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RULED MANIFOLDS WITH CONSTANT HERMITIAN SCALAR CURVATURE

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Assume that $(M:\omega_M)$ is an m-dimensional compact Kähler manifold with *constant* Hermitian scalar curvature and $\pi : E \longrightarrow M$ is a *simple* holomorphic vector bundle of rank n over M with an Einstein-Hermitian metric H_E. Let A denote the Einstein-Hermitian connection on E induced by H_E. Let $\mathbb{P}(E)$ denote the projectivization of E over M. Then $\mathbb{P}(E)$ is an $(m+n-1)$-dimensional complex manifold. Let L be the universal line bundle over $\mathbb{P}(E)$. Then the Einstein-Hermitian metric H_E induces a Hermitian metric H_{L^*} on L^* over $\mathbb{P}(E)$. Thus there is a representative

$$\left(\frac{i}{2\pi}F_{H_{L^*}}\right) = \frac{i}{2\pi}\cdot\bar{\partial}\partial\log H_{L^*},$$

of the Euler class $e\,(L^*)$ of L^* on $\mathbb{P}(E)$ induced by the Hermitian metric H_{L^*}. Note that the representative $\left(\frac{i}{2\pi}F_{H_{L^*}}\right)$ of $e\,(L^*)$ on $\mathbb{P}(E)$ induces the Fubini-Study metric on each fiber of $\check{\pi} : \mathbb{P}(E) \longrightarrow M$. Thus for each $k \in \mathbb{N}$ large enough,

$$\left(\frac{i}{2\pi}F_{H_{L^*}}\right) + k\cdot\check{\pi}^*\omega_M$$

is a Kähler form on $\mathbb{P}(E)$. The main purpose of this article is to announce the following result with a partial proof via formal power series expansion. Details will appear elsewhere.

Theorem A. *Assume that there is no nontrivial infinitesimal deformation of Kähler forms on M with constant Hermitian scalar curvature. Then, for each $k \in \mathbb{N}$ large enough, there exists a Kähler form on $\mathbb{P}(E)$ in the Kähler class*

$$\left[\left(\frac{i}{2\pi}F_{H_{L^*}}\right) + k\cdot\check{\pi}^*\omega_M\right],$$

with constant Hermitian scalar curvature.

Remarks. 1. When E is stable, it is known that there exists an Einstein-Hermitian connection on E by the results of Donaldson ([D1, D2]), and Uhlenbeck and Yau ([UY]). Conversely, when E is simple and admits an Einstein-Hermitian connection, it is known that E must be stable ([K]). 2. When

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$(M : \omega_M)$ is Einstein-Kähler, it is known by results of [C,BM] that, modulo the automorphism group of M, there is essentially only one Einstein-Kähler metric on M in the Kähler class $[\omega_M]$. In particular, when there is no nontrivial holomorphic vector field on M, there is only one Einstein-Kähler metric in $[\omega_M]$ on M.

Decomposition of the space of smooth functions on $\mathbb{P}(E)$. Note that the Einstein-Hermitian connection A on E defines one smooth distribution $\mathcal{H}$ of horizontal spaces on $\mathbb{P}(E)$:

$$T\left(\mathbb{P}(E)\right) = V \oplus \mathcal{H}.$$

Here V is the sub-bundle of $T\left(\mathbb{P}(E)\right)$ over $\mathbb{P}(E)$ consisting of tangent vectors which are tangential to the fibers of $\breve{\pi} : \mathbb{P}(E) \longrightarrow M$. Let $V^{[*]}$ denote the maximal sub-bundle of $T^*\left(\mathbb{P}(E)\right)$ over $\mathbb{P}(E)$ whose action on $\mathcal{H}$ is identically zero. Then the decomposition $T\left(\mathbb{P}(E)\right) = V \oplus \mathcal{H}$ of $T\left(\mathbb{P}(E)\right)$ over $\mathbb{P}(E)$ induces the following corresponding decomposition

$$T^*\left(\mathbb{P}(E)\right) = V^{[*]} \oplus \breve{\pi}^*\left(T^*(M)\right),$$

of $T^*\left(\mathbb{P}(E)\right)$ over $\mathbb{P}(E)$. Thus we have the following decomposition

$$\wedge^* T^*\left(\mathbb{P}(E)\right) = \mathcal{C}_V \oplus \mathcal{C}_m \oplus \mathcal{C}_M,$$

of $\wedge^* T^*\left(\mathbb{P}(E)\right)$ over $\mathbb{P}(E)$. Here $\mathcal{C}_V = \wedge^* V^{[*]}$ and $\mathcal{C}_M = \wedge^* \breve{\pi}^* T^*(M)$. Besides $\mathcal{C}_m$ is the sub-bundle of $\wedge^* T^*\left(\mathbb{P}(E)\right)$ over $\mathbb{P}(E)$ consisting of the *mixed* components of $\wedge^* T^*\left(\mathbb{P}(E)\right)$. Thus we have the following diagram of projection maps over $\mathbb{P}(E)$ such that $id. = \Pi_{\mathcal{C}_V} \oplus \Pi_{\mathcal{C}_m} \oplus \Pi_{\mathcal{C}_M}$ on $\wedge^* T^*\left(\mathbb{P}(E)\right)$:

$$\begin{tikzcd} \mathcal{C}_V & \wedge^* T^*\left(\mathbb{P}(E)\right) \arrow[l, "\Pi_{\mathcal{C}_V}"'] \arrow[r, "\Pi_{\mathcal{C}_M}"] \arrow[d, "\Pi_{\mathcal{C}_m}"] & \mathcal{C}_M \\ & \mathcal{C}_m & \end{tikzcd}$$

Let $\Gamma\left(\mathbb{P}(E) : \mathbb{R}\right)$ denote the space of smooth functions on $\mathbb{P}(E)$. We introduce one Hermitian form (metric) $\breve{\omega}$ on $\mathbb{P}(E)$ by setting

$$\breve{\omega} = \Pi_{\mathcal{C}_V}\left(\frac{i}{2\pi} F_{H_{L^*}}\right) + \breve{\pi}^* \omega_M.$$

Note that the derivation operator $d : \Gamma\left(\mathbb{P}(E) : \mathbb{R}\right) \longrightarrow \Gamma\left(\mathbb{P}(E) : T^*\left(\mathbb{P}(E)\right) \otimes \mathbb{R}\right)$ can be expressed as

$$d = d_V + d_M,$$

in which $d_V : \Gamma(\mathbb{P}(E):\mathbb{R}) \longrightarrow \Gamma\left(\mathbb{P}(E):\mathbb{R}\otimes V^{[*]}\right)$ and $d_M : \Gamma(\mathbb{P}(E):\mathbb{R}) \longrightarrow \Gamma(\mathbb{P}(E):\mathbb{R}\otimes \breve{\pi}^*(T^*(M)))$. Let d_V^* and d_M^* be respectively the adjoint operators of d_V and d_M with respect to the Hermitian form (metric) $\breve{\omega}$. Then we have

$$\Delta = d^* \circ d = \Delta_V + \Delta_M,$$

in which $\Delta_V \equiv d_V^* \circ d_V$ and $\Delta_M \equiv d_M^* \circ d_M$. Similarly we have $\bar{\partial} = \bar{\partial}_V + \bar{\partial}_M$ and $\partial = \partial_V + \partial_M$. Let Λ_V and Λ_M be respectively the adjoint operators of

$$\bullet \longmapsto \Pi_{\mathcal{C}_V}\left(\frac{i}{2\pi}F_{H_{L^*}}\right) \wedge \bullet,$$

and

$$\bullet \longmapsto \breve{\pi}^*\omega_M \wedge \bullet,$$

on $\mathbb{P}(E)$ with respect to the Hermitian form (metric) $\breve{\omega}$. Since H_E is *Einstein-Hermitian* it can be checked directly that

$$i\cdot\Lambda_V \circ \left(\Pi_{\mathcal{C}_V}\circ\bar{\partial}\partial\right) f = \frac{\Delta_V f}{2},$$

and

$$i\cdot\Lambda_M \circ \left(\Pi_{\mathcal{C}_M}\circ\bar{\partial}\partial\right) f = \frac{\Delta_M f}{2},$$

for any smooth function f on $\mathbb{P}(E)$. (Even though $\breve{\omega}$ is not necessarily Kähler.) Besides, we have

$$\Delta_M \circ \Delta_V = \Delta_V \circ \Delta_M.$$

In the following context we will sometimes use the suggestive symbols $\bar{\partial}_V\partial_V$ and $\bar{\partial}_M\partial_M$ defined as follows:

$$\bar{\partial}_V\partial_V \equiv \left(\Pi_{\mathcal{C}_V}\circ\bar{\partial}\partial\right), \quad \text{and} \quad \bar{\partial}_M\partial_M \equiv \left(\Pi_{\mathcal{C}_M}\circ\bar{\partial}\partial\right).$$

The reason for using these suggestive symbols will become clear when we express $\left(\Pi_{\mathcal{C}_V}\circ\bar{\partial}\partial\right)$ and $\left(\Pi_{\mathcal{C}_M}\circ\bar{\partial}\partial\right)$ explicitly using the special coordinate system in [H].

Since the restriction of $\left(\frac{i}{2\pi}F_{H_{L^*}}\right)$ on each fiber of $\breve{\pi} : \mathbb{P}(E) \longrightarrow M$ is simply the Fubini-Study Kähler form, there is a well-defined vector bundle W over M whose fiber over each point $z \in M$ is the eigen-space of the lowest non-zero eigenvalue of the (Fubini-Study) Laplacian on $\breve{\pi}^{-1}(z)$. On the other hand, integration along the fibers maps one smooth function on $\mathbb{P}(E)$ onto one smooth function on M. Thus for each smooth $\mathbb{R}$-valued function $f \in \Gamma(\mathbb{P}(E):\mathbb{R})$ on $\mathbb{P}(E)$ we have

$$f = \hat{\sigma}(f) \oplus \sigma(f) \oplus \tilde{\sigma}(f),$$

in which $(\hat{\sigma}(f):\sigma(f)) \in \Gamma(M:\mathbb{R}) \oplus \Gamma(M:W)$, and the restriction of $\tilde{\sigma}(f)$ on each fiber $\breve{\pi}^{-1}(z)$ of $\breve{\pi} : \mathbb{P}(E) \longrightarrow M$ is orthogonal to both the space of constant functions on $\breve{\pi}^{-1}(z)$ and the space W_z. Here $\Gamma(M:W)$ is the space of smooth sections of W over M, and W_z is the fiber of W over $z \in M$. Note that the smooth distribution $\mathcal{H}$ defines one connection $\nabla^{\mathcal{H}}$ on the fiber bundle $\breve{\pi} : \mathbb{P}(E) \longrightarrow M$, preserving the bundle W over M.

In the following section "Partial proof of theorem A via formal power series sxpansion", we will need the following result and its infinitesimal version:

Proposition A. *Assume that $\mathbb{C}^n$ and $\mathbb{P}(\mathbb{C}^n)$ are respectively endowed with the standard Hermitian metric $H = \sum \delta_{\alpha\beta} \cdot w_\alpha \cdot \bar{w}_\beta$ on $\mathbb{C}^n$ and the Fubini-Study metric on $\mathbb{P}(\mathbb{C}^n)$. Then for any traceless Hermitian quadratic form $q = \sum q_{\alpha\beta} \cdot w_\alpha \cdot \bar{w}_\beta$ on $\mathbb{C}^n$ (with $\sum q_{\gamma\gamma} = 0$), the function*

$$\frac{-4\pi \cdot (\sum \delta_{\alpha\beta} \cdot w_\alpha \cdot \bar{w}_\beta) \cdot \sum q_{\alpha\gamma} \cdot q_{\gamma\beta} \cdot w_\alpha \cdot \bar{w}_\beta + (4\pi + 2\pi n) \cdot (\sum q_{\alpha\beta} \cdot w_\alpha \cdot \bar{w}_\beta)^2}{(\sum \delta_{\alpha\beta} \cdot w_\alpha \cdot \bar{w}_\beta)^2}$$

on $\mathbb{P}(\mathbb{C}^n)$ is orthogonal to the eigen-space of the lowest non-zero eigen-value of the (Fubini-Study) Laplacian on $\mathbb{P}(\mathbb{C}^n)$.

This result can be expressed equivalently in the following way:

Let $\omega_{F\text{-}S} = -\frac{i}{2\pi}\bar{\partial}\partial \log H$ be the Fubini-Study Kähler form on $\mathbb{P}(\mathbb{C}^n)$. We define the function $Q\left(\frac{q}{H} : \frac{q}{H}\right)$ on $\mathbb{P}(\mathbb{C}^n)$ as follows:

$$Q\left(\frac{q}{H} : \frac{q}{H}\right) \cdot \frac{\omega_{F\text{-}S}^{(-1+n)}}{(-1+n)!} = i\bar{\partial}\partial\left(\frac{q}{H}\right) \wedge i\bar{\partial}\partial\left(\frac{q}{H}\right) \wedge \frac{\omega_{F\text{-}S}^{(-3+n)}}{(-3+n)!}.$$

Then the function

$$-\left(2\pi \cdot n \cdot \frac{q}{H}\right) \cdot \left(2\pi \cdot n \cdot \frac{q}{H}\right) + Q\left(\frac{q}{H} : \frac{q}{H}\right),$$

on $\mathbb{P}(\mathbb{C}^n)$ is orthogonal to the eigen-space of the lowest non-zero eigen-value of the (Fubini-Study) Laplacian on $\mathbb{P}(\mathbb{C}^n)$.

Actually, for any traceless Hermitian quadratic form $q = \sum q_{\alpha\beta} \cdot w_\alpha \cdot \bar{w}_\beta$ on $\mathbb{C}^n$, we have

$$-\left(2\pi \cdot n \cdot \frac{q}{H}\right) \cdot \left(2\pi \cdot n \cdot \frac{q}{H}\right) + Q\left(\frac{q}{H} : \frac{q}{H}\right) + 4\pi^2 \cdot \sum q_{\alpha\beta} \cdot q_{\beta\alpha} = -\pi \cdot (-4\pi \cdot id. + \Delta_{F\text{-}S})\left(\frac{q}{H} \cdot \frac{q}{H}\right).$$

Here $\Delta_{F\text{-}S}$ is the Laplacian operator induced by the Fubini-Study Kähler form $\omega_{F\text{-}S}$ on $\mathbb{P}(\mathbb{C}^n)$. Note that the constant function $4\pi^2 \cdot \sum q_{\alpha\beta} \cdot q_{\beta\alpha}$ on $\mathbb{P}(\mathbb{C}^n)$ associated with $q = \sum q_{\alpha\beta} \cdot w_\alpha \cdot \bar{w}_\beta$ only depends on the unitary group orbit of q.

Remark. Note that for any traceless Hermitian quadratic form $\sum q_{\alpha\beta} \cdot w_\alpha \cdot \bar{w}_\beta$ on $\mathbb{C}^n$ (with $\sum q_{\gamma\gamma} = 0$),

$$\frac{\sum q_{\alpha\beta} \cdot w_\alpha \cdot \bar{w}_\beta}{\sum \delta_{\alpha\beta} \cdot w_\alpha \cdot \bar{w}_\beta},$$

is an eigen-function of the lowest non-zero eigen-value $4\pi \cdot n$ of the (Fubini-Study) Laplacian on $\mathbb{P}(\mathbb{C}^n)$. Conversely, any eigen-function of the lowest non-zero eigen-value $4\pi \cdot n$ of the (Fubini-Study) Laplacian on $\mathbb{P}(\mathbb{C}^n)$ can be expressed in this way. □

Now we set

$$\Omega_M \equiv \frac{\omega_M^m}{m!},$$

and

$$\Omega \equiv \frac{\breve{\omega}^{(-1+m+n)}}{(-1+m+n)!} = \frac{\left(\Pi_{\mathcal{C}_V}\left(\frac{i}{2\pi}F_{H_{L^*}}\right)\right)^{(-1+n)}}{(-1+n)!} \wedge \frac{\breve{\pi}^*\omega_M^m}{m!}.$$

Let $\wedge^{\max}T^*(\mathbb{P}(E))$ denote the trivial bundle over $\mathbb{P}(E)$ consisting of differential forms on $\mathbb{P}(E)$ of maximal degree. Then we can identify functions on $\mathbb{P}(E)$ with sections of $\wedge^{\max}T^*(\mathbb{P}(E))$ over $\mathbb{P}(E)$ through the following map

$$f \longmapsto f \cdot \Omega.$$

Similarly, we can identify functions on M with sections of $\wedge^{\max}T^*(M)$ over M through the map

$$f \longmapsto f \cdot \Omega_M.$$

Here $\wedge^{\max}T^*(M)$ is the trivial bundle over M consisting of differential forms on M of maximal degree. Besides functions on $\mathbb{P}(E)$ which are *constant* along the fibers of $\breve{\pi} : \mathbb{P}(E) \longrightarrow M$ will be identified with functions on M. Similarly $\bullet$ can be identified with

$$\frac{\left(\Pi_{\mathcal{C}_V}\left(\frac{i}{2\pi}F_{H_{L^*}}\right)\right)^{(-1+n)}}{(-1+n)!} \wedge \breve{\pi}^*\bullet,$$

on $\mathbb{P}(E)$ for any differential form $\bullet$ on M. (We will use this convention when considering the operation "integration along the fibers".)

Partial proof of theorem A via formal power series expansion. Let K be the canonical line bundle of $\mathbb{P}(E)$. Assume that $\omega_{[k]}$ is a Kähler form on $\mathbb{P}(E)$ lying in the Kähler class

$$\left[\left(\frac{i}{2\pi}F_{H_{L^*}}\right) + k \cdot \breve{\pi}^*\omega_M\right],$$

and $\mathcal{E}_{[k]}$ is the representative of $-e(K)$ induced by $\omega_{[k]}$ on $\mathbb{P}(E)$. Then the equation of constant Hermitian scalar curvature for $\omega_{[k]}$ is as follows:

$$\text{(S)}\quad \Lambda_{[k]}\mathcal{E}_{[k]} = (-1+m+n)\cdot c_k \Longleftrightarrow$$

$$\mathcal{E}_{[k]} \wedge \frac{\omega_{[k]}^{(-2+m+n)}}{(-2+m+n)!} = (-1+m+n)\cdot c_k \cdot \frac{\omega_{[k]}^{(-1+m+n)}}{(-1+m+n)!}.$$

Since

$$\int_{\mathbb{P}(E)} \mathcal{E}_{[k]} \wedge \frac{\omega_{[k]}^{(-2+m+n)}}{(-2+m+n)!} = \int_{\mathbb{P}(E)} (-1+m+n)\cdot c_k \cdot \frac{\omega_{[k]}^{(-1+m+n)}}{(-1+m+n)!},$$

the constant c_k is pre-determined. The equation (S) is a 4-th order elliptic partial differential equation (depending on the parameter k). It is not easy to solve (S) directly. Thus we want to introduce an auxiliary equation to reduce (S) to a system of equations which is equivalent to (S). Let

$$ {}_o\omega_{[k]} \equiv \left(\frac{i}{2\pi} F_{H_{L^*}} \right) + k \cdot \check{\pi}^* \omega_M . $$

Assume that

$$ \omega_{[k]} = {}_o\omega_{[k]} + i\bar{\partial}\partial\phi_{[k]}, $$

in which the smooth $\mathbb{R}$-valued function $\phi_{[k]}$ on $\mathbb{P}(E)$ satisfies the following normalization condition

$$ \text{(N)} \qquad \int_M \hat{\sigma}\left(\phi_{[k]}\right) \cdot \Omega_M = 0 \Longleftrightarrow \int_{\mathbb{P}(E)} \phi_{[k]} \cdot \Omega = 0. $$

Let ${}_oH_{[k]}$ and $H_{[k]}$ denote the Kähler metrics on $\mathbb{P}(E)$ induced respectively by the Kähler forms ${}_o\omega_{[k]}$ and $\omega_{[k]}$ on $\mathbb{P}(E)$. Then we have

$$ \frac{i}{2\pi} \bar{\partial}\partial \log \det {}_oH_{[k]} = c_k \cdot {}_o\omega_{[k]} + {}_o\xi_{[k]}, $$

in which ${}_o\xi_{[k]}$ is a smooth $(1:1)$-form on $\mathbb{P}(E)$ satisfying the compatibility condition:

$$ \text{(C)} \qquad \int_{\mathbb{P}(E)} {}_o\xi_{[k]} \wedge \frac{{}_o\omega_{[k]}^{(-2+m+n)}}{(-2+m+n)!} = 0. $$

Thus

$$ \frac{i}{2\pi} \bar{\partial}\partial \log \det H_{[k]} = c_k \cdot \omega_{[k]} + {}_o\xi_{[k]} + i\bar{\partial}\partial\eta_{[k]}, $$

for some smooth $\mathbb{R}$-valued function $\eta_{[k]}$ on $\mathbb{P}(E)$. Then we have

$$ \frac{i}{2\pi} \bar{\partial}\partial \log \frac{\det H_{[k]}}{\det {}_oH_{[k]}} = c_k \cdot i\bar{\partial}\partial\phi_{[k]} + i\bar{\partial}\partial\eta_{[k]}, $$

and thence we may assume that

$$ \text{(A)} \qquad -\exp\left[2\pi \cdot \left(c_k \cdot \phi_{[k]} + \eta_{[k]}\right)\right] + \frac{\det H_{[k]}}{\det {}_oH_{[k]}} = 0. $$

Thus the equation (S) is reduced to

$$ \text{(B)} \quad \Lambda_{[k]}\, {}_o\xi_{[k]} + \Lambda_{[k]} \left(i\bar{\partial}\partial\eta_{[k]} \right) = 0 \Longleftrightarrow \left({}_o\xi_{[k]} + i\bar{\partial}\partial\eta_{[k]} \right) \wedge \frac{\omega_{[k]}^{(-2+m+n)}}{(-2+m+n)!} = 0. $$

Induction scheme. We will solve the system of equations (A) and (B) formally in the following context. Thus we assume that formally

$$\phi_{[k]} = \phi_0 + \sum_{\theta \in \mathbb{N}} \frac{\phi_\theta}{k^\theta},$$

and

$$\eta_{[k]} = \eta_0 + \sum_{\theta \in \mathbb{N}} \frac{\eta_\theta}{k^\theta},$$

in which, for each $\theta \geq 0$, ϕ_θ and η_θ are smooth functions on $\mathbb{P}(E)$ independent of k. Suppose that the left hand sides of (A) and (B) can be expressed respectively as follows:

$$-\exp\left[2\pi \cdot \left(c_k \cdot \phi_{[k]} + \eta_{[k]}\right)\right] + \frac{\det H_{[k]}}{\det\ {}_oH_{[k]}} = \sum_{\theta \geq 0} \frac{\mathbb{A}_\theta}{k^\theta},$$

and

$$\left({}_o\xi_{[k]} + i\bar{\partial}\partial\eta_{[k]}\right) \wedge \frac{\omega_{[k]}^{(-2+m+n)}}{(-2+m+n)!} = k^m \cdot \sum_{\theta \geq 0} \frac{\mathbb{B}_\theta}{k^\theta}.$$

Then, the system of equations (A) and (B) is formally equivalent to the following set of equations:

$$(\check{\mathrm{A}}) \qquad \mathbb{A}_\theta = 0, \qquad \forall \theta \geq 0,$$

and

$$(\check{\mathrm{B}}) \qquad \mathbb{B}_\theta = 0, \qquad \forall \theta \geq 0.$$

Now we introduce some notation which will be used later on. Assume that Z is one vector bundle over $\mathbb{P}(E)$. We will use the symbol

$$O_Z\left(k^{-\theta}\right),$$

to indicate one formal sum

$$\sum_{\nu \geq \theta} \frac{s_\nu}{k^\nu},$$

of smooth sections of Z over $\mathbb{P}(E)$. Here each s_ν is one smooth section of Z over $\mathbb{P}(E)$ independent of k. When Z is the trivial bundle $\mathbb{R} \times \mathbb{P}(E)$ we will use the symbol $O\left(k^{-\theta}\right)$ instead. In the following proof, the representative $\left(\frac{i}{2\pi} F_{H_{L^*}}\right)$ of $e\left(L^*\right)$ on $\mathbb{P}(E)$ will be fixed. Thus $e\left(L^*\right)$ will be *identified with* $\left(\frac{i}{2\pi} F_{H_{L^*}}\right)$ when there is no confusion.

Note that the Einstein-Hermitian condition implies that

$$\text{(E-H)}\quad \left(\frac{i}{2\pi}F_{H_{L^*}}\right)^{n}\wedge\omega_M^{(-1+m)}+\left(\frac{i}{2\pi}F_{H_{L^*}}\right)^{(-1+n)}\wedge\omega_M^{(-1+m)}\wedge\left(\frac{i}{2\pi}F_{H_E}\right)=0.$$

Here F_{H_E} is the curvature form on M induced by the Einstein-Hermitian connection on E. Actually the condition (E-H) is equivalent to the Einstein-Hermitian condition. Let K_M denote the canonical line bundle of M. Since H_E is *Einstein-Hermitian* it can be checked directly that

$$\begin{aligned}(1)\quad \frac{i}{2\pi}\bar{\partial}\partial\log\det {}_oH_{[k]} =&\\ n\cdot\left(\frac{i}{2\pi}F_{H_{L^*}}\right)+\check{\pi}^*\mathrm{trace}\left(\frac{i}{2\pi}F_{H_E}\right)&+\check{\pi}^*\mathrm{trace}\left(\frac{i}{2\pi}F_{H_M}\right)\\ &+O_{\mathcal{C}_M}\left(k^{-1}\right)+O_{\mathcal{C}_m}\left(k^{-2}\right)+O_{\mathcal{C}_V}\left(k^{-2}\right).\end{aligned}$$

Here H_M is the Kähler metric on M induced by the Kähler form ω_M. F_{H_M} is the curvature form of the holomorphic tangent bundle of M induced by the Kähler metric H_M on M. (Actually the reslut (1) can be calculated easily using special coordinate system [H].) Now using (E-H) it is easy to see that the rational function c_k in $\frac{1}{k}$ can be expressed as follows:

$$c_k=\frac{(-1+n)\cdot n}{(-1+m+n)}+\frac{1}{k}\cdot\frac{m}{(-1+m+n)}\cdot\frac{-e\left(K_M\right)}{\omega_M}+O\left(k^{-2}\right).$$

Here the constant $\frac{-e(K_M)}{\omega_M}=-\frac{e(K_M)}{\omega_M}$ is defined as follows:

$$\int_M\frac{-e\left(K_M\right)}{\omega_M}\cdot\frac{\omega_M^m}{m!}=\int_M\frac{-e\left(K_M\right)}{m}\wedge\frac{\omega_M^{(-1+m)}}{(-1+m)!}.$$

Besides we have

$$\begin{aligned}(2)\quad {}_o\xi_{[k]} =&\\ -n\cdot k\cdot\omega_M+\frac{m\cdot n}{(-1+m+n)}\cdot {}_o\omega_{[k]}&+\check{\pi}^*\mathrm{trace}\left(\frac{i}{2\pi}F_{H_E}\right)+\check{\pi}^*\mathrm{trace}\left(\frac{i}{2\pi}F_{H_M}\right)\\ +\frac{m}{(-1+m+n)}\cdot\frac{e\left(K_M\right)}{\omega_M}\cdot\check{\pi}^*\left(\omega_M\right)&+\frac{1}{k}\cdot\frac{m}{(-1+m+n)}\cdot\frac{e\left(K_M\right)}{\omega_M}\cdot\left(\frac{i}{2\pi}F_{H_{L^*}}\right)\\ &+O_{\mathcal{C}_M}\left(k^{-1}\right)+O_{\mathcal{C}_m}\left(k^{-2}\right)+O_{\mathcal{C}_V}\left(k^{-2}\right),\end{aligned}$$

and thence by (E-H),

$$(3)\quad {}_o\xi_{[k]}\wedge\frac{{}_o\omega_{[k]}^{(-2+m+n)}}{(-2+m+n)!}=k^m\cdot\sum_{\theta\geq 2}\frac{{}_o\mathbb{B}_\theta}{k^\theta}=k^m\cdot O_{\wedge^{\max}T^*(\mathbb{P}(E))}\left(k^{-2}\right).$$

Now we set for each integral $\theta \geq 0$

$$\zeta_\theta \equiv \left(-\frac{m\cdot n}{(-1+m+n)}\cdot\phi_\theta+\eta_\theta\right) \Longleftrightarrow \eta_\theta = \frac{m\cdot n}{(-1+m+n)}\cdot\phi_\theta+\zeta_\theta.$$

To solve $(\check{\mathrm{A}})$ and $(\check{\mathrm{B}})$ inductively we choose the following set of initial conditions:

(I) $\quad \sigma(\phi_0)=\sigma(\zeta_0)=\sigma(\zeta_1)=0=\tilde{\sigma}(\phi_0)=\tilde{\sigma}(\zeta_0)=\tilde{\sigma}(\phi_1)=\tilde{\sigma}(\zeta_1),$

$$\text{and}\quad n\cdot\hat{\sigma}(\phi_0)+\hat{\sigma}(\zeta_0)=0=\frac{(-1+n)\cdot n}{(-1+m+n)}\cdot\hat{\sigma}(\phi_0)+\hat{\sigma}(\eta_0).$$

We will solve

$$\hat{\sigma}(\phi_\theta)\oplus\sigma(\phi_{\theta+1})\oplus\tilde{\sigma}(\phi_{\theta+2}),\quad\text{and}\quad [n\cdot\hat{\sigma}(\phi_{\theta+1})+\hat{\sigma}(\zeta_{\theta+1})]\oplus\sigma(\zeta_{\theta+2})\oplus\tilde{\sigma}(\zeta_{\theta+2}),$$

using the system of equations $\hat{\sigma}(\mathbb{A}_{\theta+1})\oplus\sigma(\mathbb{A}_{\theta+2})\oplus\tilde{\sigma}(\mathbb{A}_{\theta+2})=0$ and $\hat{\sigma}(\mathbb{B}_{\theta+2})\oplus\sigma(\mathbb{B}_{\theta+2})\oplus\tilde{\sigma}(\mathbb{B}_{\theta+2})=0$ by induction on $\theta\geq 0$.

Simple calculation shows that (I) implies that $\mathbb{A}_0=0$ and $\sigma(\mathbb{A}_1)=\tilde{\sigma}(\mathbb{A}_1)=0$. Here

(4) $$\mathbb{A}_1=\left(\frac{1}{2}\right)\left(-4\pi\cdot\zeta_1+\Delta_M\phi_0+\frac{4\pi\cdot m}{(-1+m+n)}\cdot\frac{e(K_M)}{\omega_M}\cdot\phi_0+\right.$$
$$\left.(-4\pi\cdot n\cdot\phi_1+\Delta_V\phi_1)\right).$$

Besides it is easy to see that (I) implies that

$$\mathbb{B}_0=\mathbb{B}_1=0,$$

via simple calculation. Now we compute $\mathbb{B}_2$. Lengthy but direct calculation shows that

$$\mathbb{B}_2={}_o\mathbb{B}_2+\frac{n\cdot e(L^*)^{(-1+n)}\wedge\omega_M^{(-2+m)}\wedge i\bar{\partial}_M\partial_M\phi_0}{(-2+m)!\cdot n!}\wedge$$
$$\left[n\cdot\Pi_{\mathcal{C}_M}\circ e(L^*)+\check{\pi}^*\mathrm{trace}\left(\frac{i}{2\pi}F_{H_E}\right)\right]+$$
$$\frac{k^m}{k\cdot k}\cdot\frac{(-1+n)\cdot(\Pi_{\mathcal{C}_M}\circ e(L^*))\wedge e(L^*)^{(-2+n)}\wedge\omega_M^{(-1+m)}}{(-1+m)!\cdot(-1+n)!}\wedge$$
$$i\bar{\partial}_V\partial_V\left[\frac{n\cdot(-1+n)}{(-1+m+n)}\cdot\phi_1+\eta_1\right]+$$
$$\frac{k^m}{k\cdot k}\cdot\frac{(-1+n)\cdot e(L^*)^{(-2+n)}\wedge\omega_M^{(-1+m)}}{(-1+m)!\cdot(-1+n)!}\wedge\check{\pi}^*\mathrm{trace}\left(\frac{i}{2\pi}F_{H_E}\right)\wedge i\bar{\partial}_V\partial_V\phi_1+$$
$$\frac{k^m}{k\cdot k}\cdot\frac{(-1+n)\cdot e(L^*)^{(-2+n)}\wedge\omega_M^{(-1+m)}}{(-1+m+n)\cdot(-1+m)!\cdot(-1+n)!}\wedge\check{\pi}^*\mathrm{trace}\left(\frac{i}{2\pi}F_{H_M}\right)\wedge i\bar{\partial}_V\partial_V\phi_1+$$

$$\frac{k^m}{k\cdot k}\cdot\frac{e\left(L^*\right)^{(-2+n)}\wedge\omega_M^m}{(-2+n)!\cdot m!}\wedge i\bar{\partial}_V\partial_V\left(-\frac{m\cdot n}{(-1+m+n)}\cdot\phi_2+\eta_2\right)+$$

$$\frac{k^m}{k\cdot k}\cdot\frac{n\cdot(\Pi_{\mathcal{C}_M}\circ e\left(L^*\right))\wedge e\left(L^*\right)^{(-1+n)}\wedge\omega_M^{(-2+m)}}{(-2+m)!\cdot n!}\wedge$$

$$i\bar{\partial}_M\partial_M\left[\frac{(-1+n)\cdot n}{(-1+m+n)}\cdot\phi_0+\eta_0\right]+$$

$$\frac{k^m}{k\cdot k}\cdot\frac{\left[\frac{(-1+n)\cdot m\cdot n}{(-1+m)\cdot(-1+m+n)}+\frac{m\cdot n}{(-1+m+n)}\right]\cdot\frac{e(K_M)}{\omega_M}\cdot e\left(L^*\right)^{(-1+n)}\wedge\omega_M^{(-1+m)}}{(-2+m)!\cdot n!}\wedge$$

$$i\bar{\partial}_M\partial_M\phi_0+$$

$$\frac{k^m}{k\cdot k}\cdot\frac{n\cdot e\left(L^*\right)^{(-1+n)}\wedge\omega_M^{(-2+m)}}{(-2+m)!\cdot n!}\wedge i\bar{\partial}_M\partial_M\phi_0\wedge\breve{\pi}^*\mathrm{trace}\left(\frac{i}{2\pi}F_{H_M}\right)+$$

$$\frac{k^m}{k\cdot k}\cdot\frac{e\left(L^*\right)^{(-1+n)}\wedge\omega_M^{(-1+m)}}{(-1+m)!\cdot(-1+n)!}\wedge i\bar{\partial}_M\partial_M\left[\frac{n\cdot(-1+n)}{(-1+m+n)}\cdot\phi_1+\eta_1\right]+$$

$$\frac{k^m}{k\cdot k}\cdot\frac{n\cdot e\left(L^*\right)^{(-1+n)}\wedge\omega_M^{(-2+m)}}{(-2+m)!\cdot n!}\cdot i\bar{\partial}_M\partial_M\left[\eta_0+\frac{(-1+n)\cdot n}{(-1+m+n)}\cdot\phi_0\right]\wedge i\bar{\partial}_M\partial_M\phi_0+$$

$$\frac{k^m}{k\cdot k}\cdot\frac{(-2+n)\cdot e\left(L^*\right)^{(-3+n)}\wedge\omega_M^m}{(-2+n)!\cdot m!}\wedge i\bar{\partial}_V\partial_V\phi_1\wedge$$

$$i\bar{\partial}_V\partial_V\left[-\frac{m\cdot n}{(-1+m+n)}\phi_1+\eta_1\right]+$$

$$\frac{k^m}{k\cdot k}\cdot\frac{(-1+n)\cdot e\left(L^*\right)^{(-2+n)}\wedge\omega_M^{(-1+m)}}{(-1+m)!\cdot(-1+n)!}\wedge i\bar{\partial}_V\partial_V\phi_1\wedge$$

$$i\bar{\partial}_M\partial_M\left[\frac{n\cdot[-m+(-1+n)]}{(-1+m+n)}\phi_0+\eta_0\right]+$$

$$\frac{k^m}{k\cdot k}\cdot\frac{(-1+n)\cdot e\left(L^*\right)^{(-2+n)}\wedge\omega_M^{(-1+m)}}{(-1+m)!\cdot(-1+n)!}\wedge i\bar{\partial}_M\partial_M\phi_0\wedge i\bar{\partial}_V\partial_V\eta_1.$$

Thus by (E-H) and (I) we see that $\mathbb{B}_2=0$ if and only if

(5) $\quad 0=(-1+m)!\cdot(-1+n)!\cdot{}_o\mathbb{B}_2+$

$$(-1+m)\cdot e\left(L^*\right)^{(-1+n)}\wedge\omega_M^{(-2+m)}\wedge i\bar{\partial}_M\partial_M\phi_0\wedge$$

$$\left[n\cdot(\Pi_{\mathcal{C}_M}\circ e\left(L^*\right))+\breve{\pi}^*\mathrm{trace}\left(\frac{i}{2\pi}F_{H_E}\right)\right]+$$

$$\frac{(-1+n)}{(-1+m+n)}\cdot e\left(L^*\right)^{(-2+n)}\wedge\breve{\pi}^*\mathrm{trace}\left(\frac{i}{2\pi}F_{H_M}\right)\wedge\omega_M^{(-1+m)}\wedge i\bar{\partial}_V\partial_V\phi_1+$$

$$\frac{(-1+n)}{m}\cdot e\left(L^*\right)^{(-2+n)}\wedge\omega_M^m\wedge i\bar{\partial}_V\partial_V\zeta_2+$$

$$e\left(L^{*}\right)^{(-1+n)} \wedge \omega_{M}^{(-1+m)} \wedge \left[n \cdot i\bar{\partial}_{M}\partial_{M}\phi_{1} + i\bar{\partial}_{M}\partial_{M}\zeta_{1}\right] +$$
$$(-1+m) \cdot e\left(L^{*}\right)^{(-1+n)} \wedge \omega_{M}^{(-2+m)} \wedge \breve{\pi}^{*}\mathrm{trace}\left(\frac{i}{2\pi}F_{H_{M}}\right) \wedge i\bar{\partial}_{M}\partial_{M}\phi_{0} +$$
$$\frac{m \cdot (-2+m+n)}{(-1+m+n)} \cdot \frac{e\left(K_{M}\right)}{\omega_{M}} \cdot e\left(L^{*}\right)^{(-1+n)} \wedge \omega_{M}^{(-1+m)} \wedge i\bar{\partial}_{M}\partial_{M}\phi_{0}.$$

By integrating (5) along the fibers of $\breve{\pi} : \mathbb{P}(E) \longrightarrow M$ we obtain

$$0 = (-1+m)! \cdot \hat{\sigma}\left({}_{o}\mathbb{B}_{2}\right) + \omega_{M}^{(-1+m)} \wedge \left[n \cdot i\bar{\partial}_{M}\partial_{M}\hat{\sigma}\left(\phi_{1}\right) + i\bar{\partial}_{M}\partial_{M}\hat{\sigma}\left(\zeta_{1}\right)\right] + \tag{6}$$
$$(-1+m) \cdot \omega_{M}^{(-2+m)} \wedge \mathrm{trace}\left(\frac{i}{2\pi}F_{H_{M}}\right) \wedge i\bar{\partial}_{M}\partial_{M}\phi_{0} +$$
$$\frac{m \cdot (-2+m+n)}{(-1+m+n)} \cdot \frac{e\left(K_{M}\right)}{\omega_{M}} \cdot \omega_{M}^{(-1+m)} \wedge i\bar{\partial}_{M}\partial_{M}\phi_{0}.$$

Now by (4) we see that $\hat{\sigma}\left(\mathbb{A}_{1}\right) = 0$ if and only if

$$n \cdot \hat{\sigma}\left(\phi_{1}\right) + \hat{\sigma}\left(\zeta_{1}\right) = \frac{\Delta_{M}\phi_{0}}{4\pi} + \frac{m}{(-1+m+n)} \cdot \frac{e\left(K_{M}\right)}{\omega_{M}} \cdot \phi_{0}. \tag{7}$$

Thus, by substituting (7) into (6), we obtain

$$-(-1+m)! \cdot \hat{\sigma}\left({}_{o}\mathbb{B}_{2}\right) = \tag{8}$$
$$\omega_{M}^{(-1+m)} \wedge i\bar{\partial}_{M}\partial_{M}\left(\frac{\Delta_{M}\phi_{0}}{4\pi} + m \cdot \frac{e\left(K_{M}\right)}{\omega_{M}} \cdot \phi_{0}\right) +$$
$$(-1+m) \cdot \omega_{M}^{(-2+m)} \wedge \mathrm{trace}\left(\frac{i}{2\pi}F_{H_{M}}\right) \wedge i\bar{\partial}_{M}\partial_{M}\phi_{0}.$$

Note that the 4-th order linear partial differential operator

$$\frac{\omega_{M}^{(-1+m)}}{(-1+m)!} \wedge i\bar{\partial}_{M}\partial_{M}\frac{\Delta_{M}\bullet}{4\pi} + \tag{V}$$
$$\frac{(-1+m) \cdot \omega_{M}^{(-2+m)}}{(-1+m)!} \wedge \mathrm{trace}\left(\frac{i}{2\pi}F_{H_{M}}\right) \wedge i\bar{\partial}_{M}\partial_{M} \bullet +$$
$$m \cdot \frac{e\left(K_{M}\right)}{\omega_{M}} \cdot \frac{\omega_{M}^{(-1+m)}}{(-1+m)!} \wedge i\bar{\partial}_{M}\partial_{M}\bullet,$$

is nothing but the infinitesimal variational operator for the constant Hermitian scalar curvature equation on M. Thus ϕ_0 can be solved uniquely whenever the integral of the left hand side of (8) on M is *zero*. But this follows immediately from the compatibility condition (C). Hence ϕ_0 can be solved uniquely. By (7) this result implies in particular that $n \cdot \hat{\sigma}(\phi_1) + \hat{\sigma}(\zeta_1)$ is uniquely determined.

Now we solve $\sigma(\phi_1)$. Since $\sigma(\zeta_1) = 0$ and $\Delta_V \sigma(\bullet) = 4\pi \cdot n \cdot \sigma(\bullet)$ we see that $\sigma(\mathbb{B}_2) = 0$ if and only if

$$\text{(9)} \quad \left[-\frac{m}{(-1+m+n)} \cdot \frac{e(K_M)}{\omega_M} \cdot \sigma(\phi_1) + \sigma(\zeta_2) \right] = - \frac{\Delta_M \sigma(\phi_1)}{4\pi} + \text{ known terms,}$$

via simple calculation. Here we have used the fact that Δ_V and Δ_M preserve the decomposition $id. = \hat{\sigma} \oplus \sigma \oplus \tilde{\sigma}$:

$$\Delta_V \circ \hat{\sigma} \bullet \oplus \Delta_V \circ \sigma \bullet \oplus \Delta_V \circ \tilde{\sigma} \bullet = \hat{\sigma} \circ \Delta_V \bullet \oplus \sigma \circ \Delta_V \bullet \oplus \tilde{\sigma} \circ \Delta_V \bullet,$$

and

$$\Delta_M \circ \hat{\sigma} \bullet \oplus \Delta_M \circ \sigma \bullet \oplus \Delta_M \circ \tilde{\sigma} \bullet = \hat{\sigma} \circ \Delta_M \bullet \oplus \sigma \circ \Delta_M \bullet \oplus \tilde{\sigma} \circ \Delta_M \bullet\,.$$

Besides we use the phrase "known terms" to indicate those terms whose values are known. Simple calculation shows that

$$\begin{aligned} \text{(10)} \quad \mathbb{A}_2 &= (1/2)\left(\Delta_M \phi_1 + \Delta_V \phi_2 + Q_V(\sigma(\phi_1) : \sigma(\phi_1))\right) + \\ &(1/4)(\Delta_V \phi_1 \cdot \Delta_M \phi_0) + -2\pi \cdot \left(n \cdot \phi_2 + \zeta_2 + \frac{m}{(-1+m+n)} \cdot \frac{-e(K_M)}{\omega_M} \cdot \phi_1 \right) + \\ &- \left(\frac{1}{2}\right) \left[2\pi \cdot \left(n \cdot \phi_1 + \zeta_1 + \frac{m}{(-1+m+n)} \cdot \frac{-e(K_M)}{\omega_M} \cdot \phi_0 \right) \right]^2 + \text{known terms.} \end{aligned}$$

Here $Q_V(\bullet : \bullet)$ is the smooth family of fiberwise operators whose action on each fiber $\mathbb{P}(\mathbb{C}^n)$ of $\breve{\pi} : \mathbb{P}(E) \longrightarrow M$ is simply defined by $Q(\bullet : \bullet)$ on $\mathbb{P}(\mathbb{C}^n)$. Now by (7), we have

$$n \cdot \phi_1 + \zeta_1 + \frac{m}{(-1+m+n)} \cdot \frac{-e(K_M)}{\omega_M} \cdot \phi_0 = n \cdot \sigma(\phi_1) + \frac{\Delta_M \phi_0}{4\pi}.$$

Thus by (9) we have

$$\begin{aligned} \sigma(\mathbb{A}_2) &= \frac{\Delta_M \sigma(\phi_1)}{2} + \frac{\Delta_V \sigma(\phi_2)}{2} + \sigma\left(\frac{Q_V(\sigma(\phi_1) : \sigma(\phi_1))}{2} \right) + \\ &\qquad \sigma\left(\frac{\Delta_V \phi_1}{2} \cdot \frac{\Delta_M \phi_0}{2} \right) + -2\pi \cdot n \cdot \sigma(\phi_2) + \frac{\Delta_M \sigma(\phi_1)}{2} + \\ &\quad - \sigma\left(\frac{\left[2\pi \cdot \left(n \cdot \sigma(\phi_1) + \frac{\Delta_M \phi_0}{4\pi}\right)\right] \cdot \left[2\pi \cdot \left(n \cdot \sigma(\phi_1) + \frac{\Delta_M \phi_0}{4\pi}\right)\right]}{2} \right) + \\ &\qquad \text{known terms,} \\ &= \Delta_M \sigma(\phi_1) + \left[-2\pi n \cdot \sigma(\phi_1) \cdot \frac{\Delta_M \phi_0}{2} + \sigma\left(\frac{\Delta_V \phi_1}{2} \cdot \frac{\Delta_M \phi_0}{2} \right) \right] \\ &\qquad + \text{known terms,} \\ &= \Delta_M \sigma(\phi_1) + \text{known terms.} \end{aligned}$$

Here we have used Proposition A in the above calculation to obtain the middle equality. Since E is *simple* over M the operator Δ_M on $\Gamma(M:W)$ is *invertible*. (We will elaborate on this in [H].) Thus $\sigma(\phi_1)$ can be solved uniquely. Now by (9) we see that $\sigma(\zeta_2)$ is uniquely determined.

To solve $\tilde{\sigma}(\zeta_2)$ we simply note that $\tilde{\sigma}(\zeta_2)$ is the only unknown in $\tilde{\sigma}(\mathbb{B}_2)=0$:

$$\frac{\Delta_V \tilde{\sigma}(\zeta_2)}{2\cdot m}\cdot e(L^*)^{(-1+n)}\wedge\omega_M^m+\text{known terms}=0.$$

Thus $\tilde{\sigma}(\zeta_2)$ can be solved uniquely. Now we note that $\tilde{\sigma}(\phi_2)$ becomes the only unknown in $\tilde{\sigma}(\mathbb{A}_2)=0$:

$$\left(\frac{1}{2}\right)\left(-4\pi n\cdot\tilde{\sigma}(\phi_2)+\Delta_V\tilde{\sigma}(\phi_2)\right)+\text{known terms}=0$$

because the value of $\tilde{\sigma}(\zeta_2)$ has been uniquely determined. (Note that by (7) the value of $n\cdot\phi_1+\zeta_1=n\cdot\hat{\sigma}(\phi_1)+\hat{\sigma}(\zeta_1)+n\cdot\sigma(\phi_1)$ is already known.) Thus $\tilde{\sigma}(\phi_2)$ can be solved uniquely.

Now we have completed the leading step of our induction scheme. We can use the same process to deal with the later steps of our induction scheme. Actually we will encounter the same kind of system of equations at each later step of our induction scheme. Here we list some important equations. We assume that $\theta\in\mathbb{N}$ and the induction hypothesis has been satisfied. Then lengthy but direct calculation shows that

$$\begin{aligned}
\mathbb{B}_{\theta+2}=&\frac{n\cdot e(L^*)^{(-1+n)}\wedge\omega_M^{(-2+m)}\wedge i\bar{\partial}_M\partial_M\phi_\theta}{(-2+m)!\cdot n!}\wedge\\
&\left[n\cdot\Pi_{\mathcal{C}_M}\circ e(L^*)+\check{\pi}^*\text{trace}\left(\frac{i}{2\pi}F_{H_E}\right)\right]+\\
&\frac{k^m}{k\cdot k}\cdot\frac{(-1+n)\cdot(\Pi_{\mathcal{C}_M}\circ e(L^*))\wedge e(L^*)^{(-2+n)}\wedge\omega_M^{(-1+m)}}{(-1+m)!\cdot(-1+n)!}\wedge\\
&i\bar{\partial}_V\partial_V\left[\frac{n\cdot(-1+n)}{(-1+m+n)}\cdot\phi_{\theta+1}+\eta_{\theta+1}\right]+\\
&\frac{k^m}{k\cdot k}\cdot\frac{(-1+n)\cdot e(L^*)^{(-2+n)}\wedge\omega_M^{(-1+m)}}{(-1+m)!\cdot(-1+n)!}\wedge\check{\pi}^*\text{trace}\left(\frac{i}{2\pi}F_{H_E}\right)\wedge i\bar{\partial}_V\partial_V\phi_{\theta+1}+\\
&\frac{k^m}{k\cdot k}\cdot\frac{(-1+n)\cdot e(L^*)^{(-2+n)}\wedge\omega_M^{(-1+m)}}{(-1+m+n)\cdot(-1+m)!\cdot(-1+n)!}\wedge\check{\pi}^*\text{trace}\left(\frac{i}{2\pi}F_{H_M}\right)\wedge i\bar{\partial}_V\partial_V\phi_{\theta+1}+\\
&\frac{k^m}{k\cdot k}\cdot\frac{e(L^*)^{(-2+n)}\wedge\omega_M^m}{(-2+n)!\cdot m!}\wedge i\bar{\partial}_V\partial_V\left(-\frac{m\cdot n}{(-1+m+n)}\cdot\phi_{\theta+2}+\eta_{\theta+2}\right)+
\end{aligned}$$

$$\frac{k^m}{k\cdot k}\cdot\frac{n\cdot\left(\Pi_{\mathcal{C}_M}\circ e\left(L^*\right)\right)\wedge e\left(L^*\right)^{(-1+n)}\wedge\omega_M^{(-2+m)}}{(-2+m)!\cdot n!}\wedge$$
$$i\bar{\partial}_M\partial_M\left[\frac{(-1+n)\cdot n}{(-1+m+n)}\cdot\phi_\theta+\eta_\theta\right]+$$
$$\frac{k^m}{k\cdot k}\cdot\frac{\left[\frac{(-1+n)\cdot m\cdot n}{(-1+m)\cdot(-1+m+n)}+\frac{m\cdot n}{(-1+m+n)}\right]\cdot\frac{e(K_M)}{\omega_M}\cdot e\left(L^*\right)^{(-1+n)}\wedge\omega_M^{(-1+m)}}{(-2+m)!\cdot n!}\wedge$$
$$i\bar{\partial}_M\partial_M\phi_\theta+$$
$$\frac{k^m}{k\cdot k}\cdot\frac{n\cdot e\left(L^*\right)^{(-1+n)}\wedge\omega_M^{(-2+m)}}{(-2+m)!\cdot n!}\wedge i\bar{\partial}_M\partial_M\phi_\theta\wedge\breve{\pi}^*\mathrm{trace}\left(\frac{i}{2\pi}F_{H_M}\right)+$$
$$\frac{k^m}{k\cdot k}\cdot\frac{e\left(L^*\right)^{(-1+n)}\wedge\omega_M^{(-1+m)}}{(-1+m)!\cdot(-1+n)!}\wedge i\bar{\partial}_M\partial_M\left[\frac{n\cdot(-1+n)}{(-1+m+n)}\cdot\phi_{\theta+1}+\eta_{\theta+1}\right]+$$
$$\frac{k^m}{k\cdot k}\cdot\frac{n\cdot e\left(L^*\right)^{(-1+n)}\wedge\omega_M^{(-2+m)}}{(-2+m)!\cdot n!}\cdot i\bar{\partial}_M\partial_M\left[n\cdot\phi_\theta+\zeta_\theta\right]\wedge i\bar{\partial}_M\partial_M\phi_0+$$
$$\frac{k^m}{k\cdot k}\cdot\frac{n\cdot e\left(L^*\right)^{(-1+n)}\wedge\omega_M^{(-2+m)}}{(-2+m)!\cdot n!}\cdot i\bar{\partial}_M\partial_M\left[n\cdot\phi_0+\zeta_0\right]\wedge i\bar{\partial}_M\partial_M\phi_\theta+$$
$$\frac{k^m}{k\cdot k}\cdot\frac{(-2+n)\cdot e\left(L^*\right)^{(-3+n)}\wedge\omega_M^{m}}{(-2+n)!\cdot m!}\wedge i\bar{\partial}_V\partial_V\zeta_1\wedge i\bar{\partial}_V\partial_V\phi_{\theta+1}+$$
$$\frac{k^m}{k\cdot k}\cdot\frac{(-2+n)\cdot e\left(L^*\right)^{(-3+n)}\wedge\omega_M^{m}}{(-2+n)!\cdot m!}\wedge i\bar{\partial}_V\partial_V\zeta_{\theta+1}\wedge i\bar{\partial}_V\partial_V\phi_1+$$
$$\frac{k^m}{k\cdot k}\cdot\frac{(-1+n)\cdot e\left(L^*\right)^{(-2+n)}\wedge\omega_M^{(-1+m)}}{(-1+m)!\cdot(-1+n)!}\wedge i\bar{\partial}_V\partial_V\zeta_{\theta+1}\wedge i\bar{\partial}_M\partial_M\phi_0+$$
$$\frac{k^m}{k\cdot k}\cdot\frac{(-1+n)\cdot e\left(L^*\right)^{(-2+n)}\wedge\omega_M^{(-1+m)}}{(-1+m)!\cdot(-1+n)!}\wedge i\bar{\partial}_V\partial_V\phi_{\theta+1}\wedge i\bar{\partial}_M\partial_M[n\cdot\phi_0+\zeta_0]+$$
$$\frac{k^m}{k\cdot k}\cdot\frac{(-1+n)\cdot e\left(L^*\right)^{(-2+n)}\wedge\omega_M^{(-1+m)}}{(-1+m)!\cdot(-1+n)!}\wedge i\bar{\partial}_M\partial_M\left[n\cdot\phi_\theta+\zeta_\theta\right]\wedge i\bar{\partial}_V\partial_V\phi_1+$$
$$\frac{k^m}{k\cdot k}\cdot\frac{(-1+n)\cdot e\left(L^*\right)^{(-2+n)}\wedge\omega_M^{(-1+m)}}{(-1+m)!\cdot(-1+n)!}\wedge i\bar{\partial}_M\partial_M\phi_\theta\wedge i\bar{\partial}_V\partial_V\zeta_1+$$
$$\text{known terms.}$$

Thus at the θ-th step of our induction scheme we have the following analogue of (5):

$$0=(-1+m)\cdot e\left(L^*\right)^{(-1+n)}\wedge\omega_M^{(-2+m)}\wedge i\bar{\partial}_M\partial_M\hat{\sigma}\left(\phi_\theta\right)\wedge$$
$$\left[n\cdot\left(\Pi_{\mathcal{C}_M}\circ e\left(L^*\right)\right)+\breve{\pi}^*\mathrm{trace}\left(\frac{i}{2\pi}F_{H_E}\right)\right]+$$

$$\frac{(-1+n)}{(-1+m+n)} \cdot e\left(L^*\right)^{(-2+n)} \wedge \check{\pi}^* \text{trace}\left(\frac{i}{2\pi} F_{H_M}\right) \wedge \omega_M^{(-1+m)} \wedge i\bar{\partial}_V \partial_V \phi_{\theta+1} +$$
$$\frac{(-1+n)}{m} \cdot e\left(L^*\right)^{(-2+n)} \wedge \omega_M^m \wedge i\bar{\partial}_V \partial_V \zeta_{\theta+2} +$$
$$e\left(L^*\right)^{(-1+n)} \wedge \omega_M^{(-1+m)} \wedge \left[n \cdot i\bar{\partial}_M \partial_M \phi_{\theta+1} + i\bar{\partial}_M \partial_M \zeta_{\theta+1}\right] +$$
$$(-1+m) \cdot e\left(L^*\right)^{(-1+n)} \wedge \omega_M^{(-2+m)} \wedge \check{\pi}^* \text{trace}\left(\frac{i}{2\pi} F_{H_M}\right) \wedge i\bar{\partial}_M \partial_M \hat{\sigma}\left(\phi_\theta\right) +$$
$$\frac{m \cdot (-2+m+n)}{(-1+m+n)} \cdot \frac{e\left(K_M\right)}{\omega_M} \cdot e\left(L^*\right)^{(-1+n)} \wedge \omega_M^{(-1+m)} \wedge i\bar{\partial}_M \partial_M \hat{\sigma}\left(\phi_\theta\right)$$
$$+ \text{“known terms”},$$

with the “known terms” in this equation satisfying the compatibility condition

$$\int_{\mathbb{P}(E)} \text{“known terms”} = 0.$$

Note that this condition follows simply from cohomological consideration and our induction hypothesis that

$$\mathbb{B}_0 = \cdots = \mathbb{B}_{\theta+1} = 0.$$

Besides we have the following analogue of (4):

$$\mathbb{A}_{\theta+1} = \left(\frac{1}{2}\right) \left(-4\pi \cdot \zeta_{\theta+1} + \Delta_M \phi_\theta + \frac{4\pi \cdot m}{(-1+m+n)} \cdot \frac{e\left(K_M\right)}{\omega_M} \cdot \phi_\theta + \right.$$
$$\left. \left(-4\pi \cdot n \cdot \phi_{\theta+1} + \Delta_V \phi_{\theta+1}\right)\right) + \text{known terms},$$

via simple calculation.

Now we suppose that $\hat{\sigma}\left(\phi_\theta\right)$ and $n \cdot \hat{\sigma}\left(\phi_{\theta+1}\right) + \hat{\sigma}\left(\zeta_{\theta+1}\right)$ have been determined uniquely (so that ϕ_θ and η_θ have been completely determined). Then at the θ-th step of our induction scheme we have the following analogue of (10):

$$\mathbb{A}_{\theta+2} =$$
$$(1/2)\left(\Delta_M \phi_{\theta+1} + \Delta_V \phi_{\theta+2} + Q_V\left(\sigma\left(\phi_{\theta+1}\right) : \sigma\left(\phi_1\right)\right) + Q_V\left(\sigma\left(\phi_1\right) : \sigma\left(\phi_{\theta+1}\right)\right)\right) +$$
$$\frac{\Delta_V \phi_{\theta+1}}{2} \cdot \frac{\Delta_M \phi_0}{2} + -2\pi \cdot \left(n \cdot \phi_{\theta+2} + \zeta_{\theta+2} + \frac{m}{(-1+m+n)} \cdot \frac{-e\left(K_M\right)}{\omega_M} \cdot \phi_{\theta+1}\right) +$$
$$-2\pi \cdot \left[n \cdot \phi_{\theta+1} + \zeta_{\theta+1} + \frac{m}{(-1+m+n)} \cdot \frac{-e\left(K_M\right)}{\omega_M} \cdot \phi_\theta\right] \times$$
$$2\pi \cdot \left[n \cdot \phi_1 + \zeta_1 + \frac{m}{(-1+m+n)} \cdot \frac{-e\left(K_M\right)}{\omega_M} \cdot \phi_0\right] +$$
$$\text{known terms.}$$

Thus to determine $\sigma\left(\phi_{\theta+1}\right)$ we must use the infinitesimal version of Proposition A (instead of Proposition A itself).

With these equations listed the reader should have no difficulity in filling in the details. Thus the system of equations ($\check{\mathrm{A}}$) and ($\check{\mathrm{B}}$) can be solved inductively. $\square$

Though this partial proof of Theorem A does not ensure the complete validity of Theorem A it does ensure the following result:

Corollary A. *Assume that there is no nontrivial infinitesimal deformation of Kähler forms on M with constant Hermitian scalar curvature. Then for each $q \geq 0$ and each large $N \in \mathbb{N}$, there exist constants $k(q,N) \in \mathbb{N}$ and $C(q,N) > 0$ such that for $k \geq k(q,N)$, we have*

$$\left\| -(-1+m+n)\cdot c_k \cdot \frac{{}_N\omega_{[k]}^{(-1+m+n)}}{(-1+m+n)!} + {}_N\mathcal{E}_{[k]} \wedge \frac{{}_N\omega_{[k]}^{(-2+m+n)}}{(-2+m+n)!} \right\|_{\mathcal{C}^q(\mathbb{P}(E),\breve{\omega})} \leq C(q,N)\cdot k^{-N+m}.$$

Here ${}_N\mathcal{E}_{[k]}$ is the representative of $-\mathrm{e}(K)$ induced by the Kähler form

$${}_N\omega_{[k]} \equiv {}_o\omega_{[k]} + i\bar{\partial}\partial\phi_0 + \sum_{\theta=1}^{N} \frac{i\bar{\partial}\partial\phi_\theta}{k^\theta},$$

on $\mathbb{P}(E)$ with each ϕ_θ being constructed via the induction scheme. $\|\bullet\|_{\mathcal{C}^q(\mathbb{P}(E),\breve{\omega})}$ is the $\mathcal{C}^q$-norm of $\bullet$ with respect to the Hermitian form (metric) $\breve{\omega}$ on $\mathbb{P}(E)$.

This result will be used, together with certain *apriori* estimates for the constant Hermitian scalar curvature equations in [H], to show that a family of genuine solutions of (S) can be constructed via the Contraction Mapping Theorem (Implicit Function Theorem). Theorem A will then follow as an immediate corollary.

Note added in proofs. Theorem A should be stated more precisely as follows: "Assume that there is no nontrivial infinitesimal deformation of Kähler forms in the Kähler class $[\omega_M]$ on M with constant Hermitian scalar curvature ... ." Corollary C should be modified similarly. Also, Claude LeBrun has recently brought his work "*Polarized 4-Manifolds, Extremal Kähler Metrics, and Seiberg-Witten Theory*" on ruled surfaces to the author's attention. This work appeared in Mathematical Research Letters **2** (1995), 653–662.

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