INTERSECTIONAL PAIRS OF $n$-KNOTS,
LOCAL MOVES OF $n$-KNOTS, AND
THEIR ASSOCIATED INVARIANTS OF $n$-KNOTS

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1. Introduction

Our first purpose is to discuss the following problem. Let $S^{n+2}_1$ and $S^{n+2}_2$ be $(n + 2)$-spheres embedded in the $(n + 4)$-sphere $S^{n+4}$ $(n \geq 1)$ which intersect transversely. If we assume $M = S^{n+2}_1 \cap S^{n+2}_2$ is PL homeomorphic to the single standard $n$-sphere we obtain a pair of $n$-knots, $M$ in $S^{n+2}_1$ and $M$ in $S^{n+2}_2$. We consider which pairs of $n$-knots we obtain as above. That is, let $(K_1, K_2)$ be a pair of $n$-knots. Then we consider whether the pair of $n$-knots $(K_1, K_2)$ is obtained as above. We give a complete answer to this problem (Theorem 3.1).

In order to get the complete answer, we introduce a local move of $n$-knots $(n \geq 1)$. Furthermore, we show a relation between the local move and some invariants of $n$-knots (Theorem 4.1 and Corollary 4.2).

Our second purpose is to discuss the relation between the local move and the invariants of $n$-knots. In the case of 1-links, there is a great deal known about relations between local moves and knot invariants. (See [V, Wi, Ka2].) Our discussion is a high dimensional version of this theory.

2. Definitions

An (oriented) (ordered) $m$-component $n$-(dimensional) link is a smooth, oriented submanifold $L = \{K_1, ..., K_m\}$ of $S^{n+2}$, which is the ordered disjoint union of $m$ connected oriented submanifolds, each PL homeomorphic to the standard $n$-sphere. If $m = 1$, then $L$ is called a knot. (This definition is used often. See [CO, L1, L3].)

Let $L_1$ and $L_2$ be $n$-links. $L_1$ is said to be equivalent to $L_2$ if there exists an orientation preserving diffeomorphism $h$ of $S^{n+2}$ such that $h|L_1$ is an orientation preserving diffeomorphism from $L_1$ to $L_2$. We work in the smooth category.

Definition. $(K_1, K_2)$ is called a pair of $n$-knots if $K_1$ and $K_2$ are $n$-knots. $(K_1, K_2, X_1, X_2)$ is called a 4-tuple of $n$-knots and $(n + 2)$-knots or a 4-tuple of $(n, n + 2)$-knots if $(K_1, K_2)$ is a pair of $n$-knots and $X_1$ and $X_2$ are $(n + 2)$-knots diffeomorphic to the standard $(n + 2)$-sphere. $(n \geq 1)$. 

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Definition. A 4-tuple of \((n, n+2)\)-knots \((K_1, K_2, X_1, X_2)\) is said to be realizable if there exists a smooth transverse immersion \(f : S^{n+2}_1 \coprod S^{n+2}_2 \rightarrow S^{n+4}\) satisfying the following conditions: \((n \geq 1)\)

1. The intersection \(\Sigma = f(S^{n+2}_1) \cap f(S^{n+2}_2)\) is PL homeomorphic to the standard \(n\)-sphere.
2. \(f^{-1}(\Sigma)\) in \(S^{n+2}_i\) defines an \(n\)-knot \(K_i\) \((i = 1, 2)\).
3. \(f|_{S^{n+2}_i}\) is an embedding. \(f(S^{n+2}_i)\) in \(S^{n+4}\) is equivalent to \(X_i\) \((i=1,2)\).

A pair of \(n\)-knots \((K_1, K_2)\) is said to be realizable or is called an intersectional pair of \(n\)-knots if there is a realizable 4-tuple of \((n, n+2)\)-knots \((K_1, K_2, X_1, X_2)\).

3. Intersectional pair of \(n\)-knots

Our main theorem is:

**Theorem 3.1.** A pair of \(n\)-knots \((K_1, K_2)\) \((n \geq 1)\) is realizable if and only if \((K_1, K_2)\) satisfies the condition that

\[
\begin{align*}
(K_1, K_2) & \text{ is arbitrary if } n \text{ is even,} \\
\text{Arf}(K_1) = \text{Arf}(K_2) & \text{ if } n = 4m + 1, \ (m \geq 0). \\
\sigma(K_1) = \sigma(K_2) & \text{ if } n = 4m + 3,
\end{align*}
\]

There is a mod 4 periodicity in dimension. It is similar to the periodicity in knot cobordism theory ([L1]) and surgery theory (see [Br, Wa, CS, We]). We have the following result on the realization of 4-tuples of \((n, n+2)\)-knots.

**Theorem 3.2.** A 4-tuple of \((n, n+2)\)-knots \((K_1, K_2, X_1, X_2)\) is realizable if \(K_1\) and \(K_2\) are slice \((n \geq 1)\). In particular, if \(n\) is even, an arbitrary 4-tuple of \((n, n+2)\)-knots \((K_1, K_2, X_1, X_2)\) is realizable.

**Remarks.** 1. Kervaire proved that all even dimensional knots are slice ([Ke]). 2. In [O1] the author discussed the case of two 3-spheres in a 5-sphere. In [O2] the author discussed the case of the intersection of three 4-spheres.

**Problem.** Which 4-tuples of \((2n+1, 2n+3)\)-knots are realizable \((n \geq 1)\)?

4. High-dimensional pass-moves

In order to prove Theorem 3.1, we introduce a new local move for high dimensional knots, the high dimensional pass-move. Pass-moves for 1-knots are discussed in p.146 of [Ka]. We define high dimensional pass-moves for \((2k+1)\)-knots \(S^{2k+3}_i, \ (k \geq 1)\).

**Definition.** Take a trivially embedded \((2k+3)\)-ball \(B = B^{2k+2} \times [-1,1]\) in \(S^{2k+3}\). We define \(J_+, J_- \subset B\) as follows. (See Figure 4.1.) In \(\partial B^{2k+2} \times \{0\}\), take trivially embedded \(S^k_1, S^k_2\) such that \(\text{lk}(S^k_1, S^k_2) = 1\). Let \(N(S^k_i)\) be a tubular neighborhood of \(S^k_i\) in \(\partial B^{2k+2} \times \{0\}\). Let \(h^{k+1}\) be a \((2k+2)\)-dimensional \((k+1)\)-handle which is attached to \(\partial B^{2k+2} \times \{0\}\) along \(N(S^k_i)\) with the trivial framing and which is embedded trivially in \(B^{2k+2} \times \{0\}\). Let \(h^{k+1}_+\) (resp. \(h^{k+1}_-\)) be a \((2k+2)\)-dimensional \((k+1)\)-handle which is embedded in \(B = B^{2k+2} \times [0,1]\) (resp.
Let $B = B^{2k+2} \times [-1,0]$ and which is attached to $\partial B^{2k+2} \times \{0\}$ along $N(S^k_2)$ with the trivial framing. Let $h^{k+1}_+ \cap h^{k+1}_- = N(S^k_2)$. Let $h^{k+1}_+ \cap h^{k+1}_- = h^{k+1}_+ \cap h^{k+1}_- = \phi$.

Let $J_+$ be the submanifold $(\partial h^{k+1}_+) - N(S^k_1) \amalg (\partial h^{k+1}_-) - N(S^k_2)$ in $B$. Let $J_-$ be the submanifold $(\partial h^{k+1}_+) - N(S^k_1) \amalg (\partial h^{k+1}_-) - N(S^k_2)$ in $B$.

In Figure 4.1, we draw $B = B^{2k+2} \times [-1,1]$ by using the projection to $B^{2k+2} \times \{0\}$.

![Figure 4.1](image)

Let $K_+, K_- \subset S^{2k+3}$. We say that $K_+$ is obtained from $K_-$ by one high dimensional pass-move if there is a trivially embedded $(2k+3)$-ball $B \subset S^{2k+3}$ such that $K_+ \cap B$ is $J_+$ and $K_- \cap B$ is $J_-$. Let $K, K' \subset S^{2k+3}$. We say that $K$ is pass-move equivalent to $K'$ if there are $(2k+1)$-knots $K_1 = K, K_2, \ldots, K_{\mu-1}, K_\mu = K'$ ($\mu \in \mathbb{N}$) such that $K_i$ is pass-move equivalent to $K_{i+1}$.

We prove the following:

**Theorem 4.1.** For $(2k+1)$-knots $K_1$ and $K_2$, the following two conditions are equivalent: ($k \geq 1$)

1. There exists a $(2k+1)$-knot $K_3$ which is pass-move equivalent to $K_1$ and cobordant to $K_2$.

2. $K_1$ and $K_2$ satisfy the condition $\left\{ \begin{array}{ll} \text{Arf}(K_1) = \text{Arf}(K_2) & \text{when } k \text{ is even,} \\ \sigma(K_1) = \sigma(K_2) & \text{when } k \text{ is odd.} \end{array} \right.$

The $k = 0$ case of Theorem 4.1 follows from [Ka].
Corollary 4.2. Let $K_1$ and $K_2$ be $(2k+1)$-knots ($k \geq 1$). Suppose that $K_1$ is pass-move equivalent to $K_2$. Then $K_1$ and $K_2$ satisfy the condition that

$$\begin{cases}
\text{Arf}(K_1) = \text{Arf}(K_2) & \text{when } k \text{ is even}, \\
\sigma(K_1) = \sigma(K_2) & \text{when } k \text{ is odd}.
\end{cases}$$

Note. In [O3] the author proved a relation between another local move of 2-knots and other invariants of 2-knots.

5. Proof of Theorem 3.1

We prove the following lemmas by explicit construction.

Lemma 5.1. Let $K$ be an $n$-knot. Then the pair of $n$-knots $(K, K)$ is realizable ($n \geq 1$).

Lemma 5.2. Let $K_1$ and $K_2$ be $(2k+1)$-knots. Suppose that $K_1$ is pass-move equivalent to $K_2$. Then the pair of $(2k+1)$-knots $(K_1, K_2)$ is realizable ($k \geq 0$).

Lemma 5.3. Let $K_1$, $K_2$ and $K_3$ be $n$-knots ($n \geq 1$). Suppose that the pair of $n$-knots $(K_1, K_2)$ is realizable and that $K_2$ is cobordant to $K_3$. Then the pair of $n$-knots $(K_1, K_3)$ is realizable.

Theorem 3.1 is deduced from Theorem 4.1 and Lemmas 5.1, 5.2, 5.3.

6. Proof of Theorem 3.2

It suffices to prove that a 4-tuple of $(n, n+2)$-knots $(K_1, K_2, T, T')$ is realizable, where $K_1$ is a slice $n$-knot, $K_2$ is the trivial $n$-knot, $T$ is the trivial $(n+2)$-knot. Any 1-twist spun knot is unknotted ([Z]). Theorem 3.2 follows from this fact.

7. The proof of Theorem 4.1

Every $p$-knot ($p > 1$) is cobordant to a simple knot. (See [L1] for a proof and the definition of simple knots.) By using this fact, we prove that the $k \geq 1$ case of Theorem 4.1 can be deduced from Theorem 7.1.

Proposition 7.1. For simple $(2k+1)$-knots $K_1$ and $K_2$, the following two conditions are equivalent: ($k \geq 1$)

1. $K_1$ is pass-move equivalent to $K_2$.
2. $K_1$ and $K_2$ satisfy the condition
$$\begin{cases}
\text{Arf}(K_1) = \text{Arf}(K_2) & \text{when } k \text{ is even}, \\
\sigma(K_1) = \sigma(K_2) & \text{when } k \text{ is odd}.
\end{cases}$$

Proof of Proposition 7.1. $(2) \Rightarrow (1)$. $K_1$ bounds a Seifert hypersurface $V_1$ with a handle decomposition (one 0-handle) $\cup (k+1)$-handles). Take a Seifert matrix associated with $V_1$. By using high dimensional pass moves, we can change the Seifert matrix without changing the diffeomorphism type of $V_1$. Thus we obtain a $(2k+1)$-knot $K'_2$ whose Seifert matrix is same as the Seifert matrix of $K_2$ if $(2)$ holds. By the classification theorem of simple knots by [L2], $K'_2$ is equivalent to $K_2$. 
(1)⇒(2). Suppose that $(2k+1)$-knots $K_* \subset S^{2k+3}_*$ bounds a Seifert hypersurface $V_*$. Note $V_*$ are $(2k+2)$-manifolds. There is a compact oriented parallelizable $(2k+4)$-manifold $P$ whose boundary is $S^{4k+3}_1 \amalg S^{4k+3}_2$ containing compact oriented $(2k+3)$-manifold $Q$ whose boundary is $V_1 \cup (S^{2k+1} \times [1, 2]) \cup V_2$. (Here, $\partial V_*$ is $K_*$ and $S^{2k+1} \times \{\ast\}$ is $K_*$.)

We use characteristic classes and intersection products to prove (1)⇒(2).

8. Intersectional pair of submanifolds

In §1 suppose $M$ is not PL homeomorphic to the standard sphere. Then we obtain a pair of submanifolds, $M$ in $S^{n+2}_i$ ($i = 1, 2$). Let $N$ be a closed oriented manifold. $(K_1, K_2)$ is called a pair of submanifolds (diffeomorphic to $N$) if $K_i$ is a submanifold of $S^{n+2}$ diffeomorphic to $N$.

Let $(K_1, K_2)$ be a pair of submanifolds diffeomorphic to $M$. We say $(K_1, K_2)$ is an intersectional pair if the submanifold $K_i$ is equivalent to the submanifold $M = S^{n+2}_1 \cap S^{n+2}_2$ in $S^{n+2}_i$ as in §1 ($i = 1, 2$). It is natural to ask the following problem.

Problem 8.1. Which pairs of submanifolds are intersectional pairs?

The author can prove the following results. When $n$ is even, not all pair of submanifolds as above are realizable. When $n = 4m + 3$, we can define the signature as in the knot case and the signature is an obstruction. Therefore not all pairs are realizable. When $n = 3$, $(K_1, K_2)$ is realizable if and only if $\sigma(K_1) = \sigma(K_2)$. When $n \neq 3$, $\sigma(K_1) = \sigma(K_2)$ does not imply $(K_1, K_2)$ is realizable in general. When $n = 4m + 1$, there is a closed oriented manifold $M$ such that if $K_1$ and $K_2$ are PL homeomorphic to $M$, then $(K_1, K_2)$ is realizable. In other words, there is no invariant corresponding to the Arf invariant as in the knot case. Of course, not all pairs are realizable.

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