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ON THE EXPONENT OF FINITE-DIMENSIONAL HOPF ALGEBRAS

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1. Introduction

One of the classical notions of group theory is the notion of the exponent of a group. The exponent of a group is the least common multiple of orders of its elements.

In this paper we generalize the notion of exponent to Hopf algebras. We define the exponent of a Hopf algebra H (with bijective antipode) to be the smallest n such that $m_n \circ (I \otimes S^{-2} \otimes \cdots \otimes S^{-2n+2}) \circ \Delta_n = \varepsilon \cdot 1$, where m_n, Δ_n, S, 1, and ε are the iterated product and coproduct, the antipode, the unit, and the counit. If H is involutive (for example, H is semisimple and cosemisimple), the last formula reduces to $m_n \circ \Delta_n = \varepsilon \cdot 1$.

We give four other equivalent definitions of the exponent (valid for finite-dimensional Hopf algebras). In particular, we show that the exponent of H equals the order of the Drinfeld element u of the quantum double $D(H)$, and the order of $\mathcal{R}_{21}\mathcal{R}$, where $\mathcal{R}$ is the universal R-matrix of $D(H)$.

We show that the exponent is invariant under twisting. We prove that for semisimple and cosemisimple Hopf algebras H, the exponent is finite and divides $\dim(H)^3$. For triangular semisimple Hopf algebras in characteristic zero, we show that the exponent divides $\dim(H)$. These theorems are motivated by the work of Kashina [Ka1,Ka2], who conjectured that if H is semisimple and cosemisimple then (using our language) the exponent of H is always finite and divides $\dim(H)$, and showed that the order of the squared antipode of any finite-dimensional semisimple and cosemisimple Hopf algebra in the Yetter-Drinfeld category of H divides the exponent of H.

At the end we formulate some open questions, in particular suggest a formulation for a possible Hopf algebraic analogue of Sylow's theorem.

2. Definition and Elementary Properties of Exponent

Let H be a Hopf algebra over any field k, with multiplication map m, comultiplication map Δ and antipode S. We will always assume that S is bijective. Let $m_1 = I$ and $\Delta_1 = I$ be the identity map $H \to H$, and for any integer $n \geq 2$ let $m_n : H^{\otimes n} \to H$ and $\Delta_n : H \to H^{\otimes n}$ be defined by $m_n = m \circ (m_{n-1} \otimes I)$, and $\Delta_n = (\Delta_{n-1} \otimes I) \circ \Delta$. We start by making the following definition.

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Definition 2.1. *The exponent of a Hopf algebra H, denoted by* $\exp(H)$, *is the smallest positive integer n satisfying $m_n \circ (I \otimes S^{-2} \otimes \cdots \otimes S^{-2n+2}) \circ \Delta_n = \varepsilon \cdot 1$. If such n does not exist, we say that* $\exp(H) = \infty$.

Let us list some of the elementary properties of $\exp(H)$.

Proposition 2.2. *Let H be a Hopf algebra over k. Then:*

(1) *The order of any group-like element of H divides* $\exp(H)$ *(here we agree that any positive integer n divides ∞).*
(2) *For any group G,* $\exp(k[G])$ *equals the exponent of G (see e.g.* [Ro, p.12]*), i.e. the least common multiple of the orders of the elements of G.*
(3) *If* $\exp(H) = n < \infty$ *then $m_r \circ (I \otimes S^{-2} \otimes \cdots \otimes S^{-2r+2}) \circ \Delta_r = \varepsilon \cdot 1$ if and only if r is divisible by n.*
(4) *If H is finite-dimensional,* $\exp(H^*) = \exp(H)$.
(5) $\exp(H_1 \otimes H_2)$ *equals the least common multiple of* $\exp(H_1)$ *and* $\exp(H_2)$.
(6) *If* $\exp(H) = 2$ *then H is commutative and cocommutative (this generalizes the fact that a group G with $g^2 = 1$ for all $g \in G$ is abelian).*
(7) *The exponents of Hopf subalgebras and quotients of H divide* $\exp(H)$.
(8) *If $K \supseteq k$ is a field then* $\exp(H \otimes_k K) = \exp(H)$.

Proof. (1) Suppose $\exp(H) < \infty$, and set $n = \exp(H)$. Since $S^2(g) = g$ we have that $g^n = m_n \circ \Delta_n(g) = m_n \circ (I \otimes S^{-2} \otimes \cdots \otimes S^{-2n+2}) \circ \Delta_n(g) = \varepsilon(g)1 = 1$. Therefore the order of g divides n.
(6) Since $m \circ (I \otimes S^{-2}) \circ \Delta = \varepsilon \cdot 1$ is equivalent to $S^3 = I$, we have that $I : H \to H$ is an antiautomorphism of algebras and coalgebras, and the result follows.
The proofs of the other parts are obvious. □

Remark 2.3. *Part* (2) *of Proposition* 2.2 *motivated Definition* 2.1.

Example 2.4. *Let H be a finite-dimensional Hopf algebra over an algebraically closed field k of characteristic zero. Suppose that H contains a non-trivial $1 : g$ skew-primitive element x (i.e. $\Delta(x) = x \otimes 1 + g \otimes x$, where g is a group-like element, and $x \notin k[g]$). It is clear that in this case we may assume that $xg = qgx$ for some root of unity q of order dividing $|g|$. Also, $S^2(x) = qx$, $\varepsilon(x) = 0$, $\{x, gx, \ldots, g^{|g|-1}x\}$ is linearly independent, and hence*

$$m_n \circ (I \otimes S^{-2} \otimes \cdots \otimes S^{-2n+2}) \circ \Delta_n(x) = \sum_{i=0}^{n-1} q^{-i} g^i x \neq 0.$$

Hence, $\exp(H) = \infty$. *In particular, the exponent of any pointed Hopf algebra H over k (which is not a group algebra) is ∞, since by* [TW]*, H contains a non-trivial skew-primitive element.*

In the sequel, we will assume for simplicity that H is finite-dimensional. Let us formulate four equivalent definitions of $\exp(H)$. Recall that the Drinfeld double $D(H) = H^{*cop} \otimes H$ of H is a quasitriangular Hopf algebra with universal $R-$matrix $\mathcal{R} = \sum_i h_i \otimes h_i^*$, where $\{h_i\}$, $\{h_i^*\}$ are dual bases for H and H^* respectively. Let $u = m(S \otimes I)\tau(\mathcal{R}) = \sum_i S(h_i^*)h_i$, where S is the antipode of $D(H)$ and τ is the usual flip map, be the Drinfeld element of $D(H)$. By [D],

$$S^2(x) = uxu^{-1},\ x \in D(H) \ \text{ and } \ \Delta(u) = (u \otimes u)(\mathcal{R}_{21}\mathcal{R})^{-1}.$$

Theorem 2.5. *Let H be a finite-dimensional Hopf algebra over k. Then*

(1) $\exp(H)$ *equals the smallest positive integer n such that*

$$\mathcal{R}(I \otimes S^2)(\mathcal{R}) \cdots (I \otimes S^{2n-2})(\mathcal{R}) = 1.$$

(2) $\exp(H)$ *equals the order of u.*
(3) $\exp(H)$ *equals the order of $\mathcal{R}_{21}\mathcal{R}$.*
(4) $\exp(H)$ *equals the order of any non-zero element $v \in D(H)$ satisfying*

$$\Delta(v) = (v \otimes v)(\mathcal{R}_{21}\mathcal{R})^{-1}.$$

Proof. First note that since $(\Delta \otimes I)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}$, it follows that

$$(\Delta_n \otimes I)(\mathcal{R}) = \mathcal{R}_{1,n+1} \cdots \mathcal{R}_{n,n+1},$$

for all n.

Second, recall that the map $H^* \otimes H \to D(H)$, $p \otimes h \mapsto ph$ is a linear isomorphism [D].

Now we will show the equivalence of Definition 2.1 and the four definitions in the theorem.

(Definition 2.1 $\Leftrightarrow$ 1 $\Leftrightarrow$ 2) Write $\mathcal{R} = \sum_j a_j \otimes b_j$. Using the above we obtain the following equivalences:

$$m_n \circ (I \otimes S^{-2} \otimes \cdots \otimes S^{-2n+2}) \circ \Delta_n = \varepsilon \cdot 1 \iff$$

$$(m_n \circ (I \otimes S^{-2} \otimes \cdots \otimes S^{-2n+2}) \circ \Delta_n \otimes I)(\mathcal{R}) = 1 \iff$$

$$(m_n \circ (I \otimes S^{-2} \otimes \cdots \otimes S^{-2n+2}) \otimes I)(\mathcal{R}_{1,n+1} \cdots \mathcal{R}_{n,n+1}) = 1 \iff$$

$$\textstyle\sum_{i_1,\ldots,i_n} a_{i_1} S^{-2}(a_{i_2}) \cdots S^{-2n+2}(a_{i_n}) \otimes b_{i_1} \cdots b_{i_n} = 1 \iff$$

$$\mathcal{R}(I \otimes S^2)(\mathcal{R}) \cdots (I \otimes S^{2n-2})(\mathcal{R}) = 1 \iff$$

$$u^n = 1$$

(in the last step we applied $m \circ (I \otimes S)\tau$ to both sides of the equation, and used the fact that $uS^{-2}(x) = xu$, for all $x \in D(H)$).

(2 $\Leftrightarrow$ 3) Clearly if $u^n = 1$ then $(\mathcal{R}_{21}\mathcal{R})^n = 1$. In the other direction, first note that $(\mathcal{R}_{21}\mathcal{R})^n = 1$ implies that $u^n \in G(D(H))$ (where $G(A)$ is the group of

grouplike elements of a Hopf algebra A). Therefore by [R], $u^n = ab$ where $a \in G(H^*)$ and $b \in G(H)$. Regarding u as an element of $H^* \otimes H$, we have that $m(I \otimes \varepsilon)(u) = m(\varepsilon \otimes I)(u) = 1$. Hence it follows that $1 = m(I \otimes \varepsilon)(u^n) = a$ and $1 = m(\varepsilon \otimes I)(u^n) = b$, so $u^n = 1$.
($2 \Leftrightarrow 4$) First note that $v = ug$, where $g \in G(D(H))$. Since g commutes with u we have that $v^n = u^n g^n$. Therefore if $u^n = 1$ then $v^n = 1$ by parts 1 and 3 of Proposition 2.2, and if $v^n = 1$ then $u^n \in G(D(H))$, so $u^n = 1$ as explained above. □

Corollary 2.6. *Let H be a finite-dimensional Hopf algebra over k. Then*

$$\exp(H^{cop}) = \exp(H^{op}) = \exp(H).$$

Proof. Since $(D(H^{*cop}), \tilde{\mathcal{R}}) \cong (D(H)^{op}, \mathcal{R}_{21})$ as quasitriangular Hopf algebras, it follows from part 1 of Theorem 2.5 that $\exp(H^{*cop}) = \exp(H)$. Hence the result follows from part 4 of Proposition 2.2. □

3. Invariance of Exponent Under Twisting

In this section we show that $\exp(H)$ is invariant under twisting. First recall Drinfeld's notion of a twist for Hopf algebras.

Definition 3.1. *Let H be a Hopf algebra over k. A twist for H is an invertible element $J \in H \otimes H$ which satisfies*

$$(\Delta \otimes I)(J)J_{12} = (I \otimes \Delta)(J)J_{23} \ \text{and} \ (\varepsilon \otimes I)(J) = (I \otimes \varepsilon)(J) = 1.$$

Given a twist J for H, we can construct a new Hopf algebra H^J, which is the same as H as an algebra, with coproduct Δ^J given by

$$\Delta^J(x) = J^{-1}\Delta(x)J, \ x \in H.$$

If (H, R) is quasitriangular then so is H^J with the R-matrix

$$R^J = J_{21}^{-1}RJ.$$

In particular, since H is a Hopf subalgebra of $D(H)$, we can twist $D(H)$ using the twist $J \in D(H) \otimes D(H)$ and obtain $(D(H)^J, \mathcal{R}^J)$.

Proposition 3.2. *Let H be a finite-dimensional Hopf algebra over k, and let J be a twist for H. Then $(D(H^J), \mathcal{R}) \cong (D(H)^J, \mathcal{R}^J)$ as quasitriangular Hopf algebras.*

Proof. Let H_+ and H_- be the Hopf subalgebras of $D(H)^J$ generated by the left and right components of $\mathcal{R}^J$ respectively. Clearly, $H_+ \subseteq H^J$. In order to prove the theorem it is sufficient to prove that the multiplication map $H_+ \otimes H_- \to D(H)^J$ is a linear isomorphism, since then $H_+ = H^J$ (by dimension counting) and the result will follow.

Clearly, $\dim(H_+) \le \dim(H)$ and $\dim(H_-) = \dim(H_+)$, so we need to show that $H_+H_- = D(H)$. Since $J\mathcal{R}_{21}^J\mathcal{R}^J J^{-1} = \mathcal{R}_{21}\mathcal{R}$ we have that $HH_-H =$

$D(H)$ (looking at the first component). Let $A = H_+H_- = H_-H_+$, $\dim(H) = d$, $\dim(H_+) = d_+$, $\{v_1, \ldots, v_{d/d_+}\}$ with $v_1 = 1$ be a basis of H as a right H_+-module, and $\{w_1, \ldots, w_{d/d_+}\}$ with $w_1 = 1$ be a basis of H as a left H_+-module (such bases exist by the freeness theorem for Hopf algebras [NZ]). Then we get by dimension counting that $D(H) = \bigoplus_{i,j=1}^{d/d_+} v_iAw_j$. Thus, $HH_- \cap H_-H = A$, hence $H \subseteq A$ which implies that $HAH = A$, and the result follows. □

Theorem 3.3. *Let H be a finite-dimensional Hopf algebra over k, and let J be a twist for H. Then* $\exp(H) = \exp(H^J)$.

Proof. By part 3 of Theorem 2.5, and Proposition 3.2, it is sufficient to show that the order of $\mathcal{R}_{21}^J\mathcal{R}^J$ equals to the order of $\mathcal{R}_{21}\mathcal{R}$. But this is clear since they are conjugate. □

Corollary 3.4. *Let H be a finite-dimensional Hopf algebra over k. Then* $\exp(D(H)) = \exp(H)$.

Proof. By [RS], there exists $J \in D(H) \otimes D(H)$ such that

$$D(D(H)) \cong (D(H) \otimes D(H))^J$$

as Hopf algebras. Then using Theorem 3.3 we get that $\exp(D(H))$ equals the order of u in $(D(H) \otimes D(H))^J$ which equals the order of u in $D(H) \otimes D(H)$, and hence equals $\exp(H)$ (since $u_{D(H)\otimes D(H)} = u_{D(H)} \otimes u_{D(H)}$). □

4. The Exponent of a Semisimple and Cosemisimple Hopf Algebra

In this section, we will show that if H is semisimple and cosemisimple then $\exp(H)$ is finite, and give an estimate for it in terms of $\dim(H)$.

Let H be a semisimple and cosemisimple Hopf algebra over k (note that by [LR] the cosemisimplicity assumption is redundant if the characteristic of k is 0). Recall that for semisimple and cosemisimple H, $D(H)$ is also semisimple and cosemisimple [R]. Also, by [LR, EG2], $S^2 = I$ and hence u is central in $D(H)$. This implies that $\exp(H)$ equals the smallest positive integer n satisfying $m_n \circ \Delta_n = \varepsilon \cdot 1$, and also to the order of $\mathcal{R}$ (by part 1 of Theorem 2.5).

Remark 4.1. *In* [Ka1, Ka2] *Kashina studied the smallest positive integer n satisfying $m_n \circ \Delta_n = \varepsilon \cdot 1$, for any finite-dimensional Hopf algebra H. In particular she observed the analogous properties listed in Proposition* 2.2*, and proved an analogue to Corollary* 3.4 *under the assumption that this smallest n is the same for H and H^{cop}.*

Theorem 4.2. *Let (H, R) be a semisimple triangular Hopf algebra over a field k of characteristic* 0*. Then* $\exp(H)$ *divides* $\dim(H)$.

Proof. By part 8 of Proposition 2.2, we may assume that k is algebraically closed. Now, it is straightforward to check that the theorem holds for $(k[G], 1\otimes 1)$ where G is a finite group. But by [EG1, Theorem 2.1], there exist a finite group G and a twist $J \in k[G]\otimes k[G]$ such that $H \cong k[G]^J$ as Hopf algebras. Hence the result follows from Theorem 3.3. □

Theorem 4.3. *Let H be a semisimple and cosemisimple Hopf algebra over k. Then $\exp(H)$ divides $\dim(H)^3$.*

Theorem 4.3 will be proved later.

Corollary 4.4. *Let H be a semisimple and cosemisimple Hopf algebra over k, and let B be a finite-dimensional semisimple Hopf algebra in the category of Yetter-Drinfeld modules over H. Then the order of the antipode of B is finite and divides $2\dim(H)^3$, and if H is semisimple triangular and the characteristic of k is 0, then the order of the antipode of B is finite and divides $2\dim(H)$.*

Proof. Follows from Theorems 4.2 and 4.3, and [Ka1, Theorem 6]. □

Remark 4.5. *Theorem 4.3 is motivated by Vafa's theorem* [V]. *Vafa's theorem (see* [Ki] *for the mathematical exposition) states that the twists in a semisimple modular category act on the irreducible objects by multiplication by roots of unity. Thus, the fact that $u \in D(H)$ has a finite order follows from the fact that the category of representations of $D(H)$ is modular, with system of twists given by the action of the central element u (see e.g.* [EG1]*).*

Kashina conjectured the following:

Conjecture 4.6. *Let H be a semisimple and cosemisimple Hopf algebra over k. Then $\exp(H)$ is finite and divides $\dim(H)$.*

This conjecture was checked by Kashina in a number of special cases [Ka1, Ka2]. Our results presented above give a proof to the first part of the conjecture, and additional supportive evidence for its second part.

Now we will prove Theorem 4.3. In order to do this, we need a lemma.

Lemma 4.7. *Let H be a Hopf algebra of finite dimension d over k, $\mathcal{R} \in H \otimes H^{*cop} \subset D(H)\otimes D(H)$ be the universal R-matrix, and $u \in D(H)$ be the Drinfeld element. Then:*

1. *For any finite-dimensional $H-$module V_+ and finite-dimensional H^*-module V_-, one has $(\det(\mathcal{R}_{|V_+\otimes V_-}))^d = 1$.*
2. *For any finite-dimensional $D(H)-$module V, one has $(\det(u_{|V}))^{d^2} = 1$.*

Proof. (1) Recall that $(\Delta\otimes I)(\mathcal{R})=\mathcal{R}_{13}\mathcal{R}_{23}$. Apply this identity to $V_+\otimes H\otimes V_-$, where H is the regular representation of H. Since $V_+\otimes H=(\dim V_+)H$, this yields, after taking determinants:

$$(\det(\mathcal{R}_{|H\otimes V_-}))^{\dim V_+}=(\det(\mathcal{R}_{|V_+\otimes V_-}))^d(\det(\mathcal{R}_{|H\otimes V_-}))^{\dim V_+}.$$

The result follows after cancellation.
(2) We use Drinfeld's formula, $\Delta(u)=(u\otimes u)(\mathcal{R}_{21}\mathcal{R})^{-1}$. Using part 1, and the fact that $D(H)=H^*\otimes H$ as H^*-module and $H-$module, we compute:

$$\det(\Delta(u)_{|V\otimes D(H)})=(\det(u_{_{|V}}))^{d^2}(\det(u_{_{|D(H)}}))^{\dim V}.$$

Since $V\otimes D(H)=(\dim V)D(H)$, the result follows. □

Proof of Theorem 4.3. By part 8 of Proposition 2.2, we may assume that k is algebraically closed. Since u is central, we have for any irreducible $D(H)-$module V that $\det(u_{_{|V}})=\lambda(u,V)^{\dim V}$, where $\lambda(u,V)$ is the eigenvalue of u on V. So by Lemma 4.7, $\lambda(u,V)^{\dim V\cdot d^2}=1$. But by [EG1, Theorem 1.5], and in positive characteristic by [EG2, Theorem 3.7], $\dim V$ divides d, so $\lambda(u,V)^{d^3}=1$. Thus, $u^{d^3}=1$ and we are done by part 2 of Theorem 2.5. □

In the non-semisimple case, as we know, Theorem 4.3 fails, and the order of u may be infinite. The analogue of Theorem 4.3 in this case is the following theorem.

Let A be a finite-dimensional algebra. For any two irreducible $A-$modules V_1 and V_2, write $V_1\sim V_2$ if they occur as constituents in the same indecomposable $A-$module. Extend $\sim$ to an equivalence relation. For an irreducible module W, let $[W]$ be the equivalence class of W. For an indecomposable module V let $[V]$ be the equivalence class of any constituent W of V. Let $N_{_V}$ be the greatest common divisor of dimensions of elements of $[V]$.

Theorem 4.8. *Let H be a Hopf algebra of dimension d over an algebraically closed field k. Then:*

1. *For any indecomposable $D(H)-$module V, the unique eigenvalue of the central element $z=uS(u)$ on V is a root of unity of order dividing $d^2N_{_V}$.*
2. *For any indecomposable $D(H)-$module V, every eigenvalue of u on V is a root of unity of order dividing $2d^2N_{_V}$ (so the eigenvalues of u on any $D(H)$ module are roots of unity).*

Proof. (1) Recall that $z=u^2g$, where g is a grouplike element of $D(H)$. By [NZ], the order of g divides $\dim(H)=d$. Since V is indecomposable and z is central, z has a unique eigenvalue $\lambda(z,V)$ on V. For any $W\in[V]$, $\lambda(z,V)=\lambda(z,W)$, so we get, $(\det(u_{_{|W}}))^{2d}=\lambda(z,V)^{d\cdot\dim W}$, which implies by part 2 of Lemma 4.7, that $\lambda(z,V)^{d^2\cdot\dim W}=1$.

(2) Since, any eigenvalue μ of $u_{|V}$ has the property $\mu^2 = \lambda(z, V)\nu$, where ν is an eigenvalue of g^{-1}, we have by part 1 that $\mu^{2d^2 \dim W} = 1$, and the result follows. $\square$

Corollary 4.9. *If* $\exp(H) = \infty$, *then* u *is not semisimple.*

Corollary 4.10. *Let* H *be a finite-dimensional Hopf algebra over a field* k *of positive characteristic* p. *Then* $\exp(H) < \infty$.

Proof. Let u be the Drinfeld element of $D(H)$. By part 2 of Theorem 4.8, the eigenvalues of u are roots of unity. Hence there exists a positive integer a such that $u^a = 1 + n$ where $n \in D(H)$ is a nilpotent element. But then $u^{ap^b} = 1$ for a sufficiently large positive integer b, and the result follows from part 2 of Theorem 2.5. $\square$

5. Concluding Remarks

In conclusion we would like to formulate some questions.

Question 5.1. *Suppose that* H *is a semisimple and cosemisimple Hopf algebra of dimension* d *over* k. *If a prime* p *divides* d, *must it divide* $\exp(H)$*?*

We do not know the answer to this question even in characteristic zero, even for $p = 2$. However, if H is a group algebra then the answer is positive, since the statement is equivalent to (a special case of) Sylow's first theorem: a finite group whose order is divisible by p has an element of order p. So positive answer to Question 5.1 would be a "quantum Sylow theorem".

Question 5.2. *Let* H *be a semisimple and cosemisimple Hopf algebra over* k *whose exponent is a power of a prime* p. *Must the dimension of* H *be a power of the same prime?*

This is a special case of Question 5.1, but we still do not know the answer, except for the case $\exp(H) = 2$, when the answer is trivially positive. For group algebras, the statement is equivalent to the well-known group-theoretical result that a finite group where orders of all elements are powers of p is a p-group (a special case of Sylow's theorem).

Question 5.3. *Let* H *be a finite-dimensional Hopf algebra over* k *such that the element* $u \in D(H)$ *is semisimple in the regular representation. Does it follow that* H *and* H^* *are semisimple*

(1) *In characteristic zero?*
(2) *In positive characteristic* p*?*

By Theorem 4.8, part (1) of Question 5.3 is equivalent to the question whether for a finite-dimensional Hopf algebra H in characteristic 0, $\exp(H) < \infty$ implies that H is semisimple.

A positive answer to part (2) of Question 5.3 implies a positive answer to Question 5.1 for involutive Hopf algebras defined over $\mathbb{Z}$ and free as $\mathbb{Z}$-modules (which includes group algebras, i.e. this would generalize Sylow's theorem). Indeed, if H is such a Hopf algebra then for any prime p dividing the dimension of H, either H/pH or $(H/pH)^*$ is not semisimple (as $tr(S^2) = 0$), and hence $D(H)$ is not semisimple. If the answer to part (2) of Question 5.3 is positive, then this would imply that u is not semisimple over F_p, i.e. the order of u is divisible by p, as desired.

For group algebras, the answer to part (2) of Question 5.3 is positive: in this case semisimplicity of u is equivalent to semisimplicity of $R = \sum g \otimes \delta_g$, which implies that all group elements g are semisimple. This would imply that their orders are not divisible by p, which by Sylow's theorem implies that p does not divide the order of the group.

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