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HEAT KERNEL ON CONNECTED SUMS OF RIEMANNIAN MANIFOLDS

Alexander Grigor'yan and Laurent Saloff-Coste

1. Introduction

This note is about the heat kernel on a connected sum M of non-compact manifolds M_1, M_2, ... ,M_k assuming that one knows enough about the heat kernels for each M_i individually (which is the case when M_i are complete manifolds of non-negative Ricci curvature). We announce here matching uniform upper and lower bounds for the heat kernel on such manifolds M. The proofs will be given elsewhere.

For an arbitrary Riemannian manifold M, denote by Δ the Laplace operator of the Riemannian metric of M and by $p(t,x,y)$ the heat kernel, that is, the smallest positive fundamental solution to the heat equation $u_t = \Delta u$ on $\mathbb{R}_+ \times M$ (here $x, y \in M$ and $t > 0$). Alternatively, $p(t,x,y)$ can be defined as the kernel of the semigroup $\exp(t\Delta)$. Yet another definition of $p(t,x,y)$ is that it is the density of the transition probability of the Brownian motion on M generated by the operator Δ.

In $\mathbb{R}^n$, the heat kernel is given by the classical formula

$$p(t,x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right). \tag{1}$$

On a general manifold M, one may hope to obtain some estimates of the heat kernel related to the geometry of M. In the past two decades, this question has received considerable attention from many authors. This was partially motivated by various important applications of heat kernel bounds although the question of understanding the interplay between the heat kernel and the geometry is interesting by itself. The celebrated theorem of Li and Yau [16] says that if M is a geodesically complete manifold with non-negative Ricci curvature then

$$\frac{c}{V(x,\sqrt{t})} \exp\left(-\frac{d^2(x,y)}{ct}\right) \le p(t,x,y) \le \frac{C}{V(x,\sqrt{t})} \exp\left(-\frac{d^2(x,y)}{Ct}\right). \tag{2}$$

Here $V(x,r)$ is the Riemannian volume of the geodesic ball $B(x,r)$ of radius r centered at $x \in M$; $d(x,y)$ is the geodesic distance between the points $x, y \in M$;

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C, c are positive constants. There are many other settings when certain heat kernel bounds are known — see for example [1], [3], [4], [7], [8], [11], [12], [18], [17].

In most known estimates the heat kernel is controled by two multiplicative terms: a Gaussian factor $\exp(-\frac{d^2(x,y)}{Ct})$ and a long time decay term which is some function of time t as in (1) and (2). The reason is that in most situations considered previously the manifold M possesses a certain degree of homogeneity so that $p(t,x,y)$ depends on x and y mainly via the distance $d(x,y)$. At the present time, there are no methods yielding satisfactory heat kernel bounds for highly inhomogeneous manifolds having substantially different geometric properties in different parts. The manifolds with ends considered in this note can be regarded as the simplest case of such non-homogeneous manifolds.

The main results are Theorems 1 and 4. Since the results are not simple, we first explain them for connected sums of Euclidean spaces (Section 2). The general case is considered in Sections 3 and 4.

2. Connected sums of Euclidean spaces

Let M_1, M_2, ... , M_k be a finite family of geodesically complete non-compact Riemannian manifolds. We say that a manifold M is a connected sum of M_i's and write

$$M = M_1 \# M_2 \# \cdots \# M_k$$

if, for some compact $K \subset M$ (which will be called *the central part* of M), the exterior $M \setminus K$ is a disjoint union of open sets E_1, E_2, ... , such that each E_i is isometric to $M_i \setminus K_i$, for some compact $K_i \subset M_i$ (in fact, we will identify E_i and $M_i \setminus K_i$). In other words, in order to obtain M, one cuts out holes K_i in each M_i and glues together the rest of M_i, that is E_i, over some compact K. Of course, a connected sum is not uniquely defined. When writing $M_1 \# M_2 \# \cdots$, we refer to any manifold which can be obtained in this way.

Another point of view is to regard M as a manifold with ends E_i assuming that each end E_i can be closed to a complete manifold M_i. Sometimes it is convenient to say that M_i's are the ends of M.

The connected sum $\mathbb{R}^n \# \mathbb{R}^n$ was considered by Kuz'menko and Molchanov [15] as early as in 1979 as an example of a manifold where the Liouville property for bounded harmonic functions fails. However, even for this simple manifold, no uniform bounds of the heat kernel was known, except for partial estimates [2], [19]. Our main results — Theorems 1 and 4, provide sharp uniform heat kernel estimates for a rather wide class of connected sums $M_1 \# M_2 \# \cdots \# M_k$. In particular, they cover (but are not restricted to) the case when M_i are manifolds with non-negative Ricci curvature.

In this section, we will explain our results in the situation when each M_i is similar, at infinity, to an Euclidean space. To be precise, let us fix some (large) integer N and define the manifold $\mathcal{R}^n := \mathbb{R}^n \times \mathbb{S}^{N-n}$, for all integers n from 1 to N. Clearly, $\mathcal{R}^n$ has topological dimension N whereas, on a large scale, it

resembles $\mathbb{R}^n$. It is natural to say that *the asymptotic dimension* of $\mathcal{R}^n$ is n. It makes sense to consider the connected sums of $\mathcal{R}^n$'s with different n since all those manifolds have the same topological dimension N. Hence, the main example of this section is the manifold $M = \mathcal{R}^{n_1}\#\mathcal{R}^{n_2}\#\cdots\#\mathcal{R}^{n_k}$.

E.B. Davies [9] considered certain one-dimensional model spaces with ends having asymptotic dimensions n_1, n_2, ... and observed some interesting phenomena for the heat kernel long time behavior. For example, he showed that, if all $n_i \neq 2$ and some $n_i > 2$ then, for a fixed point x and for $t \to \infty$,

$$p(t,x,x) \leq Ct^{-n/2}$$

where

$$n = \min_i \{2 + |n_i - 2|\}.$$

In particular, the contributions to n of the ends with the asymptotic dimensions 1 and 3 are equal. Indeed, if $n_i = 1$ or 3 then $2 + |n_i - 2| = 3$. E.B. Davies conjectured that the same is true for connected sums of manifolds, and the positive answer to his conjecture follows from our results (see Corollary 5). Moreover, one can see that there is a certain duality between the manifolds of asymptotic dimensions less than 2 and greater than 2.

For the manifold $M = \mathcal{R}^n$, the following estimate of the heat kernel is well-known:

$$p(t,x,y) \asymp \frac{1}{\min\left(t^{N/2}, t^{n/2}\right)} \exp\left(-\frac{d^2(x,y)}{ct}\right), \tag{3}$$

$d(x,y)$ being the geodesic distance on M. Here the sign $\asymp$ means that the ratio of the right-hand side and of the left-hand side is bounded from above and from below by some positive constants, for all values of $x, y \in M$ and $t > 0$. We always follow the convention that the constant c in the Gaussian exponential term may be different for the upper and for the lower bounds.

Consider now the manifold $M = \mathcal{R}^n\#\mathcal{R}^m$ where both $n, m > 2$. If $t \leq 1$ then the estimate (3) holds again. Assume in the sequel $t > 1$. For any point $x \in M$, denote

$$|x| := \sup_{z\in K} d(x,z).$$

The point $x \in M$ lies either in the central part K or in one of two ends $\mathcal{R}^n \setminus K$, $\mathcal{R}^m \setminus K$. Let $x \in \mathcal{R}^n \setminus K$ and $y \in \mathcal{R}^m \setminus K$. Then we claim that

$$p(t,x,y) \asymp \left(\frac{1}{t^{m/2}\,|x|^{n-2}} + \frac{1}{t^{n/2}\,|y|^{m-2}}\right) \exp\left(-\frac{d^2(x,y)}{ct}\right), \tag{4}$$

which is a special case of Theorem 1 below. However, even (4) exhibits some new phenomena. Let us consider two asymptotic regimes.

(a) *Long time regime* when $t \to \infty$ whereas $|x|$ and $|y|$ stay bounded. Clearly, (4) implies

$$p(t,x,y) \asymp \frac{1}{t^{\min(n,m)/2}}$$

so that the long time behaviour of the heat kernel is determined by the end with the smallest asymptotic dimension.

(b) *Medium time regime* when $t \to \infty$ and $|x| \asymp |y| \asymp \sqrt{t}$. Then (4) implies

$$p(t,x,y) \asymp \frac{1}{t^{n/2+m/2-1}}. \tag{5}$$

In particular, for the manifold $M = \mathbb{R}^n \# \mathbb{R}^n$, in the medium time regime, we have

$$p(t,x,y) \asymp \frac{1}{t^{n-1}}. \tag{6}$$

Note that even this estimate is new (see [2] for an upper bound and [19] for a lower bound in the medium time regime). For comparison, if x and y stay in *the same* end and $|x| \asymp |y| \asymp \sqrt{t}$ then $p(t,x,y) \asymp t^{-n/2}$ as for the heat kernel of $\mathbb{R}^n$ (see (8) below). Since $n-1 > n/2$, the estimate (6) reflects *a bottleneck effect*: for $|x| \asymp |y| \asymp \sqrt{t}$, the heat kernel $p(t,x,y)$ is substantially smaller if x, y are in different ends than if they are in the same end.

Each term in (4) has a clear probabilistic meaning. Let X_t be the Brownian motion on M, that is, the diffusion generated by the Laplace operator Δ. Then $p(t,x,y)$ is the density of the probability that X_t hits y at time t after starting at x. Roughly speaking, there are two possibilities for a continuous path to get from x to y: either X_t spends most time in $\mathcal{R}^m$ or in $\mathcal{R}^n$. The probability of the former is of the order $\frac{1}{t^{m/2}|x|^{n-2}}$ and of the latter is of the order $\frac{1}{t^{n/2}|y|^{m-2}}$.

Another new effect can be observed when considering connected sums of at least *three* manifolds. Let

$$M = \mathcal{R}^{n_1} \# \mathcal{R}^{n_2} \# \cdots \# \mathcal{R}^{n_k}$$

where all $n_i > 2$. Denote

$$n = \min_i n_i \,.$$

Assuming as above that $t > 1$, $x \in \mathcal{R}^{n_i} \setminus K$ and $y \in \mathcal{R}^{n_j} \setminus K$ where $i \neq j$, we have the following estimate

$$p(t,x,y) \asymp \left(\frac{1}{t^{n_j/2}\,|x|^{n_i-2}} + \frac{1}{t^{n_i/2}\,|y|^{n_j-2}} + \frac{1}{t^{n/2}\,|x|^{n_i-2}\,|y|^{n_j-2}} \right) e^{-\frac{d^2(x,y)}{ct}}. \tag{7}$$

The first two terms here have the same interpretation as above. The third term is dominated by one of the first two if $n = n_i$ or $n = n_j$ (which is the case, in particular, when $k = 2$). However, if $n < n_i$ and $n < n_j$ (which may happen only if $k \geq 3$) then the third term dominates, for t large enough. It roughly

corresponds to the probability that X_t goes from x to y spending a long time in the end with the smallest asymptotic dimension n. Hence, the most probable way of getting from x to y is by looping through the smallest end!

If both points x, y belong to the same end $\mathcal{R}^{n_i} \setminus K$, then we have, for all $t > 1$,

$$p(t,x,y) \asymp \frac{1}{t^{n/2}\,|x|^{n_i-2}\,|y|^{n_i-2}} e^{-\frac{|x|^2+|y|^2}{ct}} + \frac{1}{t^{n_i/2}} e^{-\frac{d^2(x,y)}{ct}}. \tag{8}$$

Let us emphasize that in all the estimates above, we have assumed $n_i > 2$ for all i. If $n_i \leq 2$ for some i then the heat kernel behaves differently as will be discussed in Section 4 — see Theorem 4 and Corollary 5.

The condition $n_i > 2$ is equivalent to *the non-parabolicity* of $\mathcal{R}^{n_i}$. A manifold M is called *parabolic* if the Brownian motion X_t on M is recurrent, and non-parabolic otherwise. The non-parabolicity is equivalent to the existence of a positive fundamental solution of the Laplace operator (see the survey [13] for detailed account of parabolicity).

In our main results — Theorems 1 and 4 below, we will distinguish the cases when all the ends M_i are non-parabolic or not.

3. The case when all the ends are non-parabolic

For any manifold M_i, denote by d_i the geodesic distance on M_i , by $B_i(x,r)$ the geodesic ball on M_i of the radius r centered at x, and by $V_i(x,r)$ the Riemannian volume of that ball. Also, let p_i be the heat kernel of M_i. Note that the non-parabolicity of M_i is equivalent to

$$\int^{\infty} p_i(t,x,x)dt < \infty, \tag{9}$$

for some/all $x \in M_i$. If M_i is geodesically complete and, for some $x \in M_i$,

$$\int^{\infty} \frac{dt}{V_i(x,\sqrt{t})} = \infty \tag{10}$$

then M_i is parabolic — see [5], [10], [14], [20].

Let us introduce the following hypotheses, which in general may be true or not.

(i) *The volume doubling condition*: for some $C > 0$ and for all $r > 0$, $x \in M_i$,
$$V_i(x,2r) \leq CV_i(x,r).$$

(ii) *The on-diagonal upper bound of the heat kernel*: for some $C > 0$ and for all $t > 0$, $x \in M_i$,
$$p_i(t,x,x) \leq \frac{C}{V_i(x,\sqrt{t})}. \tag{11}$$

(iii) *The off-diagonal lower bound of the heat kernel* : for some $C > 0$ and for all $t > 0$, $x, y \in M$,

$$p_i(t,x,y) \geq \frac{C^{-1}}{V_i(x,\sqrt{t})} \exp\left(-C\frac{d_i^2(x,y)}{t}\right). \tag{12}$$

For example, all the hypotheses (i), (ii) and (iii) hold if M_i has non-negative Ricci curvature — see [16]. Note that, under the hypotheses (i) and (ii), the condition (10) is not only sufficient but also necessary for the parabolicity of M_i. Let us also mention that (i) and (ii) imply an off-diagonal upper bound for the heat kernel matching (12) (see [12]) as well as an on-diagonal lower bound matching (11) (see [6]). The necessary and sufficient conditions for (i) and (ii) are given in terms of a relative Faber-Krahn inequality — see [12]. The hypotheses (i),(ii) *and* (iii) are equivalent to a parabolic Harnack inequality, which, in turn, is equivalent to the volume doubling (i) and a Poincaré inequality — see [17] and [11].

In order to state the heat kernel upper bound on the manifold M, let us introduce the following notation. Denote as above

$$|x| = \sup_{z\in K} d(x,z),$$

and define the new distances:

1. $d_+(x,y)$ is the infimum of lengths of smooth curves connecting x and y in M and *intersecting* K;
2. $d_\emptyset(x,y)$ is the infimum of lengths of smooth curves connecting x and y in M and *not intersecting* K.

Clearly, both $d_+(x,y)$ and $d_\emptyset(x,y)$ are larger than or equal to the geodesic distance $d(x,y)$, and one of them is equal to $d(x,y)$. If x and y are at different ends then $d_\emptyset(x,y) = \infty$.

For each $i \geq 1$, fix a reference point $o_i \in \partial K \cap M_i$, set

$$W_i(r) = V_i(o_i, r), \tag{13}$$

$$W_{\min}(r) = \min_i W_i(r), \tag{14}$$

and define the function $W(x,r)$ by

$$W(x,r) = \begin{cases} W_i(r), & x \in E_i, \\ W_{\min}(r), & x \in K. \end{cases} \tag{15}$$

In words, $W_i(r)$ is the volume growth function for the manifold M_i, $W_{\min}(r)$ is the smallest volume growth, and $W(x,r)$ switches between W_i and $W_{\min}$ depending on the location of x.

Finally, let us define the function $H(x,t)$, for all $t > 0$ and $x \in M$, by

$$H(x,t) = \frac{|x|^2}{W_i(|x|)} + \left(\int_{|x|^2}^t \frac{ds}{W_i(\sqrt{s})}\right)_+, \qquad \text{if } x \in E_i, \tag{16}$$

$$H(x,t) = 1, \qquad \text{if } x \in K. \tag{17}$$

Theorem 1. *Assume that all M_i, $i = 1, 2, \dots, k$, are complete non-compact* ***non-parabolic*** *manifolds satisfying* (i), (ii), *and* (iii). *Then the heat kernel $p(t, x, y)$ on the connected sum $M = M_1 \# M_2 \# \cdots \# M_k$ admits the following estimate: for all $t > 0$ and $x, y \in M$,*

$$p(t,x,y) \asymp \left(\frac{H(x,t)}{W(y,\sqrt{t})} + \frac{H(y,t)}{W(x,\sqrt{t})} + \frac{H(x,t)H(y,t)}{W_{\min}(\sqrt{t})} \right) e^{-\frac{d_+^2(x,y)}{ct}} \tag{18}$$

$$+ \frac{1}{\sqrt{V_i(x,\sqrt{t})V_i(y,\sqrt{t})}} e^{-\frac{d_\emptyset^2(x,y)}{ct}}, \tag{19}$$

where the term in (19) *refers to the case when both x, y belong to the same end E_i (otherwise $d = \infty$). If all M_i satisfy only* (i) *and* (ii) *(but not necessarily* (iii)*,) then the upper bound in* (18)–(19) *is still valid.*

In particular, if $x \in E_i$ and $y \in E_j$ where $i \neq j$ then (18)–(19) yield

$$p(t,x,y) \asymp \left(\frac{H(x,t)}{W_j(\sqrt{t})} + \frac{H(y,t)}{W_i(\sqrt{t})} + \frac{H(x,t)H(y,t)}{W_{\min}(\sqrt{t})} \right) e^{-\frac{d^2(x,y)}{ct}}, \tag{20}$$

because in this case $d_\emptyset(x,y) = \infty$ and $d_+(x,y) = d(x,y)$. If x and y lie at the same end, that is, $i = j$ then we obtain

$$p(t,x,y) \asymp \frac{H(x,t)H(y,t)}{W_{\min}(\sqrt{t})} e^{-\frac{|x|^2+|y|^2}{ct}} + \frac{1}{V_i(x,\sqrt{t})} e^{-\frac{d^2(x,y)}{ct}}. \tag{21}$$

Let us show examples of computation of the function H.

Example 1. Suppose that, for all i, the volume growth on M_i is given by

$$W_i(r) \asymp r^{n_i}, \qquad \text{for } r > 1,$$

where all $n_i > 2$. The latter implies, in particular, (9) whence M_i is non-parabolic. Next, we have

$$W_{\min} \asymp r^n,$$

where $n = \min_i n_i$. Assuming $x \in E_i$ and $|x| > 1$, we compute $H(x,t)$ by

$$H(x,t) \asymp |x|^{2-n_i} + \left(\int_{|x|^2}^t \frac{ds}{s^{n_i/2}} \right)_+ \asymp |x|^{2-n_i}.$$

Hence, the estimates (20) and (21) become (7) and (8), respectively.

Example 2. Suppose now that, for all i, the volume growth function is as follows

$$W_i(r) \asymp r^2 \log^{\alpha_i} r, \qquad r > 2,$$

where $\alpha_i > 1$ (which ensures the non-parabolicity of M_i). Then

$$W_{\min} \asymp r^2 \log^{\alpha} r$$

where

$$\alpha = \min_i \alpha_i.$$

Assuming $x \in E_i$ and $|x| > 2$, we compute $H(x,t)$ by

$$H(x,t) = \log^{-\alpha_i}|x| + \left(\int_{|x|^2}^t \frac{ds}{s\log^{\alpha_i} s}\right)_+ \asymp \log^{-\alpha_i}|x| + \left(\log^{1-\alpha_i}|x|^2 - \log^{1-\alpha_i} t\right)_+.$$

If $|x| \geq \sqrt{t}$ then $H(x,t) \asymp \log^{-\alpha_i}|x|$. Assuming $y \in E_j$ and $|x| \asymp |y| \asymp \sqrt{t}$ (where $i \neq j$) we obtain from (20) the following medium time asymptotic

$$p(t,x,y) \asymp \frac{1}{t\log^{\alpha_i+\alpha_j} t}.$$

If $|x| << \sqrt{t}$ then $H(x,t) \asymp \log^{1-\alpha_1}|x|$. Assuming that $|x|, |y|$ remain bounded while $t \to \infty$, we obtain the long time asymptotic

$$p(t,x,y) \asymp \frac{1}{t\log^{\alpha} t}.$$

4. The mixed case with both parabolic and non-parabolic ends

We will need here the following additional hypothesis.

(iv) *Connectedness of the annuli:* for all R large enough, the annulus $B_i(o_i, 4R) \setminus B_i(o_i, R/4)$ is connected [1].

So far we have not required that the end E_i is connected. Clearly, the connectedness of the annuli is even stronger assumption than the connectedness of E_i. Basically, the hypothesis (iv) prohibits the situation when M_i consists of two "nearly" disjoint unbounded parts connected only by a rare sequence of small tubes.

It is easy to see that any two-dimensional convex surface in $\mathbb{R}^3$ satisfies all the hypotheses (i), (ii), (iii) and (iv), provided it is complete and unbounded. Also, all manifolds $\mathcal{R}^n$ introduced above satisfy these hypotheses.

We start with the following lemma.

Lemma 2. *Suppose that all M_i, $i = 1, 2, \ldots, k$ are geodesically complete manifolds satisfying* (i), (ii), (iii), *and* (iv). *Assume that* ***at least one*** *of them is*

[1] This hypothesis can be slightly relaxed as follows: for any two points x, y both at distance R from o_i, there is a continuous path connecting x to y and staying in the annulus $B_i(o_i, 4R) \setminus B_i(o_i, R/4)$.

non-parabolic. Then there exists a positive harmonic function $h(x)$ *on* $M = M_1 \# M_2 \# \cdots \# M_k$ *such that, for all* $x \in M$,

$$h(x) \asymp 1 + \left(\int_1^{|x|^2} \frac{ds}{W(x, \sqrt{s})} \right)_+$$

where W *is defined by* (13)–(15).

In particular, $h(x) \asymp 1$ if x stays in a non-parabolic end whereas $h(x) \to \infty$ if $x \to \infty$ within a parabolic end.

For any smooth positive function $h(x)$ on M, one may consider the *weighted* manifold $\widetilde{M} = (M, h^2)$ which means that, in addition to the Riemannian structure, $\widetilde{M}$ is endowed with the measure $\widetilde{\mu}$ defined by $d\widetilde{\mu} = h^2 d\mu$ where μ is the Riemannian measure on M. There is a natural weighted Laplace operator on $\widetilde{M}$ defined by

$$\widetilde{\Delta} = h^{-2} \mathrm{div}(h^2 \nabla),$$

which is formally self-adjoint with respect to $\widetilde{\mu}$. Denote by $\widetilde{p}$ the heat kernel of $\widetilde{\Delta}$. If $h(x)$ is, in addition, harmonic then it is easy to see that

$$\widetilde{\Delta} = h^{-1} \circ \Delta \circ h,$$

which implies

$$\widetilde{p}(t, x, y) = \frac{p(t, x, y)}{h(x)h(y)}. \tag{22}$$

Let us define the function $h_i(x)$ on M_i as follows: $h_i(x) = h(x)$ if $x \in E_i$ and h_i is extended to K_i somehow, to remain smooth and positive. Clearly, we have, for all $x \in M_i$,

$$h_i(x) \asymp 1 + \left(\int_1^{|x|^2} \frac{ds}{W_i(\sqrt{s})} \right)_+ . \tag{23}$$

Consider the weighted manifold $\widetilde{M_i} = (M_i, h_i^2)$ and introduce similarly $\widetilde{\mu}_i$, $\widetilde{\Delta}_i$ and $\widetilde{p}_i$ (since h_i is not necessarily harmonic on M_i, the kernel $\widetilde{p}_i$ does not necessarily satisfies the analogue of (22)). Denote $\widetilde{V}_i(x, r) = \widetilde{\mu}_i(B_i(x, r))$.

Lemma 3. *Suppose that all* M_i, $i = 1, 2, \ldots, k$ *are geodesically complete manifolds satisfying* (i), (ii), (iii) *and* (iv). *Assume that at least one of them is non-parabolic. Let* h *and* h_i *be the functions defined as above. Then each* $\widetilde{M_i}$ *is non-parabolic and satisfies the conditions* (i), (ii) *and* (iii) (where one should naturally replace V_i by $\widetilde{V}_i$ and p_i by $\widetilde{p}_i$).

In fact, Theorem 1 remains true for weighted manifolds. Therefore, by Lemma 3, one can apply Theorem 1 to the above setting, estimate $\widetilde{p}$, and then recover p by (22). To state the result which we obtain in this way, we need the following notation. Let us set

$$\widetilde{W}_i(r) := \widetilde{V}_i(o_i, r)$$

and define $\widetilde{W}_{\min}(r)$, $\widetilde{W}(x,r)$ and $\widetilde{H}(x,t)$ analogously to (14), (15) and (16)–(17) where W_i must be replaced everywhere by $\widetilde{W}_i$.

Theorem 4. *Assume that all M_i, $i = 1, 2, \dots, k$, are complete non-compact manifolds satisfying* (i), (ii), (iii), (iv), *and* ***at least one*** *of M_i's is non-parabolic. Then the heat kernel $p(t,x,y)$ on the connected sum $M = M_1 \# M_2 \# \dots \# M_k$ admits the following estimate: for all $t > 0$ and $x, y \in M$,*

$$p(t,x,y) \asymp h(x)h(y) \left(\frac{\widetilde{H}(x,t)}{\widetilde{W}(y,\sqrt{t})} + \frac{\widetilde{H}(y,t)}{\widetilde{W}(x,\sqrt{t})} + \frac{\widetilde{H}(x,t)\widetilde{H}(y,t)}{\widetilde{W}_{\min}(\sqrt{t})} \right) e^{-\frac{d_+^2(x,y)}{ct}} + \frac{h(x)h(y)}{\sqrt{\widetilde{V}_i(x,\sqrt{t})\widetilde{V}_i(y,\sqrt{t})}} e^{-\frac{d_\emptyset^2(x,y)}{ct}},$$

where the last term refers to the case when both x, y belong to the same end E_i (otherwise $d = \infty$).

Denote

$$\eta_i(r) := 1 + \left(\int_1^{r^2} \frac{ds}{W_i(\sqrt{s})} \right)_+ . \tag{24}$$

Then (23) implies $h_i(x) \asymp \eta_i(|x|)$. It is possible to prove that

$$\widetilde{W}_i(r) \asymp \eta_i^2(r) W_i(r) \tag{25}$$

and, if $x \in E_i$,

$$\widetilde{H}(x,t) \asymp \frac{|x|^2}{\eta_i^2(|x|) W_i(|x|)} + \frac{1}{\eta_i(|x|)\, \eta_i(\sqrt{t})} \left(\int_{|x|^2}^t \frac{ds}{W_i(\sqrt{s})} \right)_+ . \tag{26}$$

Let us show some examples how to apply Theorem 4 using (24)–(26).

Example 3. Let $M_1 = \mathcal{R}^1 := \mathbb{R}_+ \times \mathbb{S}^{N-1}$, $M_2 = \mathcal{R}^3$ and consider the connected sum $M = \mathcal{R}^1 \# \mathcal{R}^3$. We have, for $r > 1$,

$$W_1(r) \asymp r \quad \text{and} \quad W_2(r) \asymp r^3.$$

By (24), we obtain, for $r > 1$,

$$\eta_1(r) \asymp r, \quad \text{and} \quad \eta_2(r) \asymp 1.$$

Then we compute $\widetilde{W}_i$ by (25)

$$\widetilde{W}_1(r) \asymp r^3, \quad \text{and} \quad \widetilde{W}_2(r) \asymp r^3,$$

whence

$$\widetilde{W}_{\min}(r) = \min(\widetilde{W}_1(r), \widetilde{W}_2(r)) \asymp r^3.$$

Finally, let us compute $\widetilde{H}(x,t)$ and $\widetilde{H}(y,t)$ assuming $x \in E_1$, $y \in E_2$ and $|x|, |y|, t > 1$. By (26), we obtain

$$\widetilde{H}(x,t) \asymp |x|^{-1}, \quad \text{and} \quad \widetilde{H}(y,t) \asymp |y|^{-1}.$$

Hence, Theorem 4 yields, for the chosen x, y and t,

$$p(t,x,y) \asymp \frac{1}{t^{3/2}} \left(1 + \frac{|x|}{|y|}\right) e^{-\frac{d^2(x,y)}{ct}}. \tag{27}$$

We see that the contributions of both ends $\mathcal{R}^1$ and $\mathcal{R}^3$ to the long time behavior of the heat kernel on M are of the same order $t^{-3/2}$. This may seem surprising, in view of the heat estimate (4) for the manifold $\mathcal{R}^n \# \mathcal{R}^m$ with $n, m > 2$, which contains both terms $t^{-n/2}$ and $t^{-m/2}$. The explanation is that what counts for the manifold $\mathcal{R}^1 \# \mathcal{R}^3$ is the heat kernel long time behavior on $\widetilde{\mathcal{R}}^1$ rather than that on $\mathcal{R}^1$. However, $\widetilde{\mathcal{R}}^1$ has the volume growth $\widetilde{W}_1(r) \asymp r^3$ and, therefore, the heat kernel of the order $t^{-3/2}$. This effect was first observed by E.B. Davies [9] in a model situation of a one-dimensional complex.

Example 4. Let us take $M_1 = \mathcal{R}^2$ and $M_2 = \mathcal{R}^3$. We compute as in the previous example, for $r > 2$ and $|x| > 2$,

$$W_1(r) \asymp r^2, \quad \text{and} \quad W_2(r) \asymp r^3,$$

$$h_1(x) \asymp \log|x|, \quad \text{and} \quad h_2(x) \asymp 1,$$

$$\widetilde{W}_1(r) \asymp r^2 \log^2 r, \quad \text{and} \quad \widetilde{W}_2(r) \asymp r^3.$$

If $x \in E_1$ and $y \in E_2$ then, for $t > 2$, we have $\widetilde{H}(y,t) \asymp |y|^{-1}$ whereas

$$\widetilde{H}(x,t) \asymp \frac{1}{\log^2|x|} + \left(\frac{1}{2\log|x|} - \frac{1}{\log t}\right)_+.$$

Hence, for the chosen x, y, t, we obtain

$$p(t,x,y) \asymp \left(\frac{\log|x|}{|y|\, t \log^2 t} + \frac{1}{t^{3/2}}\left[\frac{1}{\log|x|} + \left(\frac{1}{2} - \frac{\log|x|}{\log t}\right)_+\right]\right) e^{-\frac{d^2(x,y)}{ct}}$$

In particular, the long time asymptotic is as follows

$$p(t,x,y) \asymp \frac{1}{t \log^2 t},$$

whereas the medium time asymptotic (when $|x| \asymp |y| \asymp \sqrt{t}$) is given by

$$p(t,x,y) \asymp \frac{1}{t^{3/2}\log t}\,.$$

In certain situations, the formulas (24), (25) and (26) can be further simplified. Namely, let us assume that, for r large enough,

$$\int_1^{r^2} \frac{ds}{W_i(\sqrt{s})} \asymp \frac{r^2}{W_i(r)}. \tag{28}$$

Then we have the following relations, for r large enough,

$$\begin{aligned} \eta_i(r) &\asymp \frac{r^2}{W_i(r)}, \\ \widetilde{W}_i(r) &\asymp \frac{r^4}{W_i(r)}, \\ \widetilde{H}(x,t) &\asymp |x|^{-2}\, W_i(|x|), \end{aligned}$$

where in the last line $x \in E_i$ and $|x|$, t are large enough. For example, (28) holds if $W_i(r) \asymp r^\alpha$ with $0 < \alpha < 2$. For this W_i, we obtain

$$\begin{aligned} \eta_i(r) &\asymp r^{2-\alpha}, \\ \widetilde{W}_i(r) &\asymp r^{4-\alpha} = r^{\alpha^*}, \\ \widetilde{H}(x,t) &\asymp |x|^{\alpha-2} = |x|^{2-\alpha^*}, \end{aligned}$$

where

$$\alpha^* := 4 - \alpha.$$

We see that $\widetilde{W}_i(r)$ and $\widetilde{H}(x,t)$ look like the corresponding functions on a *non-parabolic* manifold with volume growth r^{α^*}. Hence, to some extent, the parabolic manifold with volume growth r^α can be regarded as dual to the non-parabolic manifold with volume growth r^{α^*}.

Corollary 5. *Under the hypotheses of* Theorem 4*, assume in addition that each manifold M_i has volume growth of the order r^{n_i} where $n_i \neq 2$. Denote*

$$n_i^* := \begin{cases} 4 - n_i, & n_i < 2, \\ n_i, & n_i > 2, \end{cases}$$

and

$$n := \min_{1 \le i \le k} n_i^*.$$

If $x \in E_i$, $y \in E_j$, $i \neq j$, and $|x|, |y|, t$ are large enough then

$$p(t,x,y) \asymp \left(\frac{1}{t^{n_j^*/2}\,|x|^{n_i^*-2}} + \frac{1}{t^{n_i^*/2}\,|y|^{n_j^*-2}} + \frac{1}{t^{n/2}\,|x|^{n_i^*-2}\,|y|^{n_j^*-2}} \right) |x|^{(2-n_i)_+}\,|y|^{(2-n_j)_+}\,e^{-\frac{d^2(x,y)}{ct}}. \tag{29}$$

In particular, (29) contains the estimate (7) for the manifold $\mathcal{R}^{n_1}\#\mathcal{R}^{n_2}\#\cdots\#\mathcal{R}^{n_k}$ when all n_i are larger than 2. The estimate (27) for the manifold $\mathcal{R}^1\#\mathcal{R}^3$ is also a straightforward consequence of (29).

The long time asymptotic in (29) is determined by the term $t^{-n/2}$. This was noticed by Davies [9] for an one-dimensional complex, modelling manifolds with ends. If all $n_i > 2$ then $n/2$ can be interpreted as the exponent of the largest heat kernel of M_i's. However, in general this is not true. It turns out that $t^{-n_i^*/2}$ is the rate of decay of the *Dirichlet* heat kernel of E_i, that is, the heat kernel on E_i with the vanishing boundary condition on ∂E_i. Therefore, the term $t^{-n/2}$ is determined in general by the largest Dirichlet heat kernel on E_i's. In fact, precise estimates of the Dirichlet heat kernel on E_i are crucial tools for the proof of the results described above.

Assume $n_i < 2$, $n_j > 2$ and consider the medium time asymptotic regime when $|x| \asymp \sqrt{t}$ and $t \to \infty$, in which case (29) implies

$$p(t,x,y) \asymp \frac{1}{t^{n_j^*/2}} + \frac{1}{t\,|y|^{n_j^*-2}}. \tag{30}$$

If also $|y| \asymp \sqrt{t}$ then the first term in (30) is leading. Hence, the medium time asymptotic is determined only by the larger end, in contrast to the case when both ends are non-parabolic (c.f., (5)).

However, the most interesting paradoxical effect occurs if we let $|y| \asymp 1$ in (30). In this case,

$$p(t,x,y) \asymp t^{-1}, \tag{31}$$

regardless of the exponents n_i, n_j! Therefore, if x moves away at the rate $\sqrt{t}$ and y stays near the central part, then the heat kernel $p(t,x,y)$ given by (31) is *larger* than in the long time regime when both x and y stay near the central part.

The explanation is that if x and y are close to the central part and t is large then the process X_t started at x tends to escape to infinity within the larger end so that its chances to loop back to y are relatively small. On the contrary, if X_t starts at the point x located at the smaller end at the distance $\sqrt{t}$ from the central part, then it cannot escape to infinity within this end because of its *parabolicity*. Hence, it moves towards the central part and hits y in time t with a higher probability.

The hypothesis of Theorem 4 and Corollary 5 that at least one end M_i is non-parabolic is equivalent to the non-parabolicity of M — see [10]. If all M_i's are

parabolic and, thus, M is also parabolic, then the estimate (29) cannot be true. Indeed, the long time decay $t^{-n/2}$ given by (29) implies (9), which contradicts the parabolicity of M.

The case when all M_i are parabolic will be addressed elsewhere.

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Department of Mathematics, 180 Queen's Gate, Huxley Building, Imperial College, London SW7 2BZ, United Kingdom

E-mail address: a.grigoryan@ic.ac.uk

CNRS, Toulouse, France,

Department of Mathematics, Malott Hall, Cornell University, Ithaca, NY 14853-4201

E-mail address: lsc@math.cornell.edu