MAXIMAL FUNCTIONS AND HILBERT
TRANSFORMS ALONG VARIABLE FLAT CURVES

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Abstract. In this work we establish $L^p$ boundedness for maximal functions and Hilbert transforms along variable curves in the plane, via $L^2$ estimates for certain singular integral operators with oscillatory terms.

§1. Introduction

In this paper, we study the $L^p(\mathbb{R}^2)$ boundedness for the maximal function $\mathcal{M}$ and the Hilbert transform $\mathcal{H}$ along variable curves. In our discussions, these are defined a priori on functions in $C^\infty_0(\mathbb{R}^2)$ by

$$\mathcal{M}f(x) = \sup_{0<h<\infty} \frac{1}{h} \left| \int_0^h f(x_1 - t, x_2 - S(x_1, x_1 - t)) dt \right|$$
and

$$\mathcal{H}f(x) = \text{p.v.} \int_{-\infty}^\infty f(x_1 - t, x_2 - S(x_1, x_1 - t)) \frac{dt}{t},$$

where $S(x, y)$ is a suitable real-valued function vanishing on the diagonal.

We shall also consider the singular integral operators $T_\lambda$ (acting on functions on the real line) which are of the form

$$T_\lambda f(x) = \lim_{\epsilon \to 0} \int_{|x-y| \geq \epsilon} e^{i\lambda S(x, y)} (x - y)^{-1} f(y) dy.$$

Local versions of the operators $T_\lambda$ have been studied by Phong and Stein [PS], and by Pan [P] who proved the $L^p$ boundedness of $T_\lambda$ with bounds independent of $\lambda$ when the mixed derivative $S''_{xy}$ does not vanish to infinite order at any diagonal point $(x_0, x_0)$. In [S] Seeger showed that for a certain class of phases $S$ without this finite type property, the associated operators $T_\lambda$ are also uniformly bounded on $L^2$. Here we extend the Seeger-type result, for a different, closely related (but not always directly comparable) class of phases, to all other $p$, $1 < p < \infty$. 

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Theorem 1. Let $S$ be an antisymmetric function in $C^3(\mathbb{R}^2)$. Let $g: \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing function satisfying that there exists a constant $B \geq 1$ such that
\[
g(Bt) \geq 2g(t),
\]
and suppose that for some $E \geq 1$ and $A \geq 1$,
\[
\begin{align*}
A^{-1}g(|x - y|) &\leq |S'_1(x, y)| \leq Ag(E|x - y|), \\
A^{-1}g(|x - y|) &\leq |S'_2(x, y)| \leq Ag(E|x - y|).
\end{align*}
\]
Suppose furthermore that $S''_{112}$ is single-signed on $\mathbb{R}^2$. Then $M$ is bounded on $L^p$ for all $1 < p \leq \infty$, and $H$ is bounded on $L^p$ for all $1 < p < \infty$.

A local version of the theorem where the hypotheses are assumed on $S$ in a neighbourhood of the diagonal and $M$ and $H$ are suitably modified also holds. Included in this setting are examples such as $S(x, y) = e^{-2g(x, y)}$, $g(x, x) \neq 0$, or $S(x, y) = e^{-h(x, y)}$, $h(x, x) > 0$, (defined thus for $y > x$ and extended to be antisymmetric) where $g$ and $h$ are smooth. Thus the theorem covers certain “flat” curves which is a point of principal interest.

It is well-known that estimates for $H$ yield uniform estimates for $T_\lambda$. Indeed, if $F_2$ denotes the Fourier transform in the second variable then $F_2Hf(x_1, \lambda) = T_\lambda(F_2f(\cdot, \lambda))(x_1)$. By applying Plancherel’s theorem we see that
\[
\sup_{\lambda \in \mathbb{R}} \|T_\lambda\|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} = \|H\|_{L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)}.
\]
Moreover, a variant of de Leeuw’s theorem implies that if $H$ is bounded on $L^p(\mathbb{R}^2)$ then
\[
\sup_{\lambda \in \mathbb{R}} \|T_\lambda\|_{L^p(\mathbb{R}) \to L^p(\mathbb{R})} \leq \|H\|_{L^p(\mathbb{R}^2) \to L^p(\mathbb{R}^2)}.
\]
Thus, an immediate consequence of Theorem 1 is the following corollary:

Corollary 2. If $S$ satisfies the same hypotheses as in Theorem 1, then for any $p$, $1 < p < \infty$, we have
\[
\sup_{\lambda \in \mathbb{R}} \|T_\lambda\|_{L^p(\mathbb{R}) \to L^p(\mathbb{R})} \leq C.
\]

In the work of Seeger [S], $S$, in addition to (1), is assumed to satisfy a condition on its second order derivatives (related to the so-called $k$-quasimonotonicity condition.) We, in contrast, demand a condition on the third-order derivative $S''_{112}$.

In the translation-invariant setting, when $S(x, y) = \gamma(x - y)$, our condition reduces to $\gamma'' \geq 0$ (see [DR]) which implies the infinitesimal doubling (see [CCVWW]) and hence doubling of $\gamma'$, (see [NVWW1], [NVWW2] and [CCC-DRVWW]). On the other hand, the single-signedness of $S''_{112}$ is the minimal
hypothesis for our proof — based upon smoothing estimates for relatives of $T_\lambda$ where the Hilbert singularity is replaced by a smooth cut-off function — to work according to what is currently known about smoothing for oscillatory integral operators. (See for example [PS1] and [CCW]). A general result for more general versions of $H$ in which the variable curve has rotational curvature not vanishing to infinite order has recently been established in [CNSW]. In our setting, that condition reduces to $S''_{x,y}$ not vanishing to infinite order on the diagonal. The first results for the non-translation invariant flat case (i.e. in which the rotational curvature may vanish to infinite order on the diagonal) were obtained in [CWW2].

The idea of the proof is to consider pieces of the operators like the following

$$S_j f(x) = \frac{1}{2j+1} \int_{2^j < |t| \leq 2^{j+1}} f(x_1 - t, x_2 - S(x_1, x_1 - t)) dt,$$

and

$$T_j f(x) = \int_{2^j < |t| \leq 2^{j+1}} f(x_1 - t, x_2 - S(x_1, x_1 - t)) \frac{dt}{t}.$$

Thus $H f(x) = \sum_j T_j f(x)$ and $M$ is bounded for a non-negative function $b$ by $M f(x) \leq C \sup_j |S_j f(x)|$. And then we can apply the following generalization of the Cotlar-Stein lemma, see [C].

**Proposition 3. (Almost-Orthogonality Principle)** Assume that $\{Q_j\}$ satisfies $\sum_{j \in \mathbb{Z}} Q_j = I$. Assume that

$$\|Q_j^* Q_k\|_{2-2} + \|Q_j Q_k^*\|_{2-2} \leq C 2^{-|j-k|},$$

and that

$$\|\sum_j \pm Q_j\|_{p_0 - p_0} + \|\sum_j \pm Q_j^*\|_{p_0 - p_0} \leq C,$$

for some $p_0 \in (1,2)$. Suppose that $Q_j = P_j - P_{j+1}$ where $P_j \geq 0$ and $\|\sup_j |P_j f|\|_r \leq C_r \|f\|_r$ for $p_0 \leq r \leq 2$. Assume also that

$$\|\left(\sum_k |Q_k g_k|^2\right)^{1/2}\|_{p_0} \leq C\left(\sum_k |g_k|^2\right)^{1/2}\|_{p_0}.$$

Suppose that $\{T_j\}, \{S_j\}$ satisfy

$$|T_j f| \leq S_j |f|$$

where $S_j \geq 0$. Assume that $\|S_j\|_{r-2} \leq C$ for $p_0 \leq r \leq 2$. Moreover, assume that

$$\|(S_j - P_j)Q_{j+k}^*\|_{2-2} + \|(S_j - P_j)^* Q_{j+k}\|_{2-2} \leq C 2^{-|j|};$$
and
\[ \|T_j Q_{j+k}^*\|_{2-2} + \|T_j^* Q_{j+k}\|_{2-2} \leq C 2^{-\epsilon |k|}. \]

Then \( f \to \sup_j |S_j f(x)| \) and \( \sum_j T_j \) are bounded on \( L^p \), \( p_0 < p \leq 2 \).

In order to define the appropriate Littlewood-Paley decomposition \( I = \sum_j Q_j \)
we define the dilations,
\[ A(t) = \begin{pmatrix} t & 0 \\ 0 & G(t) \end{pmatrix} \quad \text{with} \quad G(t) = \int_0^t g(s) ds. \]

Similar dilations were first explicitly used in the flat translation invariant case
in [CCVWW]. The collection \( \{A(t)\} \) satisfies the Rivièrè condition
\[ \|A(s)^{-1} A(t)\| \leq C \left( \frac{t}{s} \right)^\epsilon \quad \text{for} \quad s \geq t, \quad \text{for some} \quad \epsilon > 0. \]

In fact in this case it is true with \( \epsilon = 1 \) and it is enough to show that \( \frac{G(t)}{G(s)} \leq C \frac{t}{s} \)
for \( s \geq t \). Observe that if \( s \geq t \) there exists a natural number \( k \) such that
\( B^k t \leq s \leq B^{k+1} t \) and we have
\[ \frac{1}{t} G(t) = \frac{1}{t} \int_0^t g(u) du \leq \frac{1}{2t} \int_0^t g(Bu) du \leq \cdots \leq \frac{1}{2^{k+1}t} \int_0^t g(B^{k} u) du \]
\[ = \frac{1}{B^{k+1} 2^k t} \int_0^{B^k t} g(u) du \leq \frac{1}{B^{k+1} 2^k t} G(s) = \frac{s}{B^{k+1} 2^k t} \frac{G(s)}{s} \leq B \frac{G(s)}{s}. \]

Let \( \phi \) be a nonnegative \( C_0^\infty(\mathbb{R}^2) \) function such that \( \int \phi = 1 \). We set the initial
averaging operator
\[ P_0 f(x) = \int \phi(x - y) f(y) dy; \]
then an approximation of the identity (with \( P_j \to I \) as \( j \to -\infty \) and \( P_j \to 0 \) as \( j \to \infty \)) is given by
\[ P_j f(x) = \int (\det A_j)^{-1} \phi(A_j^{-1} (x - y)) f(y) dy \quad \text{with} \quad A_j = A(2^j). \]

The natural Littlewood-Paley difference operators \( Q_j \) are then \( Q_j = P_j - P_{j+1} \).

According to [CVWW] and [CWW1], the conditions on the operators \( P_j \) and
\( Q_j \) in the almost orthogonality lemma are satisfied for any \( p_0, 1 < p_0 < \infty \);
just the Rivièrè condition is required. Therefore, subject to having verified (2)
and (3), Proposition 3 shows that \( \mathcal{M} \) and \( \mathcal{H} \) are bounded on \( 1 < p \leq 2 \). But
the maximal function is trivially bounded on \( L^\infty \), thus it maps continuously \( L^p \)
into \( L^p \) for \( 1 < p \leq 2 \). And for \( \mathcal{H} \) we notice that the original problem itself is
selfadjoint, so the boundedness of \( \mathcal{H} \) for \( 1 < p \leq 2 \) implies its boundedness for
\( 2 \leq p < \infty \).

It hence remains to prove (2) and (3).
2. The curves and their normalization

Let \( S(x, y) \) be an antisymmetric function. Then it is easy to see that
\[
S'_1(z, x) = -S'_2(x, z), \quad S''_1(z, x) = -S''_2(x, z), \quad S''_2(z, x) = -S''_1(x, z)
\]
(2) whenever \( S'_2(x, x) = 0 \), and \( S'_{12}(z, x) = -S'_{22}(x, x) \). Since we also assume that
\( S''_{12} \) does not change sign then \( S''_{12} \) does not change its sign either and its sign is opposite to that of \( S''_{12} \). On the other hand, by applying the mean value theorem we get
\[
\text{sgn} S'_{12}(z, x) = \text{sgn} S''_{12}(z, x).
\]
(4)

Finally, by using \( S'_1(x, x) = S'_2(x, x) = 0 \) one may see that
\[
\text{sgn} S'_1 = -\text{sgn} S''_{112} \quad \text{and} \quad \text{sgn} S'_2 = \text{sgn} S''_{112},
\]
(5)
since \( S'_1(x, x) = S'_2(x, x) \) for some \( \nu \in \overline{xy} \) (the line segment joining \( z \) to \( x \)) and so \( \text{sgn} S'_1(x, x) = \text{sgn} S''_{112} \text{sgn} (z - \nu) \text{sgn} (x - z) \).

Lemma 4. If \( S''_{112} \) is single-signed and \( S \) is antisymmetric then for any \( x \) and \( y \)
\[
|S'_1(x, y)| \leq |x - y||S'_{12}(x, y)| \quad \text{and} \quad |S'_2(x, y)| \leq |x - y||S'_{12}(x, y)|.
\]

Proof. We use the mean value theorem:
\[
|S'_1(x, y)| = |S'_1(x, y) - S'_1(x, x)| = |S'_{12}(x, u)||x - y| \quad \text{for some} \quad u \in \overline{x y}
\]
If \( x < y \) then \( x < u < y \) and \( |S''_{12}(x, u)| = -\text{sgn} S''_{112} S''_{12}(x, u) \). Since \( \text{sgn} S''_{112} = -\text{sgn} S''_{12} \) this function is increasing in \( u \) and so \( |S''_{12}(x, u)| \leq |S''_{12}(x, y)| \). When \( x > y \), \( |S''_{12}(x, u)| = \text{sgn} S''_{112} S''_{12}(x, u) \) is decreasing in \( u \) and thus also \( |S''_{12}(x, u)| \leq |S''_{12}(x, y)| \).

To prove the estimate for \( |S'_2(x, y)| \) we can repeat the proof, or realize that
\[
|S'_2(x, y)| = |S'_1(y, x)| \leq |y - x||S'_{12}(y, x)| = |y - x||S'_{12}(x, y)|, \quad \text{since} \quad S'_1 \text{ is also antisymmetric.}
\]

In our development we shall need to work with normalized versions of \( S(., .) \), that is, for fixed \( j \)
\[
\tilde{S}(x, y) = \frac{S(2^j x, 2^j y)}{G(2^j)}.
\]
It is easy to check several facts concerning them that we shall need later on.
First,
\[
|\tilde{S}'_i(x, y)| \geq A^{-1} \text{ whenever } |x - y| \geq 1.
\]
(6)

To see this observe that
\[
|\tilde{S}'_i(x, y)| = \frac{2^j |S'_i(2^j x, 2^j y)|}{G(2^j)} \geq \frac{A^{-1} 2^j g(2^j x - 2^j y)}{G(2^j)} \geq \frac{A^{-1} 2^j g(2^j)}{G(2^j)} \geq A^{-1},
\]
where the last inequality is true because $g$ is a non-decreasing function.

By Lemma 4 whenever $|x - y| \leq C_0$ then

$$|\tilde{S}_1'(x, y)| \leq C_0|\tilde{S}'_{12}(x, y)| \quad \text{and} \quad |\tilde{S}_2'(x, y)| \leq C_0|\tilde{S}''_{12}(x, y)|$$

since

$$|\tilde{S}_1'(x, y)| = 2^j \left| \frac{|S_1'(2^j x, 2^j y)|}{G(2^j)} \right| \leq 2^j |2^j x - 2^j y| \left| \frac{|S''_{12}(2^j x, 2^j y)|}{G(2^j)} \right| = |x - y| |\tilde{S}''_{12}(x, y)|.$$ 

With this observation we can prove the following lemma:

**Lemma 5.** If $S$ is antisymmetric and $S''_{12}$ is single-signed then, for any $x$, $y$ and $z$ such that either $-C_1 \leq z - x, z - y \leq 0$ with $x \leq y$, or $0 \leq z - x, z - y \leq C_1$ for $x \geq y$, there exists a constant $C$ such that

$$\frac{|\tilde{S}_1'(z, y)| + |\tilde{S}_1'(z, x)| + |\tilde{S}_2'(z, x)|}{|\tilde{S}_1'(z, x) - \tilde{S}_1'(z, y)|} \leq \frac{C}{|x - y|}.$$ 

**Proof.** We consider the case $-C_1 \leq z - x, z - y \leq 0$ which implies $|x - y| \leq 2C_1$ with $x \leq y$ (the proof for the case $0 \leq z - x, z - y \leq C_1$ for $x \geq y$ is a repetition of the following arguments). We have that

$$|\tilde{S}_1'(z, x) - \tilde{S}_1'(z, y)| = \int_x^y - \text{sgn} \tilde{S}''_{12} \cdot \tilde{S}''_{12}(z, u) du$$

$$\geq - \text{sgn} \tilde{S}''_{12} \cdot \tilde{S}''_{12}(z, x)(y - x) = |\tilde{S}''_{12}(z, x)||y - x|,$$

where we have used that the function inside the integral is increasing in $u$. By (7), as $|z - x| \leq C_1$ then

$$|\tilde{S}_1'(z, x) - \tilde{S}_1'(z, y)| \geq c|\tilde{S}'_1(z, x)||y - x|$$

and

$$|\tilde{S}_1'(z, x) - \tilde{S}_1'(z, y)| \geq c|\tilde{S}'_2(z, x)||y - x|.$$ 

Then we just need to prove the lemma for $|\tilde{S}_1'(z, y)|$. If $|\tilde{S}_1'(z, x)| \geq \frac{1}{2} |\tilde{S}_1'(z, y)|$ then with the previous estimate we get also $|\tilde{S}_1'(z, x) - \tilde{S}_1'(z, y)| \geq c|\tilde{S}_1'(z, y)||y - x|$. But otherwise $|\tilde{S}_1'(z, x) - \tilde{S}_1'(z, y)| \geq \frac{1}{2} |\tilde{S}_1'(z, y)| \geq c|\tilde{S}_1'(z, y)||y - x|$, since $|y - x| \leq 2C_1$. \(\Box\)

3. The heart of the proof

If $T$ is an integral operator on $\mathbb{R}^n$ with distribution kernel $K(x, y)$, and $A \in \text{GL}(n, \mathbb{R})$, we let $A_T$ be the operator whose kernel is $(\text{det} A)^{-1} K(A^{-1} x, A^{-1} y)$. Thus $\|A_T\|_{p-p} = \|T\|_{p-p}$ for all $1 \leq p \leq \infty$. In the case that $T$ is the Hilbert
transform along a curve $\Gamma(x,t)$, then $A_\ast T$ becomes the Hilbert transform along the curve $A_\ast \Gamma$, where $(A_\ast \Gamma)(x,t) = A[\Gamma(A^{-1}x,t)]$.

We just need to prove estimates (2) and (3). By the essential self-adjointness of the problem, it suffices to prove either the first or the second inequalities in (2) and (3). For $k > 0$, they are a direct consequence of the smoothness of $\{P_j\}$, the support properties of $\{T_j,S_j,P_j\}$, and the fact that $T_j 1 = T_j^\ast 1 = (S_j - P_j) 1 = (S_j - P_j)^\ast 1 = 0$. For instance, we indicate how to prove that $\|T_j^* Q_j + k\|_{2-2} \leq C 2^{-ck}$; for this it suffices to show that $\|T_j^* P_j + k\|_{2-2} \leq C 2^{-ck}$.

Moreover, by setting $T_{jk} f(x) = A_{j+k}^{-1} T_j^* f(x)$ it is equivalent to the estimate $\|T_{jk}^* P_j 0\|_{2-2} \leq C 2^{-ck}$. To prove that we just need the cancellation property $T_{jk} 1 = 0$ and that $T_{jk}^*$ has its distribution kernel supported in $\{(x,y) : |x-y| \leq C2^{-ck}\}$. This reduces to seeing that if $|t| \leq 2^{j+1}$, $|x - A_{j+k}^{-1} \Gamma(x,t)| \leq C 2^{-ck}$.

Now

$$|x - A_{j+k}^{-1} \Gamma(x,t)| \leq \left| \frac{t}{2^{j+k}} \right| + \left| \frac{S(2^{j+k} x_1 + t, 2^{j+k} x_1)}{G(2^{j+k})} \right|,$$

To handle the second term, notice that since $S_1'(x,x) = 0$ then

$$|S(2^{j+k} x_1 + t, 2^{j+k} x_1)| = \left| \int_0^t S_1'(2^{j+k} x_1 + s, 2^{j+k} x_1) ds \right| \leq \int_0^{|t|} A g(E s) ds \leq C G(E |t|),$$

which is smaller than or equal to $C G(E 2^{j+1})$. The support condition now follows from the Riviére property. (The estimate $G(E 2^{j+1})/G(2^{j+k}) \leq C 2^{-ck}$ holds for any $k \geq k_0$ with $k_0$ such that $2^{k_0 - 1} \geq E$, but otherwise $\|T_j^* Q_j + k\|_{2-2} \leq C \leq C 2^{-ck}$.)

When $k \leq 0$, since $\|Q_j + k P_j^\ast\|_{2-2} \leq C 2^{ck}$ then $\|Q_j + k (S_j - P_j)^\ast\|_{2-2} \leq C 2^{ck}$ is equivalent to $\|Q_j + k S_j^\ast\|_{2-2} \leq C 2^{ck}$, and the bound for $\|Q_j + k T_j^\ast\|_{2-2} \leq C 2^{ck}$ will follow exactly the same argument.

Now we have to break up the operator $S_j^\ast$ into two pieces determined by whether or not $t$ is positive, and we work with $(\tilde{S}_j^\ast) = A_j^{-1} S_j^\ast$. Then we set the normalized “positive” part of the operator $S_j^\ast$ as follows

$$(\tilde{S}_j^\ast)^+ f(x) = \int f(x_1 + t, x_2 + \tilde{S}(x_1 + t, x_1)) \alpha^+ (t) dt, \text{ with } \tilde{S}(x,y) = \frac{S(2^j x, 2^j y)}{G(2^j)},$$

where $\alpha^+$ is a real-valued smoothed-out version of $\chi_{[1,2]}$. (The corresponding kernel for the case $(S_j^\ast)^-$ is with $\alpha^-$ being a smoothed-out version of $\chi_{[-2,-1]}$).

We write $\tilde{Q}_j + k = A_j^{-1} \tilde{Q}_j$. Therefore, we need to show that

$$\|\tilde{Q}_j + k (\tilde{S}_j^\ast)^\ast\|_{2-2} \leq C 2^{ck} \quad \text{and} \quad \|\tilde{Q}_j + k (\tilde{S}_j^\ast)^-\|_{2-2} \leq C 2^{ck}.$$
Since the two estimates are similar we concentrate only on the first.

Let \( K : \mathbb{R}^m \to C \) be a kernel, and \( A : \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^q \) be a function. Let

\[
T f(x) = \int_{\mathbb{R}^m} f(x_1 - y_1, x_2 - A(x_1, y_1)) K(y) dy
\]

where \((x_1, x_2) \in \mathbb{R}^p \times \mathbb{R}^q\) and \((y_1, y_2) \in \mathbb{R}^p \times \mathbb{R}^m\). Define \(T_\lambda h(x) = \int_{\mathbb{R}^m} h(x - y_1) e^{i\lambda A(x,y)} K(y) dy\) where now \(x \in \mathbb{R}^p\). Then \((TS^*)_\lambda = T_\lambda S^*_\lambda\), and Plancherel's theorem in the \(x_2 \in \mathbb{R}^q\) variable shows that \(\|T\|_{2-2} = \sup_\lambda \|T_\lambda\|_{2-2}\). Thus, in our case at hand, it suffices to prove

\[
\|((\tilde{Q}_{j+k})_\lambda((\widetilde{S}_j^\ast)_\lambda)\|_{2-2} \leq C 2^{ek},
\]

uniformly in \(\lambda\), or indeed

\[
\|((\tilde{Q}_{j+k})_\lambda((\widetilde{S}_j^\ast)_\lambda)\|_{2-2} \leq C 2^{ek}.
\]

Now the convolution kernel of \(\tilde{P}_{j+k}\) can be written as

\[
\frac{1}{2^k} \Phi_1 \left( \frac{x_1}{2^k} \right) \frac{G(2^j)}{G(2^{j+k})} \Phi_2 \left( \frac{G(2^j)}{G(2^{j+k})} \right),
\]

for some even functions \(\Phi_1\) and \(\Phi_2\) such that \(\Phi_1\) is supported in \([-2, 2]\), and \(\Phi_2\) is such that \(\tilde{\Phi}_2\) is identically one in \([-1, 1]\) and also supported in \([-2, 2]\). By taking the Fourier transform in the second variable we have

\[
K_{\lambda} (\tilde{Q}_{j+k})(x_1) = \frac{1}{2^k} \Phi_1 \left( \frac{x_1}{2^k} \right) \tilde{\Phi}_2 \left( \frac{G(2^{j+k})}{G(2^j)} \right) - \frac{1}{2^{k+1}} \Phi_1 \left( \frac{x_1}{2^{k+1}} \right) \tilde{\Phi}_2 \left( \frac{G(2^{j+k+1})}{G(2^{j+1})} \right)
\]

\[
= \frac{1}{2^{k+1}} \Phi_1 \left( \frac{x_1}{2^{k+1}} \right) \left[ \tilde{\Phi}_2 \left( \frac{G(2^{j+k})}{G(2^j)} \right) - \tilde{\Phi}_2 \left( \frac{G(2^{j+k+1})}{G(2^{j+1})} \right) \right]
\]

\[
= \frac{1}{2^k} \Phi_1 \left( \frac{x_1}{2^k} \right) \tilde{\Phi}_2 \left( \frac{G(2^{j+k})}{G(2^j)} \right)
\]

\[
= I_{\lambda}(x_1) + II_{\lambda}(x_1)
\]

where \(\Psi(x) = \Phi_1(x) - \frac{1}{2} \Phi_1(\frac{x}{2})\) and so \(\int \Psi = 0\).

Since \(\tilde{\Phi}_2\) is identically one in \([-1, 1]\), \(I_{\lambda}(x_1) = 0\) unless \(\frac{G(2^{j+k+1})}{G(2^{j+1})} \geq 1\), and as \(\left| \tilde{\Phi}_2 \left( \frac{G(2^{j+k})}{G(2^j)} \right) - \tilde{\Phi}_2 \left( \frac{G(2^{j+k+1})}{G(2^{j+1})} \right) \right| \leq 2\), the estimate (8) when we consider the part \(I_{\lambda}\) of the \(Ker(\tilde{Q}_{j+k})_\lambda\) follows from \(\|((\tilde{Q}_{j+k})_\lambda\|_{2-2} \leq C\)

and

\[
\|((\widetilde{S}_j^\ast)_\lambda)\|_{2-2} \leq C 2^{ek}, \quad \text{when } \frac{G(2^{j+k+1})}{G(2^{j+1})} \geq 1.
\]

It is not difficult to see that the kernel of \(((\widetilde{S}_j^\ast)_\lambda)((\tilde{S}_j^\ast)_\lambda)\) is, as a function of \(x\) and \(y\),

\[
K^+_\lambda(x,y) = \int e^{i\lambda[S(z,y) - S(z,x)]} \alpha^+(z-y) \alpha^+(z-x) dz.
\]
To show (9), since $K^+_{\lambda}$ is supported in $|x - y| \leq 5$, it suffices to prove that 
\[ \int |K^+_{\lambda}(x, y)|^2 dx \leq C/|\lambda| \] uniformly in $y$, since both the Riviére condition and $G(2^{j+k+1})|\lambda|/G(2^j) \geq 1$ then imply $\int |K^+_{\lambda}(x, y)|^2 dx \leq C2^{\epsilon k}$. In order to do that we first observe that by Van der Corput's lemma $|K_{\lambda}(x, y)| \leq C/(|\lambda||x - y|)$. Indeed, set $u(z) = \tilde{S}(z, y) - \tilde{S}(z, x)$ then $u'(z) = \tilde{S}'_1(z, y) - \tilde{S}'_1(z, x)$ and $u''(z) = \tilde{S}''_{11}(z, y) - \tilde{S}''_{11}(z, x)$ for fixed $x$ and $y$ and since $\tilde{S}''_{11}$ is single-signed, $u''$ is single-signed and $|u'(z)| = |\tilde{S}'_1(z, y) - \tilde{S}'_1(z, x)| \geq C|x - y||\tilde{S}'_1(z, x)| \geq c|x - y|$ (see Lemma 5 and (6)). Then,

\[ \int |K^+_{\lambda}(x, y)|^2 dx \leq \int_{\{x:|x - y| < \delta\}} Cdx + \int_{\{x:|x - y| > \delta\}} \frac{1}{\lambda^2|x - y|^2} dx \]
\[ \leq C\delta + C\frac{1}{\lambda^2\delta} \leq C\frac{1}{|\lambda|}, \]

by taking $\delta = 1/|\lambda|$.

Thus the contribution to (8) arising from $I_{\lambda}$ is under control.

Now we need to consider, for technical reasons, separately the cases $\chi_{x \geq y}$ and $\chi_{x \leq y}$. Let $A$ be the operator with kernel $K^+_{\lambda}(x, y)\chi_{\{x \geq y\}}$ and let $B$ be the operator with kernel $K^+_{\lambda}(x, y)\chi_{\{x \leq y\}}$; since $K^+_{\lambda}(y, x) = K^+_{\lambda}(x, y)$ we have that $A^* = B$, and in order to prove (8) it is enough to prove it either for $A$ or for $B$. Since we have a trivial estimate for the $L^\infty$ operator norm it suffices to show that the $L^1$ norm of the operator has the decay we want, and in fact it is enough to show that

\[ \int \left| \int II_{\lambda}(x - x')K'_\lambda(x', y)dx' \right| dx \leq C2^{\epsilon k} \]

uniformly in $y \in \mathbb{R}$, $\lambda \in \mathbb{R}$, and $K'_\lambda$ denotes $K^+_{\lambda}$ restricted to $x \geq y$. But $\int II_{\lambda}(x - x')K'_\lambda(x', y)dx' = C\Psi_k *_1 K'_\lambda(\cdot, y)$ ($*_1$ means convolution in the first variable and $\Psi_k(x) = \frac{1}{2\pi} \Psi\left(\frac{x}{2\pi}\right)$, and therefore since $\int \Psi_k = 0$ the following lemma finishes the proof.

**Lemma 6.** For $K'_\lambda(x, y) = K^+_{\lambda}(x, y)\chi_{\{x \geq y\}}(y)$, we have

\[ \int |K'_\lambda(x + h, y) - K'_\lambda(x, y)| dx \leq C|h|^{\frac{1}{2}}. \]

**Proof.** Let us assume $|h| \leq \frac{1}{4}$ otherwise the conclusion is clear, and let us assume
for simplicity that \( h > 0 \). Then

\[
\int |K'_\lambda(x + h, y) - K'_\lambda(x, y)| \, dx \\
\leq \int \left| \int e^{i\lambda \bar{S}(z,y) - \bar{S}(z,x+h)} - e^{i\lambda \bar{S}(z,y) - \bar{S}(z,x)} \right| \alpha^+(z-y) \alpha^+(z-x-h) \, dx \\
+ \int \left| \int e^{i\lambda \bar{S}(z,y) - \bar{S}(z,x)} \left( \alpha^+(z-y) \alpha^+(z-x) - \alpha^+(z-y) \alpha^+(z-x-h) \right) \right| \, dx \\
= I + II.
\]

The second term is fine because, since we are working with normalized pieces of curves the regions of integration are finite, and the function \( \alpha^+ \) is smooth enough, so it is clearly \( O(|h|) \). The first term satisfies

\[
I = \int \left| \int \int_0^h \frac{\partial}{\partial t} e^{i\lambda \bar{S}(z,y) - \bar{S}(z,x+t)} \, dt \right| \alpha^+(z-y) \alpha^+(z-x-h) \, dx
\]

\[
= \int \left| \int \int_0^h i\lambda \bar{S}'_2(z, x + t) e^{i\lambda \bar{S}(z,y) - \bar{S}(z,x+t)} \, dt \right| \alpha^+(z-y) \alpha^+(z-x-h) \, dx
\]

\[
\leq |h| \sup_{0 \leq t \leq h} \left| \int \lambda \bar{S}'_2(z, x + t) e^{i\lambda \bar{S}(z,y) - \bar{S}(z,x+t)} \alpha^+(z-y) \alpha^+(z-x-h) \, dx \right|
\]

\[
= |h| \sup_{0 \leq t \leq h} \left| \int \lambda \bar{S}'_2(z, x) e^{i\lambda \bar{S}(z,y) - \bar{S}(z,x)} \alpha^+(z-y) \alpha^+(z-x-h+t) \, dx \right|.
\]

Then, it suffices to show that

\[
\int_{|x-y| > \delta} \left| \lambda \bar{S}'_2(z, x) e^{i\lambda \bar{S}(z,y) - \bar{S}(z,x)} \alpha^+(z-y) \alpha^+(z-x-h+t) \right| \, dx \leq \frac{C}{\delta},
\]

independently of \( 0 \leq t \leq h \), because then

\[
\int |K'_\lambda(x + h, y) - K'_\lambda(x, y)| \, dx \leq C\delta + C \frac{|h|}{\delta} \leq C|h|^{\frac{1}{2}},
\]

by taking \( \delta = |h|^{\frac{1}{2}} \). Now we integrate by parts with respect to \( z \) and obtain

\[
\int \lambda \bar{S}'_2(z, x) e^{i\lambda \bar{S}(z,y) - \bar{S}(z,x)} \alpha^+(z-y) \alpha^+(z-x-h+t) \, dz
\]

\[
= -\frac{1}{i} \int \frac{\partial}{\partial z} \left( \frac{\bar{S}'_2(z,x)}{\bar{S}'_1(z,y) - \bar{S}'_1(z,x)} \right) e^{i\lambda \bar{S}(z,y) - \bar{S}(z,x)} \alpha^+(z-y) \alpha^+(z-x-h+t) \, dz
\]

\[
- \frac{1}{i} \int \frac{\partial}{\partial z} \left( \alpha^+(z-y) \alpha^+(z-x-h) \right) \frac{\bar{S}'_2(z,x)}{\bar{S}'_1(z,y) - \bar{S}'_1(z,x)} e^{i\lambda \bar{S}(z,y) - \bar{S}(z,x)} \, dz.
\]
Since for $K_\lambda'(x, y) = K_\lambda^+(x, y)\chi_{(x \geq y)}(y)$ we are under the hypothesis of Lemma 5 and then we get $|\tilde{S}'_1(z, y) - \tilde{S}'_1(z, x)| \geq C|x - y||\tilde{S}''_2(z, x)|$. We shall be finished if we can show

$$\int_{x : |x - y| > \delta} \left| \frac{\partial}{\partial z} \left( \frac{\tilde{S}'_2(z, x)}{\tilde{S}'_1(z, y) - \tilde{S}'_1(z, x)} \right) \right| dz dx \leq \frac{C}{\delta}$$

for $1 < z - y < 2$ and $0 < z - x < 3$ (recall that $0 < t < h$ and $h \leq \frac{1}{4}$). But

$$\frac{\partial}{\partial z} \frac{\tilde{S}'_2(z, x)}{\tilde{S}'_1(z, y) - \tilde{S}'_1(z, x)} = \frac{-\tilde{S}'_1''(z, x)\tilde{S}'_1(z, x) + \tilde{S}'_1''(z, x)\tilde{S}'_1(z, y)}{[\tilde{S}'_1(z, y) - \tilde{S}'_1(z, x)]^2} + \frac{\tilde{S}'_1''(z, x)[\tilde{S}'_1(z, y) - \tilde{S}'_1(z, x)]}{[\tilde{S}'_1(z, y) - \tilde{S}'_1(z, x)]^2}$$

$$= M + N + L.$$

Now, it is very important to check that each of the terms has single sign and that $\text{sgn } M = \text{sgn } L$; fortunately we know the signs precisely in terms of the sign of $\tilde{S}'_{112}$ (we use that $\tilde{S}'_{11}(z, y) - \tilde{S}'_{11}(z, x) = \tilde{S}'_{112}(z, \nu)(y - x)$.) Indeed then, by (4) and (5)

$$\text{sgn } M = \text{sgn } (-\tilde{S}'_{112}(z, x)\tilde{S}'_1(z, x)) = \text{sgn } \tilde{S}'_{112} \text{sgn } (x - z)(-\text{sgn } \tilde{S}'_{112})$$

$$= \text{sgn } (z - x),$$

$$\text{sgn } N = \text{sgn } (\tilde{S}'_{112}(z, x)\tilde{S}'_1(z, y)) = -\text{sgn } \tilde{S}'_{112} \text{sgn } (x - z)(-\text{sgn } \tilde{S}'_{112})$$

$$= \text{sgn } (z - x),$$

$$\text{sgn } L = -\text{sgn } \tilde{S}'_{112} \text{sgn } \tilde{S}'_{112} \text{sgn } (y - x)$$

$$= \text{sgn } (x - y).$$

So since $z - x > 0$ then $M$ and $N$ have single sign. Also since we need to prove the lemma only for $K_\lambda'(x, y) = K_\lambda^+(x, y)\chi_{(x \geq y)}(y)$ then $\text{sgn } M = \text{sgn } L$. Therefore, we can use that

$$\iint |M + N + L|dz dx \leq \iint M dz dx + \iint L dz dx + \iint N dz dx.$$

The double integral of $N$ is, for some boundary points $x^*$ and $x^{**}$, by using Lemma 5 and the fact that we are always integrating over bounded intervals, controlled by

$$\left| \iint N dz dx \right| \leq \iint \left| \frac{\tilde{S}'_{1}(z, y)}{\tilde{S}'_{1}(z, y) - \tilde{S}'_{1}(z, x)} \right|^{x^{**}} dz \leq \frac{C}{\delta} \iint dz \leq \frac{C}{\delta}.$$
Now we integrate $M$, first in the variable $x$ and then with respect to $z$,
\[
\iint M \, dx \, dz = \int \frac{-S'_1(z,x)}{S'_1(z,y) - S'_1(z,x)} \, dz \, x^* + \int \frac{S''_{12}(z,x)}{S'_1(z,y) - S'_1(z,x)} \, dx \, dz.
\]
In the same way, but first in the variable $z$ and then with respect to $x$,
\[
\iint L \, dz \, dx = \int \frac{-S'_1(z,x)}{S'_1(z,y) - S'_1(z,x)} \, dz \, x^* - \int \frac{S''_{12}(z,x)}{S'_1(z,y) - S'_1(z,x)} \, dx \, dz,
\]
for suitable boundary points $z^*$ and $z^{**}$. Therefore, again Lemma 5 gives
\[
\left| \int (M + L) \, dz \, dx \right| = \left| \int \frac{-S'_1(z,x)}{S'_1(z,y) - S'_1(z,x)} \, dz \, x^* + \int \frac{S''_{12}(z,x)}{S'_1(z,y) - S'_1(z,x)} \, dz \, x^* \right| \leq C \delta,
\]
as required. \qed

References


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