Mathematical Research Letters **5**, 675–683 (1998)

CONSTRUCTIBLE SHEAVES ON SIMPLICIAL COMPLEXES AND KOSZUL DUALITY

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Abstract. We obtain a linear algebra data presentation of the category $\mathcal{SH}_c(X,\delta)$ of sheaves constant along the perverse simplices on a finite simplicial complex X. We also establish Koszul duality between $\mathcal{SH}_c(X,\delta)$ and the category $\mathcal{M}_c(X,\delta)$ of perverse sheaves constructible with respect to the triangulation.

Introduction

Let X be a finite simplicial complex. There is a well known linear algebra data description of (constructible with respect to the triangulation) sheaves of vector spaces on X. A sheaf corresponds to a gadget (called simplicial sheaf) which assigns vector spaces to simplices and linear maps to pairs of incident simplices. In this paper we present two generalizations of this result to the case of an arbitrary perversity.

Given a perversity (i.e., some function from integers to integers) one can construct certain subsets of X called perverse simplices. The first version of such subsets was introduced in [GM80]. One could say that the role of perverse simplices in the intersection homology theory is the same as the role of usual simplices in the usual homology theory.

Let us fix a perversity δ. One can consider two categories: (1) the category of perverse sheaves (homologically) constructible with respect to the triangulation, and (2) the category of sheaves constant along the perverse simplices (δ-sheaves). The linear algebra data description of the first category was obtained by R. MacPherson in [Mac93, Mac94]. Our main result (Theorem B) gives a linear algebra data description of the category of δ-sheaves. Our linear algebra gadgets assign vector spaces to perverse simplices and linear maps to pairs of "incident" perverse simplices. The category of linear algebra data in the first (resp. second) case is denoted by $\mathcal{P}(X,\delta)$ (resp. $\mathcal{R}(X,\delta)$).

It is useful to identify an abelian category with the category of modules over some underlying algebra. In some important cases underlying algebras turn out to be Koszul. Koszulity was established for algebras underlying perverse sheaves (middle perversity) on certain algebraic varieties in [BGSo96] and [PS95]. The Koszulity of the algebras underlying $\mathcal{P}(X,\delta)$ and $\mathcal{R}(X,\delta)$ was conjectured by

Received March 19, 1998.
The author is an Alfred P. Sloan Doctoral Dissertation Fellow.

the author and established by A. Polishchuk in [Pol97]. Moreover, it was shown in [Vyb97] that the categories $\mathcal{P}(X,\delta)$ and $\mathcal{R}(X,\delta)$ are (in some sense) Koszul dual to each other. This implies the Koszul duality between the category of perverse sheaves and the category of δ-sheaves. Therefore, we could say that the two categories carry the same amount of information about the topology of X.

The category of δ-sheaves should be considered for more complex stratified objects and it may prove to be as important for the study of singular spaces as the corresponding category of perverse sheaves. The principal technical advantage of considering δ-sheaves vs. perverse sheaves is that we do not have to deal with the derived category explicitly. The price we have to pay for this is more complicated topology.

The paper is organized as follows. In part 1 we recall some notions from [GM80], [Mac93], [Vyb97], and [BBD82]. In part 2 we state the theorems. In part 3 we outline the proof of our main theorem. The complete proof, as well as the relationship between Koszul duality (for sheaves and cosheaves) and Verdier duality, will appear in a subsequent paper. Needless to say, we assume the complete responsibility for all possible errors whenever we refer to unpublished results.

Throughout the paper $\mathbb{F}$ stands for a (commutative) field, $\operatorname{char}\mathbb{F} = 0$. All vector spaces are considered to be over $\mathbb{F}$.

1. Preliminaries

1.1. Perversities and perverse skeleta.

In this paper X is a finite connected simplicial complex. (For the definition, see [KS90, 8.1.1].) We assume that $\dim X = n$. If two simplices Δ, and Δ' are incident, we write $\Delta \leftrightarrow \Delta'$. The first barycentric subdivision of X is denoted by $\hat{X}$.

Definition 1.1.1([Mac93]). A *perversity* $\delta : \mathbb{Z}_{\geq 0} \to \mathbb{Z}$ is a function from the non-negative integers $\mathbb{Z}_{\geq 0}$ to the integers $\mathbb{Z}$ such that $\delta(0) = 0$ and δ takes every interval $\{0, 1, \dots, k\} \subset \mathbb{Z}_{\geq 0}$ bijectively to an interval $\{a, a+1, \dots, a+k\} \subset \mathbb{Z}$ for some $a \in \mathbb{Z}_{\leq 0}$. In other words, a perversity is such a function δ that $\delta(0) = 0$, and for $k \in \mathbb{Z}_{\geq 0}$,

$$\delta(k) = \begin{cases} \text{either} & \max\limits_{i\in[0,k-1]} \delta(i) + 1, \\ \text{or} & \min\limits_{i\in[0,k-1]} \delta(i) - 1. \end{cases}$$

Definition 1.1.2([Mac93], cf. [GM80]). Given a perversity δ, we define $\delta(\Delta) = \delta(\dim\Delta)$, where Δ is a simplex of X. Given a simplex $\hat{\Delta} = \{c_0, c_1, \dots, c_s\}$ of $\hat{X}$ (where $\{c_0, c_1, \dots, c_s\}$ are barycenters of $\Delta_1, \Delta_2, \dots, \Delta_s$), we denote by $\max\hat{\Delta}$ such vertex c_i that $\delta(\Delta_i) = \max\{\delta(\Delta_0), \delta(\Delta_1), \dots, \delta(\Delta_s)\}$. Given a simplex Δ with the barycenter c we define the corresponding *perverse* *simplex*:

$$^{\delta}\Delta = \bigsqcup_{\max \hat{\Delta} = c} \hat{\Delta}.$$

We define the *k-th perverse skeleton* X^{δ}_k, $\min_{[0,n]} \delta \leq k \leq \max_{[0,n]} \delta$, as follows:

$$X^{\delta}_k = \bigsqcup_{\delta(\Delta) \leq k} {}^{\delta}\Delta \subset X.$$

Thus, we have a filtration:

$$X^{\delta}_i \subset X^{\delta}_{i+1} \subset \ldots \subset X^{\delta}_{i+n-1} \subset X^{\delta}_{i+n} = X,$$

where $i = \min_{[0,n]} \delta$. It is easy to see that perverse simplices are connected components of $X^{\delta}_k - X^{\delta}_{k-1}$, $\min_{[0,n]} \delta \leq k \leq \max_{[0,n]} \delta$. The decomposition of X into a disjoint union of perverse simplices is called *δ-perverse triangulation* of X.

From now on we will fix a perversity δ.

1.2. Quiver algebras and categories of their modules.

In this section we recall some definitions from [Vyb97].

Definition 1.2.1. $\mathrm{Q}(X,\delta)$ is a quiver (i.e. finite simple oriented tree) whose vertices are indexed by simplices of X, and there is an arrow from Δ to Δ' if and only if $\delta(\Delta) = \delta(\Delta') + 1$ and $\Delta \leftrightarrow \Delta'$. There is a standard construction of the *quiver algebra* $\mathbb{F}\,\mathrm{Q}(X,\delta)$ associated to $\mathrm{Q}(X,\delta)$.

Definition 1.2.2. The algebra $A(X,\delta)$ is the quotient of $\mathbb{F}\,\mathrm{Q}(X,\delta)$ by the "chain complex" relations:

If Δ' is any simplex such that $\delta(\Delta') = k+1$ and Δ'' is any simplex such that $\delta(\Delta'') = k-1$, then

$$\sum_{\substack{\Delta\,:\,\delta(\Delta)=k,\\ \Delta' \leftrightarrow \Delta \leftrightarrow \Delta''}} a(\Delta,\Delta'') \cdot a(\Delta',\Delta) = 0,$$

where $a(\Delta,\Delta'')$, $a(\Delta',\Delta)$ are the generators of $A(X,\delta)$.

Definition 1.2.3. The algebra $B(X,\delta)$ is the quotient of $\mathbb{F}\,\mathrm{Q}(X,\delta)$ by the "equivalence" relations:

Suppose that Δ', Δ'', Δ_1 and Δ_2 are simplices of X such that:

(1) $\delta(\Delta') = k+1$, $\delta(\Delta'') = k-1$ and $\delta(\Delta_1) = \delta(\Delta_2) = k$,
(2) $\Delta' \leftrightarrow \Delta_1 \leftrightarrow \Delta''$, and $\Delta' \leftrightarrow \Delta_2 \leftrightarrow \Delta''$.

Then

$$b(\Delta_1, \Delta'') \cdot b(\Delta', \Delta_1) = b(\Delta_2, \Delta'') \cdot b(\Delta', \Delta_2),$$

where $b(\Delta_1, \Delta'')$, $b(\Delta', \Delta_1)$, $b(\Delta_2, \Delta'')$, and $b(\Delta', \Delta_2)$ are the generators of $B(X, \delta)$.

The category of left finite dimensional modules over $A(X, \delta)$ will be denoted by $\mathcal{P}(X, \delta)$ and the category of left finite dimensional modules over $B(X, \delta)$ will be denoted by $\mathcal{R}(X, \delta)$. The category $\mathcal{P}(X, \delta)$ (with an extra axiom, for an arbitrary finite regular cell complex) was introduced by R. MacPherson in [Mac93]. It is called the category of *simplicial perverse sheaves*. When $\delta(k) = -k$ for $k \geq 0$, is the bottom perversity, we denote the corresponding categories by $\mathcal{P}(X)$ and $\mathcal{R}(X)$.

1.3. Sheaves.

Definition 1.3.1. A sheaf $\mathbf{A}$ of $\mathbb{F}$-vector spaces is called *constant along the* δ-*perverse simplices* if for all simplices Δ, $i^*_\Delta \mathbf{A}$ is a constant sheaf on ${}^\delta\Delta$ associated to a finite dimensional vector space over $\mathbb{F}$, where $i_\Delta : {}^\delta\Delta \hookrightarrow X$ is the inclusion of the corresponding perverse simplex ${}^\delta\Delta$.

We will denote the category of all sheaves constant along the $(-\delta)$-perverse simplices (δ-sheaves) by $\mathcal{SH}_c(X, \delta)$. In the case of the bottom perversity, $(-\delta)$-perverse simplices become usual simplices and the corresponding category of sheaves is denoted by $\mathcal{SH}_c(X)$.

Definition 1.3.2(cf. [GM80], [BBD82]). A *"classical" perversity* $p : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\leq 0}$ is a function such that $p(0) = 0$, $0 \leq p(m) - p(n) \leq n - m$ for $m \leq n$.

It is easy to see that there is a one-to-one correspondence between the set of all perversities and the set of all "classical" perversities (cf. [Pol97, section 1]). Let p be a classical perversity corresponding to our fixed perversity δ. We will denote by $\mathcal{M}_c(X, \delta)$ the category of (homologically) constructible with respect to the triangulation *p-perverse sheaves* of vector spaces introduced in [BBD82] (cf. [GM83]).

2. Theorems

Theorem A. *The following categories,*

$$\mathcal{R}(X) \simeq \mathcal{SH}_c(X)$$

are equivalent.

This result is rather well known. A version of it was presented in [Mac93], another version is offered as Exercise VIII.1 in [KS90]. A detailed discussion will appear in [GMMV]. The primary goal of this paper is to generalize this result to the case of an arbitrary perversity.

Theorem B. *The following categories,*

$$\mathcal{R}(X,\delta) \simeq \mathcal{SH}_c(X,\delta)$$

are equivalent.

The proof is outlined in part 3.

The following result was obtained by R. MacPherson ([Mac93, Mac95]). It may be considered as another generalization of the Theorem A.

Theorem C. *The following categories,*

$$\mathcal{P}(X,\delta) \simeq \mathcal{M}_c(X,\delta),$$

are equivalent.

Proof. It follows from [Pol97, section 4] that $\mathcal{M}_c(X,\delta)$ is equivalent to the category of modules over the quadratic dual to $B(X,\delta)^{\mathrm{opp}}$. Such a quadratic dual algebra is exactly $A(X,\delta)$. □

Theorem D (cf. [Pol97], [Vyb97]).
(a) The category $\mathcal{SH}_c(X,\delta)$ is Koszul.
(b) The category $\mathcal{M}_c(X,\delta)$ is Koszul.
(c) There exists a functor

$$K : D^b(\mathcal{M}_c(X,\delta)) \to D^b(\mathcal{SH}_c(X,-\delta)),$$

which is an equivalence of triangulated categories.

Proof. The proof of the fact that $B(X,\delta)$ is Koszul is given in section 3 of [Pol97]. Then (a) is implied by Theorem B. Since the quadratic dual $A(X,\delta)^! = B(X,-\delta)$ it follows that $A(X,\delta)$ is Koszul as the quadratic dual of a Koszul algebra. Therefore (b) is implied by Theorem C. Quadratic duality of $A(X,\delta)$ and $B(X,-\delta)$ and the fact that $B(X,\delta) = B(X,-\delta)^{\mathrm{opp}}$ implies that $E(A(X,\delta)) = B(X,\delta)$ by Theorem 2.10.1 of [BGSo96], where $E(A(X,\delta))$ is the Koszul dual of $A(X,\delta)$. Using the construction of [BGSo96], it is not hard to see (cf. [Vyb97]) that the Koszul duality functor restricts to the equivalence functor $K : D^b(\mathcal{P}(X,\delta)) \to D^b(\mathcal{R}(X,-\delta))$. Thus, (c) follows from Theorem B and Theorem C. □

Remarks. 1. Note that the Koszul duality functor transforms simple objects in $\mathcal{M}_c(X,\delta)$ (intersection (co)homology sheaves on closed simplices) to indecomposable projective objects in $\mathcal{SH}_c(X,-\delta)$ (constant sheaves on certain subsets of X).

2. For the general Koszul theory we refer the reader to [BGSc88].

3. Proof of theorem B

3.1. Linear algebra data.

We will redefine the category $\mathcal{R}(X,\delta)$ in terms of linear algebra data. We leave it to the reader to check that the two definitions coincide.

Definition 3.1.1. An object $\mathbf{S}$ of $\mathcal{R}(X, \delta)$ is the following data:

(1) (Stalks) For every simplex Δ in X, a finite dimensional vector space $S(\Delta)$ called the *stalk* of $\mathbf{S}$ at Δ,
(2) (Restriction maps) For every pair of simplices Δ and Δ' in X such that $\delta(\Delta) = \delta(\Delta') + 1$, and $\Delta \leftrightarrow \Delta'$, a linear map $s(\Delta, \Delta') : S(\Delta) \to S(\Delta')$ called the *restriction map*,

subject to the following "equivalence axiom":

Suppose that Δ', Δ'', Δ_1 and Δ_2 are simplices of X such that:

(1) $\delta(\Delta') = k + 1$, $\delta(\Delta'') = k - 1$, and $\delta(\Delta_1) = \delta(\Delta_2) = k$,
(2) $\Delta' \leftrightarrow \Delta_1 \leftrightarrow \Delta''$, and $\Delta' \leftrightarrow \Delta_2 \leftrightarrow \Delta''$.

Then

$$s(\Delta_1, \Delta'') \circ s(\Delta', \Delta_1) = s(\Delta_2, \Delta'') \circ s(\Delta', \Delta_2).$$

The morphisms in this category are stalkwise linear maps commuting with the restriction maps.

We now define another category $\mathcal{S}(X, \delta)$ of linear algebra data.

Definition 3.1.2. The category $\mathcal{S}(X, \delta)$ is a full subcategory of the abelian category $\mathcal{R}(\hat{X})$. An object $\mathbf{T}$ of $\mathcal{R}(\hat{X})$ belongs to $\mathcal{S}(X, \delta)$ if for any $\hat{\Delta}, \hat{\Delta}' \subseteq {}^{-\delta}\Delta$ we have:

(1) $T(\hat{\Delta}) = T(\hat{\Delta}')$,
(2) if $\hat{\Delta}$ is a codim 1 face of $\hat{\Delta}'$ then $t(\hat{\Delta}, \hat{\Delta}') = \mathrm{Id}_{T(\hat{\Delta})}$.

3.2. $\mathcal{R}(X, \delta) = \mathcal{S}(X, \delta)$.

Definition 3.2.1. We will consider a partial order on the set of simplices of X, which is defined as follows: $\Delta \geq \Delta'$ if there exists a sequence of simplices $\Delta = \Delta_0, \Delta_1, \Delta_2, \ldots, \Delta_r = \Delta'$, $r \geq 0$, such that for $i \geq 0$,

(1) Δ_i and Δ_{i+1} are incident,
(2) $\delta(\Delta_i) = \delta(\Delta_{i+1}) + 1$.

It is easy to see that if $\Delta \leftrightarrow \Delta'$ and $\delta(\Delta) \geq \delta(\Delta')$, then $\Delta \geq \Delta'$.

Definition 3.2.2. Let $\Delta \geq \Delta'$ be two simplices of X. Let $\mathbf{S}$ be an object of $\mathcal{R}(X, \delta)$. We define a linear map $s(\Delta, \Delta') : S(\Delta) \to S(\Delta')$ in the following way:

(1) if $\Delta = \Delta'$ then $s(\Delta, \Delta') = \mathrm{Id}_{S(\Delta)}$,
(2) if $\Delta > \Delta'$ then by Definition 3.2.1 there exists a sequence of simplices $\Delta = \Delta_0, \Delta_1, \Delta_2, \ldots, \Delta_r = \Delta'$. We set $s(\Delta, \Delta') = s(\Delta_{r-1}, \Delta_r) \circ \cdots \circ s(\Delta_0, \Delta_1)$.

It is easy to see that $s(\Delta, \Delta')$ does not depend on the choice of the sequence. Moreover, if $\Delta_1 \geq \Delta_2 \geq \Delta_3$, then

$$s(\Delta_1, \Delta_3) = s(\Delta_2, \Delta_3) \circ s(\Delta_1, \Delta_2). \tag{3.2.2}$$

Lemma 3.2.3. *Let $\hat{\Delta}, \hat{\Delta}'$ of $\hat{X}$ be such that $\hat{\Delta}$ is a* codim 1 *face of $\hat{\Delta}'$. Let $\hat{\Delta} \subseteq {}^{-\delta}\Delta$, $\hat{\Delta}' \subseteq {}^{-\delta}\Delta'$. Then $\Delta \leftrightarrow \Delta'$ and $\Delta \geq \Delta'$.*

Theorem 3.2.4. *The category $\mathcal{R}(X, \delta)$ is isomorphic to the category $\mathcal{S}(X, \delta)$:*

$$\mathcal{R}(X, \delta) = \mathcal{S}(X, \delta).$$

Proof. (a) The functor $\Phi : \mathcal{R}(X, \delta) \to \mathcal{S}(X, \delta)$. If $\mathbf{S}$ is an object of $\mathcal{R}(X, \delta)$ then $\mathbf{T} = \Phi(\mathbf{S})$ is constructed as follows:

$$T(\hat{\Delta}) = S(\Delta) \text{ for } \hat{\Delta} \subseteq {}^{-\delta}\Delta.$$

If $\hat{\Delta}'$ is a codim 1 face of $\hat{\Delta}''$ then:

$$t(\hat{\Delta}', \hat{\Delta}'') = s(\Delta', \Delta'') \text{ for } \hat{\Delta}' \subseteq {}^{-\delta}\Delta' \text{ and } \hat{\Delta}'' \subseteq {}^{-\delta}\Delta''.$$

(By Lemma 3.2.3 it follows that $\Delta' \geq \Delta''$.) The equivalence axiom is implied by Lemma 3.2.3 and (3.2.2). Φ on morphisms is defined in the obvious way.

(b) The functor $\Psi : \mathcal{S}(X, \delta) \to \mathcal{R}(X, \delta)$. If $\mathbf{T}$ is an object of $\mathcal{S}(X, \delta)$ then $\mathbf{S} = \Psi(\mathbf{T})$ is constructed as follows:

$$S(\Delta) = T(\hat{\Delta}) \quad \text{for any} \quad \hat{\Delta} \subseteq {}^{-\delta}\Delta.$$

$S(\Delta)$ is well defined due to the definition of $\mathcal{S}(X, \delta)$. Let Δ' and Δ'' be two incident simplices of X such that $\delta(\Delta') = \delta(\Delta'') + 1$. We have to construct the map $s(\Delta', \Delta'')$. Let c' be a barycenter of Δ' and c'' be a barycenter of Δ''. We set:

$$s(\Delta', \Delta'') = t(c', \{c', c''\}),$$

where $\{c', c''\}$ is a simplex of $\hat{X}$. The equivalence axiom follows from the definitions. Ψ on morphisms is defined in the obvious way.

(c) It follows from our explicit construction that $\Psi \circ \Phi = \mathrm{Id}$. Using the definitions and (3.2.2) it is easy to see that $\Phi \circ \Psi = \mathrm{Id}$. □

3.3. $\mathcal{S}(X, \delta) \simeq \mathcal{SH}_c(X, \delta)$**.**

Definition 3.3.1. Let $\mathbf{S}$ be an object of $\mathcal{R}(\hat{X})$. Let $i_A : A \hookrightarrow X$ be the inclusion of a closed union of simplices of $\hat{X}$. We define a functor $i_A^* : \mathcal{R}(\hat{X}) \to \mathcal{R}(A)$ as follows. If $\mathbf{T} = i_A^*\mathbf{S}$ then:

(1) $\mathbf{T}(\hat{\Delta}) = \mathbf{S}(\hat{\Delta})$ for $\hat{\Delta} \subseteq A$,
(2) $t(\hat{\Delta}', \hat{\Delta}'') = s(\hat{\Delta}', \hat{\Delta}'')$ for $\hat{\Delta}', \hat{\Delta}'' \subseteq A$.

Lemma 3.3.2. *The following functorial diagram commutes (i.e. two possible compositions of functors are isomorphic):*

$$\begin{array}{ccc} \mathcal{R}(\hat{X}) & \xleftarrow{\sim} & \mathcal{SH}_c(\hat{X}) \\ {\scriptstyle i_A^*}\downarrow & & {\scriptstyle i_A^*}\downarrow \\ \mathcal{R}(A) & \xleftarrow{\sim} & \mathcal{SH}_c(A). \end{array}$$

Here $i_A^* : \mathcal{SH}_c(\hat{X}) \to \mathcal{SH}_c(A)$ *is the standard sheaf theory functor.*

Theorem 3.3.3. *The category* $\mathcal{S}(X, \delta)$ *is equivalent to the category* $\mathcal{SH}_c(X, \delta)$*:*

$$\mathcal{S}(X, \delta) \simeq \mathcal{SH}_c(X, \delta).$$

Proof. By definitions and functoriality of i^* the category $\mathcal{SH}_c(X, \delta)$ is a full subcategory of $\mathcal{SH}_c(\hat{X})$. Lemma 3.3.2 implies that the equivalence functor $\mathcal{R}(\hat{X}) \simeq \mathcal{SH}_c(\hat{X})$ restricts to an equivalence functor $\mathcal{S}(X, \delta) \simeq \mathcal{SH}_c(X, \delta)$. $\square$

Theorem 3.2.4 and Theorem 3.3.3 imply Theorem B.

Acknowledgments

I am very grateful to Mark Goresky for numerous conversations and steady encouragement, and to Robert MacPherson, who invented the subject of cellular perverse sheaves, and from whom I learned about it. I am also grateful to the Alfred P. Sloan Foundation for financial support.

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