffeomorphic to the standard $(n+2)$-sphere. $(n \geqq 1)$.

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Definition. A 4-tuple of $(n, n+2)$-knots (K_1, K_2, X_1, X_2) is said to be *realizable* if there exists a smooth transverse immersion $f : S_1^{n+2} \coprod S_2^{n+2} \looparrowright S^{n+4}$ satisfying the following conditions: $(n \geqq 1)$

(1) The intersection $\Sigma = f(S_1^{n+2}) \cap f(S_2^{n+2})$ is PL homeomorphic to the standard n-sphere.
(2) $f^{-1}(\Sigma)$ in S_i^{n+2} defines an n-knot K_i $(i = 1, 2)$.
(3) $f|S_i^{n+2}$ is an embedding. $f(S_i^{n+2})$ in S^{n+4} is equivalent to X_i (i=1,2).

A pair of n-knots (K_1, K_2) is said to be *realizable* or is called an *intersectional pair of n-knots* if there is a realizable 4-tuple of $(n, n+2)$-knots (K_1, K_2, X_1, X_2).

3. Intersectional pair of n-knots

Our main theorem is:

Theorem 3.1. *A pair of n-knots (K_1, K_2) $(n \geqq 1)$ is realizable if and only if (K_1, K_2) satisfies the condition that*

$$\begin{cases} (K_1, K_2) \text{ is arbitrary} & \text{if } n \text{ is even,} \\ \mathrm{Arf}(K_1)=\mathrm{Arf}(K_2) & \text{if } n = 4m+1, \quad (m \geqq 0). \\ \sigma(K_1)=\sigma(K_2) & \text{if } n = 4m+3, \end{cases}$$

There is a mod 4 periodicity in dimension. It is similar to the periodicity in knot cobordism theory ([L1]) and surgery theory (see [Br,Wa,CS,We]). We have the following result on the realization of 4-tuples of $(n, n+2)$-knots.

Theorem 3.2. *A 4-tuple of $(n, n+2)$-knots (K_1, K_2, X_1, X_2) is realizable if K_1 and K_2 are slice $(n \geqq 1)$. In particular, if n is even, an arbitrary 4-tuple of $(n, n+2)$-knots (K_1, K_2, X_1, X_2) is realizable.*

Remarks. 1. Kervaire proved that all even dimensional knots are slice ([Ke]). 2. In [O1] the author discussed the case of two 3-spheres in a 5-sphere. In [O2] the author discussed the case of the intersection of three 4-spheres.

Problem. Which 4-tuples of $(2n+1, 2n+3)$-knots are realizable $(n \geqq 1)$?

4. High-dimensional pass-moves

In order to prove Theorem 3.1, we introduce a new local move for high dimensional knots, the *high dimensional pass-move.* Pass-moves for 1-knots are discussed in p.146 of [Ka]. We define high dimensional pass-moves for $(2k+1)$-knots $\subset S^{2k+3}$, $(k \geq 1)$.

Definition. Take a trivially embedded $(2k+3)$-ball $B = B^{2k+2} \times [-1, 1]$ in S^{2k+3}. We define $J_+, J_- \subset B$ as follows. (See Figure 4.1.) In $\partial B^{2k+2} \times \{0\}$, take trivially embedded S_1^k, S_2^k such that $\mathrm{lk}(S_1^k, S_2^k) = 1$. Let $N(S_*^k)$ be a tubular neighborhood of S_*^k in $\partial B^{2k+2} \times \{0\}$. Let h^{k+1} be a $(2k+2)$-dimensional $(k+1)$-handle which is attached to $\partial B^{2k+2} \times \{0\}$ along $N(S_1^k)$ with the trivial framing and which is embedded trivially in $B^{2k+2} \times \{0\}$. Let h_+^{k+1} (resp. h_-^{k+1}) be a $(2k+2)$-dimensional $(k+1)$-handle which is embedded in $B = B^{2k+2} \times [0, 1]$ (resp.

$B = B^{2k+2} \times [-1, 0]$) and which is attached to $\partial B^{2k+2} \times \{0\}$ along $N(S^k_2)$ with the trivial framing. Let $h^{k+1}_+ \cap h^{k+1}_- = N(S^k_2)$. Let $h^{k+1}_+ \cap h^{k+1} = h^{k+1}_- \cap h^{k+1} = \phi$. Let J_+ be the submanifold $\overline{(\partial h^{k+1}) - N(S^k_1)} \amalg \overline{(\partial h^{k+1}_+) - N(S^k_2)}$ in B. Let J_- be the submanifold $\overline{(\partial h^{k+1}) - N(S^k_1)} \amalg \overline{(\partial h^{k+1}_-) - N(S^k_2)}$ in B.

In Figure 4.1, we draw $B = B^{2k+2} \times [-1, 1]$ by using the projection to $B^{2k+2} \times \{0\}$.

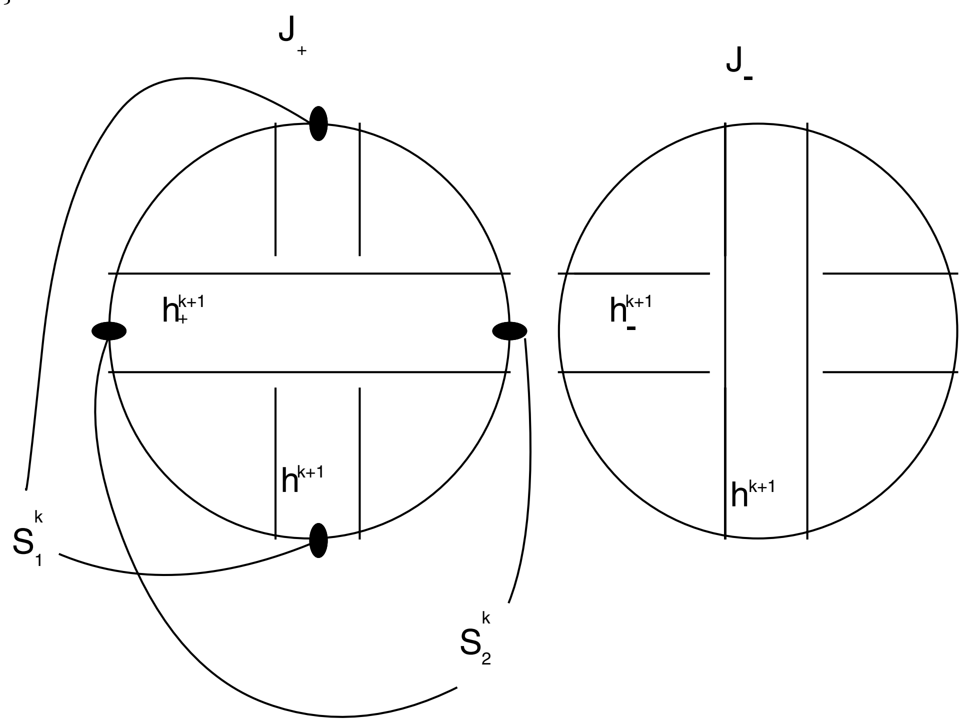

Figure 4.1

Let K_+, K_- be $(2k+1)$-knots $\subset S^{2k+3}$. We say that K_+ is obtained from K_- by one *high dimensional pass-move* if there is a trivially embedded $(2k+3)$-ball $B \subset S^{2k+3}$ such that $K_+ \cap B$ is J_+ and $K_- \cap B$ is J_-.

Let K, K' be $(2k+1)$-knots $\subset S^{2k+3}$. We say that K is *pass-move equivalent* to K' if there are $(2k+1)$-knots $K_1 = K, K_2, \ldots, K_{\mu-1}, K_\mu = K'$ ($\mu \in \mathbb{N}$) such that K_i is pass-move equivalent to K_{i+1}.

We prove the following:

Theorem 4.1. *For $(2k+1)$-knots K_1 and K_2, the following two conditions are equivalent: ($k \geqq 1$)*

(1) *There exists a $(2k+1)$-knot K_3 which is pass-move equivalent to K_1 and cobordant to K_2.*

(2) *K_1 and K_2 satisfy the condition* $\begin{cases} \mathrm{Arf}(K_1) = \mathrm{Arf}(K_2) & \textit{when } k \textit{ is even,} \\ \sigma(K_1) = \sigma(K_2) & \textit{when } k \textit{ is odd.} \end{cases}$

The $k = 0$ case of Theorem 4.1 follows from [Ka].

Corollary 4.2. *Let K_1 and K_2 be $(2k+1)$-knots $(k \geqq 1)$. Suppose that K_1 is pass-move equivalent to K_2. Then K_1 and K_2 satisfy the condition that*

$$\begin{cases} \mathrm{Arf}(K_1) = \mathrm{Arf}(K_2) & \textit{when } k \textit{ is even,} \\ \sigma(K_1) = \sigma(K_2) & \textit{when } k \textit{ is odd.} \end{cases}$$

Note. In [O3] the author proved a relation between another local move of 2-knots and other invariants of 2-knots.

5. Proof of Theorem 3.1

We prove the following lemmas by explicit construction.

Lemma 5.1. *Let K be an n-knot. Then the pair of n-knots (K, K) is realizable $(n \geqq 1)$.*

Lemma 5.2. *Let K_1 and K_2 be $(2k+1)$-knots. Suppose that K_1 is pass-move equivalent to K_2. Then the pair of $(2k+1)$-knots (K_1, K_2) is realizable $(k \geqq 0)$.*

Lemma 5.3. *Let K_1, K_2 and K_3 be n-knots $(n \geqq 1)$. Suppose that the pair of n-knots (K_1, K_2) is realizable and that K_2 is cobordant to K_3. Then the pair of n-knots (K_1, K_3) is realizable.*

Theorem 3.1 is deduced from Theorem 4.1 and Lemmas 5.1, 5.2, 5.3.

6. Proof of Theorem 3.2

It suffices to prove that a 4-tuple of $(n, n+2)$-knots (K_1, K_2, T, T) is realizable, where K_1 is a slice n-knot, K_2 is the trivial n-knot, T is the trivial $(n+2)$-knot. Any 1-twist spun knot is unknotted ([Z]). Theorem 3.2 follows from this fact.

7. The proof of Theorem 4.1

Every p-knot $(p > 1)$ is cobordant to a simple knot. (See [L1] for a proof and the definition of simple knots.) By using this fact, we prove that the $k \geqq 1$ case of Theorem 4.1 can be deduced from Theorem 7.1.

Proposition 7.1. *For simple $(2k+1)$-knots K_1 and K_2, the following two conditions are equivalent: $(k \geqq 1)$*

(1) *K_1 is pass-move equivalent to K_2.*

(2) *K_1 and K_2 satisfy the condition* $\begin{cases} \mathrm{Arf}(K_1) = \mathrm{Arf}(K_2) & \textit{when } k \textit{ is even,} \\ \sigma(K_1) = \sigma(K_2) & \textit{when } k \textit{ is odd.} \end{cases}$

Proof of Proposition 7.1. (2)⇒(1). K_1 bounds a Seifert hypersurface V_1 with a handle decomposition (one 0-handle)∪((k+1)-handles). Take a Seifert matrix associated with V_1. By using high dimensional pass moves, we can change the Seifert matrix without changing the diffeomorphism type of V_1. Thus we obtain a $(2k+1)$-knot K_2' whose Seifert matrix is same as the Seifert matrix of K_2 if (2) holds. By the classification theorem of simple knots by [L2], K_2' is equivalent to K_2.

(1)⇒(2). Suppose that $(2k+1)$-knots $K_* \subset S_*^{2k+3}$ bounds a Seifert hypersurface V_*. Note V_* are $(2k+2)$-manifolds. There is a compact oriented parallelizable $(2k+4)$-manifold P whose boundary is $S_1^{4k+3} \amalg S_2^{4k+3}$ containing compact oriented $(2k+3)$-manifold Q whose boundary is $V_1 \cup (S^{2k+1} \times [1,2]) \cup V_2$. (Here, ∂V_* is K_* and $S^{2k+1} \times \{*\}$ is K_*.) We use characteristic classes and intersection products to prove (1)⇒(2).

8. Intersectional pair of submanifolds

In §1 suppose M is not PL homeomorphic to the standard sphere. Then we obtain a pair of submanifolds, M in S_i^{n+2} $(i = 1, 2)$. Let N be a closed oriented manifold. (K_1, K_2) is called *a pair of submanifolds (diffeomorphic to N)* if K_i is a submanifold of S^{n+2} diffeomorphic to N.

Let (K_1, K_2) be a pair of submanifolds diffeomorphic to M. We say (K_1, K_2) is an *intersectional pair* if the submanifold K_i is equivalent to the submanifold $M = S_1^{n+2} \cap S_2^{n+2}$ in S_i^{n+2} as in §1 $(i = 1, 2)$. It is natural to ask the following problem.

Problem 8.1. Which pairs of submanifolds are intersectional pairs?

The author can prove the following results. When n is even, not all pair of submanifolds as above are realizable. When $n = 4m + 3$, we can define the signature as in the knot case and the signature is an obstruction. Therefore not all pairs are realizable. When $n = 3$, (K_1, K_2) is realizable if and only if $\sigma(K_1) = \sigma(K_2)$. When $n \neq 3$, $\sigma(K_1) = \sigma(K_2)$ does not imply (K_1, K_2) is realizable in general. When $n = 4m + 1$, there is a closed oriented manifold M such that if K_1 and K_2 are PL homeomorphic to M, then (K_1, K_2) is realizable. In other words, there is no invariant corresponding to the Arf invariant as in the knot case. Of course, not all pairs are realizable.

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