CONSTRUCTION OF VALUATIONS FROM $K$-THEORY

IDO EFRAT

ABSTRACT. In this expository paper we describe and simplify results of Arason, Elman, Hwang, Jacob, and Ware on the construction of valuations on a field using $K$-theoretic data.

Introduction

Several recent developments in arithmetic geometry are based on the construction of valuations on a field just from the knowledge of its absolute Galois group. For instance, this is a main ingredient in Pop’s proof of the 0-dimensional case of Grothendieck’s “anabelian conjecture”, saying that any two fields which are finitely generated over $\mathbb{Q}$ and which have isomorphic absolute Galois groups are necessarily isomorphic; see [P2]–[P4], [S]. Other examples are the characterization of the fields with a $p$-adic absolute Galois group as the $p$-adically closed fields ([E], [K]; see also [N], [P1]), and the analogous result for local fields of positive characteristic [EF].

In the earlier approaches to such results, valuations were detected by means of various local-global principles for Brauer groups (or higher cohomology groups) — often in combination with model-theoretic tools (c.f., [N], [P1]–[P3], [S]). A different approach is introduced in [E]: there one uses an explicit and elementary construction of valuations which emerged in the mid-1970’s in the theory of quadratic forms. It originates from Bröcker’s “trivialization of fans” theorem on strictly-pythagorean fields [Br], i.e., real fields $K$ such that $K^2 + aK^2 \subseteq K^2 \cup aK^2$ for all $a \in K \setminus (-K^2)$. By Bröcker’s result, such a field has a valuation with very special properties: e.g., its value group is non-2-divisible, its residue field is real, and its principal units are squares. An explicit construction of these valuations was given by Jacob [J] (in the more general context of fans on pythagorean fields). This construction was extended to arbitrary fields by Ware [Wr], and later by Arason, Elman, and Jacob [AEJ]; see [En] for a related result. Roughly speaking, all these results show that if the quadratic forms over the field “behave” as if it possesses a valuation with non-2-divisible value group, residue characteristic $\neq 2$, and such that its principal units are squares, then (apart from a few obvious exceptions) such a valuation actually exists.
In the case of an odd prime number $p$ and a field $K$ of characteristic $\neq p$ containing a primitive $p$th root of unity, Hwang and Jacob [HJ] give an analogous construction of valuations with non-$p$-divisible value group, residue characteristic $\neq p$, and for which the principal units are $p$th powers. Here the role of quadratic forms is played by certain cohomological structures: the symbolic pairings $K^\times/p \otimes_\mathbb{Z} K^\times/p \to \mathbb{F}_p\text{Br}(K)$, where $\mathbb{F}_p\text{Br}(K)$ is the $p$-torsion part of the Brauer group of $K$ (see also [Bo] and [K] for related constructions).

In this expository paper we give a unified and somewhat simplified presentation of these important constructions. Our approach is completely elementary; in particular, we do not use cohomology, nor non-commutative division rings. Further, we do not assume the existence of primitive $p$th roots of unity in the field. The cohomological structures above are replaced here by the second Milnor $K$-group $K_2^M(K)$ of $K$, i.e., the quotient of the $\mathbb{Z}$-algebra $K^\times \otimes_\mathbb{Z} K^\times$ by the ideal generated by all elements of the form $x \otimes (1-x)$, where $0, 1 \neq x \in K$, and the natural projection $K^\times \otimes_\mathbb{Z} K^\times \to K_2^M(K), x \otimes y \mapsto \{x, y\}$.

**Main Theorem.** Let $p$ be a prime number, let $K$ be a field of characteristic $\neq p$, and let $T$ be a subgroup of $K^\times$ containing $(K^\times)^p$ and $-1$. Suppose that:

(i) if $x \in K^\times \setminus T$ and $y \in T \setminus K^p$ then $\{x, y\} \neq 0$;
(ii) if the cosets of $x, y \in K^\times$ in $K^\times/T$ are $\mathbb{F}_p$-linearly independent then $\{x, y\} \neq 0$.

Then there exists a valuation ring $O$ on $K$ with value group $\Gamma$, maximal ideal $m$, and residue field $\bar{K}$ such that $(\Gamma : p\Gamma) \geq (K^\times : T)/p, 1 - m \subseteq K^p$, and $\text{char}\bar{K} \neq p$. Furthermore, if $\bar{K} = K^p$ then $(\Gamma : p\Gamma) \geq (K^\times : T)$.

For a somewhat stronger result see Theorem 4.1.

Needless to say, most ingredients of the proof herein presented already appear in the above-mentioned works. The novelty of this note is mainly in the different organization of the material. We hope that it will make this powerful construction more easily accessible to Galois-theorists. In particular, the construction in this form is already used in [EF].

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1. The sets $O^+, O^-$

From now on we fix a field $K$ and a subgroup $T$ of $K^\times$. Let $$A = \{x \in K^\times \mid T - xT \not\subseteq T \cup -xT\},$$ and let $B = \langle -1, A \rangle$ be the subgroup of $K^\times$ generated by $-1$ and $A$.

**Remark 1.1.**

(i) If $x \in T$ then $0 \in T - xT$ while $0 \not\in T \cup -xT$. Thus $T \subseteq A$.
(ii) For $x \in K^\times$ one has $x \in A$ if and only if $x^{-1} \in A$.
(iii) If $x \in K^\times \setminus T$ and $1 - x \not\in T \cup -xT$ then $1 - x \in A$: indeed, $x \in T - (1-x)T$ but $x \not\in T \cup -(1-x)T$. 
Given a subgroup $S$ of $K^\times$ we denote $O^-(S) = (1 - T) \setminus S$.

**Lemma 1.2.** If $z, w \in O^-(B)$ then either $zw \in 1 - T$ or $1 - zw \in zT = wT$.

*Proof.* One has $-z, -w \not\in B$, so

$$1 - zw = (1 - z) + z(1 - w) \in T + zT \subseteq T \cup zT$$

$$1 - zw = (1 - w) + w(1 - z) \in T + wT \subseteq T \cup wT.$$

\[\square\]

**Proposition 1.3.** Suppose that there exist $a, b \in O^-(B)$ with $1 - ab \not\in T$. Then:

(a) $O^-(\langle B, a \rangle)O^-(\langle B, a \rangle) \subseteq 1 - T$;
(b) $A = T$;
(c) $T - a^2 T \not\subseteq T \cup a^2 T$.

*Proof.* (a) Let $H = \langle B, a \rangle$. Lemma 1.2 implies that $1 - ab \in aT = bT$, whence $b \in H$. Suppose that $0 \neq x, y \in O^-(H)$ but $xy \not\in 1 - T$. As $a, b \in H$, $x, y \not\in H$, and $T \subseteq H$, Lemma 1.2 implies that $ax, by \in 1 - T$. Furthermore, $ax, by \not\in H$, so $ax, by \in O^-(B)$. As $ay^{-1} \not\in H$, also $ay^{-1} \not\in A$. Hence one of the following cases holds:

**CASE (i):** $ay^{-1} \in 1 - T$. Then $ay^{-1} \in O^-(B)$ and $(ay^{-1})(by) = ab \not\in 1 - T$. By Lemma 1.2, $1 - ab \in ay^{-1}T$, contrary to $1 - ab \in aT$ and $y \not\in H$.

**CASE (ii):** $a^{-1}y \in 1 - T$. Then $a^{-1}y \in O^-(B)$ and $xy = (ax)(a^{-1}y) \not\in 1 - T$. By applying Lemma 1.2 twice we obtain $1 - xy \in xT \cap axT$, contrary to $a \not\in B$.

(b) By Remark 1.1 (i), $T \subseteq A$. Conversely, take $x \in A$. Suppose $x \not\in T$. After replacing $x$ by an appropriate element of $xT$, we may assume that $1 - x \not\in T \cup -xT$.

By Remark 1.1 (iii), $1 - x \in A \subseteq B$. Since $x \in B$ and $a \not\in B$ we have $xa, -(1 - x)a \not\in B$. In particular, $xa, -(1 - x)a \not\in A$. Therefore

$$1 - xa \in T - xaT \subseteq T \cup -xaT$$

$$1 - xa \in T + (1 - x)aT \subseteq T \cup (1 - x)aT.$$

By the choice of $x$, the cosets $-xaT$ and $(1 - x)aT$ are disjoint. Hence $1 - xa \in T$, so $xa \in O^-(B)$.

Since also $x^{-1} \in A$ (Remark 1.1 (ii)) and since $1 - x^{-1} \not\in T \cup -x^{-1}T$, the same argument (with $x, a$ replaced by $x^{-1}, b$) shows that $x^{-1}b \in O^-(B)$. As $ab = (xa)(x^{-1}b) \not\in 1 - T$, Lemma 1.2 implies that $1 - ab \in aT \cap xaT$. This contradicts $x \not\in T$.

(c) As already noted, $1 - ab \in aT = bT$ and $a \not\in T$. Hence $1 - ab \in T - a^2 T$ but $1 - ab \not\in T \cup a^2 T$.

Next we define a group $H$ as follows:

- If $O^-(B)O^-(B) \subseteq 1 - T$ then we take $H = B$;
- If $O^-(B)O^-(B) \not\subseteq 1 - T$ then we choose $a \in O^-(B)$ such that $aO^-(B) \not\subseteq 1 - T$ and set $H = \langle B, a \rangle$.

\[\square\]
Thus \( \pm T \leq \pm A \leq B \leq H \). We abbreviate \( O^- = O^-(H) \), and let
\[
O^+ = \{ x \in H \mid xO^- \subseteq O^- \}.
\]

**Proposition 1.4.**

(a) \( O^- O^- \subseteq 1 - T \).
(b) \( 1 - O^- \subseteq O^+ \).
(c) \( O^- O^- \subseteq 1 - O^+ \).
(d) \( (1 - O^+) \cap H \subseteq O^+ \).
(e) \((1 - O^+) \setminus H \subseteq O^- \).

**Proof.** (a) follows from Proposition 1.3 (a). For \( y \neq 1 \) let \( \tilde{y} = y/(y - 1) \). Then \( y \mapsto \tilde{y} \) maps \( K \setminus \{0, 1\} \) onto itself. Moreover, \( y \in O^- \) if and only if \( \tilde{y} \in O^- \). We use the identity
\[
1 - xy = (1 - (1 - x)\tilde{y})(1 - y),
\]
for \( y \neq 1 \).

(b) Take \( x \in 1 - O^- \) and \( y \in O^- \). By (a) and (a), \( 1 - xy \in (1 - O^- O^-)T \subseteq TT = T \). Since \( x \in T \leq H \) and \( y \notin H \) this implies \( xy \in O^- \). Conclude that \( x \in O^+ \).

(c) Let \( x, y \in O^- \). By (a) and (b),
\[
1 - xy \in (1 - (1 - O^-)O^-)(1 - O^-) \subseteq (1 - O^+ O^-)O^+ \subseteq (1 - O^-)O^+ \subseteq O^+ \subseteq O^+ \subseteq O^+.
\]

(d) Suppose that \( x \in (1 - O^+) \cap H \) and \( y \in O^- \). By (a),
\[
1 - xy \in (1 - O^+ O^-)(1 - O^-) \subseteq (1 - O^-)(1 - O^-) \subseteq TT = T.
\]
As \( xy \notin H \), this shows that \( xy \in O^- \), whence \( x \in O^+ \).

(e) If \( x \in (1 - O^+) \setminus H \) then \( x \notin A \), so \( 1 - x \in H \cap (T \cup -xT) = T \). Conclude that \( x \in O^- \). \[\square\]

**2. The valuation \( O \)**

Let \( A, H, O^-, O^+ \) be as in \( \S 1 \), and let \( O = O^- \cup O^+ \).

**Proposition 2.1.** \( O \) is a valuation ring on \( K \).

**Proof.** We apply (a)–(e) of Proposition 1.4.

By definition, \( O^+ O^- \subseteq O^- \) and \( O^+ O^+ \subseteq O^+ \). As \( O^- O^- \subseteq 1 - T \) also \( O^- O^- \setminus H \subseteq O^- \). Finally, \( O^- O^- \cap H \subseteq (1 - O^+) \cap H \subseteq O^+ \). Conclude that \( OO \subseteq O \).
Next we show that for every $0 \neq x \in K$ either $x \in O$ or $x^{-1} \in O$. Indeed, if $x \notin H$ then $x \notin A$, so either $1 - x \in T$ or $1 - x^{-1} \in T$. Thus either $x \in O^-$ or $x^{-1} \in O^-$ in this case. If $x \in H \setminus O^+$ then there exists $y \in O^-$ such that $xy \notin O^-$. By what we have just seen, $(xy)^{-1} \in O^-$. Consequently, $x^{-1} = (xy)^{-1}y \in O^+O^- \subseteq OO \subseteq O$, as desired. In particular, $\pm 1 \in O$.

As $1 - O^- \subseteq O^+$, $(1 - O^+) \cap H \subseteq O^+$, and $(1 - O^+) \setminus H \subseteq O^-$, we have $1 - O \subseteq O$.

For $0 \neq x, y \in O$ we show that $x + y \in O$. By symmetry we may assume that $-x^{-1}y \in O$. Then $1 + x^{-1}y \in 1 - O \subseteq O$. Therefore $x + y = x(1 + x^{-1}y) \in OO \subseteq O$.

The assertion follows. \hfill $\Box$

**Proposition 2.2.** $O^\times \subseteq H$.

*Proof.* Otherwise there exists $x \in O^\times \setminus H$. In particular, $x \in O^-$, so $1 - x \in T$. Hence $1 - x^{-1} \in -x^{-1}T$, and therefore $1 - x^{-1} \notin T$. Conclude that $x^{-1} \notin O^-$, contrary to $x \in O^\times \setminus H$. \hfill $\Box$

We denote the maximal ideal of the valuation $O$ by $m$.

**Proposition 2.3.** $1 - m \subseteq T$.

*Proof.* By definition, $1 - O^- \subseteq T$. So let $x \in O^+ \cap m$; we show that $x \in 1 - T$. As $x^{-1} \in H \setminus O^+$ we have $x^{-1}y \notin O^-$ for some $y \in O^-$. Since $x^{-1}y \notin H$ this implies $x^{-1}y \notin O$. Hence $xy^{-1} \in O \setminus H = O^-$. By Proposition 1.4 (a), $x = (xy^{-1})y \in O^-O^- \subseteq 1 - T$. \hfill $\Box$

Fix a prime number $p$.

**Lemma 2.4.** If $1 - (m \setminus H) \subseteq (K^\times)^p$ then $1 - m \subseteq (K^\times)^p$.

*Proof.* Take $m \in m \cap H$. Since $m^{-1} \notin O^+$ there exists $y \in O^-$ such that $m^{-1}y \notin O^-$. As $m^{-1}y \notin H$ this means that $m^{-1}y \notin O$. Then $y, y^{-1}m \in O \setminus H \subseteq m$, by Proposition 2.2. By Proposition 2.3, $1 + y^{-1}m - m \in 1 - m \subseteq T \subseteq H$. Since $y \in m \setminus H$ this implies $y + m - ym \in m \setminus H$. By assumption, $(1 - y)(1 - m) = 1 - (y + m - ym) \in (K^\times)^p$. Also, $1 - y \in 1 - (m \setminus H) \subseteq (K^\times)^p$. Hence $1 - m \in (K^\times)^p$. \hfill $\Box$

**Corollary 2.5.** Suppose that for every $x \in K^\times \setminus H$ and every $y \in T \setminus (K^\times)^p$ one has $\{x, y\} \neq \emptyset$. Then $1 - m \subseteq (K^\times)^p$.

*Proof.* Let $x \in m \setminus H$. Then $x \in O^-$, so $1 - x \in T$. As $\{x, 1 - x\} = 0$ we have $1 - x \in (K^\times)^p$. Now apply Lemma 2.4. \hfill $\Box$

**Lemma 2.6.** Suppose that $p \in m$ and $1 - m \subseteq (K^\times)^p$. Then $m \setminus pm \subseteq (K^\times)^p$.

*Proof.* Given $x \in 1 - m$, we may write $x = y^p$ with $y \in O^\times$. The residues $\bar{x}, \bar{y}$ then satisfy $\bar{1} = \bar{x} = \bar{y}^p$. Since $\text{char } O/m = p$, necessarily $\bar{y} = \bar{1}$, i.e., $y \in 1 - m$. Thus $1 - m = (1 - m)^p$. \hfill $\Box$
Now let \( a \in m \setminus pm \). By what we have just seen, there exists \( b \in m \) such that \( 1 + a = (1 + b)^p \in 1 + b^p - pm \). Since \( a \notin pm \) this implies \( a \in b^p(1 - m) \subseteq (K^\times)^p \).

From now on we assume that \((K^\times)^p \leq T\).

**Corollary 2.7.** If \( 1 - m \subseteq (K^\times)^p \) and \( \text{char } K \neq p \) then \( p \notin m \).

**Proof.** Suppose \( p \in m \). Lemma 2.6 then shows that \( p \in m \setminus pm \subseteq (K^\times)^p \leq H \). Since \( p^{-1} \notin O^+ \), there exists \( a \in O^- \) such that \( p^{-1}a \notin O \). By Proposition 2.2, \( O^\times \leq H \), so \( a \in m \setminus pm \). Lemma 2.6 once again gives \( a \in (K^\times)^p \leq H \), a contradiction. \( \Box \)

## 3. The size of \( H \)

In order to prove the non-triviality of \( O \) in various situations one needs an estimate on the size of \( (H : T) \). This is obtained in Corollary 3.3 below. For its proof we need two technical facts.

**Lemma 3.1.** Let \( \Delta \) be an elementary abelian \( p \)-group and let \( \omega: \Delta \to \mathbb{Z}/p \) be a map such that:

(i) if \( a, b \in \Delta \) are \( \mathbb{F}_p \)-linearly independent and at least one of \( \omega(a), \omega(b) \) is non-zero then \( \omega(ab) = \omega(a)\omega(b) \);

(ii) there exist \( \mathbb{F}_p \)-linearly independent \( a, b \in \Delta \) such that \( \omega(a), \omega(b) \neq 0 \).

Then \( 1 \in \text{Im}(\omega) \).

**Proof.** Take \( a, b \) as in (ii). From (i) we obtain inductively that

\[
\omega(a^ib) = \omega(a)^i\omega(b) \neq 0,
\]

\( i = 1, \ldots, p-1 \). Since \( (\mathbb{Z}/p)^\times \) has order \( p - 1 \) this gives in particular \( \omega(a^{p-1}b) = \omega(b) \). Moreover, \( \omega(a^{p-1})\omega(b) = \omega(a^{p-1}b) \) by (i). Hence \( \omega(a^{p-1}) = 1 \).

**Proposition 3.2.** Assume that for every \( x \in K^\times \setminus T \) one has \( 1 - x \in \bigcup_{i=0}^{p-1} x^iT \). Suppose that the cosets of \( a, b \in K^\times \) in \( K^\times / T \) are \( \mathbb{F}_p \)-linearly independent. Then \( 1 - a \in T \cup aT \) or \( 1 - b \in T \cup bT \).

**Proof.** For every \( x \in K^\times \setminus T \) there exists by assumption a unique \( 0 \leq i \leq p - 1 \) such that \( 1 - x \in x^iT \). When \( i \neq 0 \) let \( 0 \leq \omega(x) \leq p - 1 \) be the unique integer such that \( w(x) \equiv 1 - i^{-1} \mod p \). When \( i = 0 \) we set \( \omega(x) = 0 \). Note that \( \omega(x) = 0 \) if and only if \( 1 - x \in T \cup xT \). Also, \( 1 \notin \text{Im}(\omega) \).

We apply Lemma 3.1 with \( \Delta = K^\times / T \). It suffices to show that if the cosets of \( a, b \in K^\times \) in \( K^\times / T \) are \( \mathbb{F}_p \)-linearly independent and at least one of \( \omega(a), \omega(b) \) is non-zero then \( \omega(ab) \equiv \omega(a)\omega(b) \mod p \).

Take \( 0 \leq i, j, r \leq p - 1 \) such that \( 1 - a \in a^iT, 1 - b \in b^iT, 1 - ab \in (ab)^rT \). The assumptions imply that \( i \neq 1 \) or \( j \neq 0 \). Hence

\[
1 - ab = (1 - a) + a(1 - b) \in a^i(T - a^{1-i}b^jT) \subseteq \bigcup_{k=0}^{p-1} a^i(a^{1-i}b^j)^kT.
\]
Therefore, $(ab)^rT \cap a^i(a^{1-i}b^j)^kT \neq \emptyset$ for some $0 \leq k \leq p-1$. Since $a,b$ are independent modulo $T$ one has $r \equiv i + (1-i)k \equiv jk \pmod{p}$. Then $r(i+j-1) \equiv jk(i+j-1) \equiv ij \pmod{p}$.

If $r \neq 0$ then also $i,j \neq 0$ and $1-r^{-1} \equiv (1-i^{-1})(1-j^{-1}) \pmod{p}$; i.e., $\omega(ab) \equiv \omega(a)\omega(b) \pmod{p}$, as required.

If $r = 0$ then either $i = 0$ or $j = 0$, so either $\omega(ab) = \omega(a) = 0$ or $\omega(ab) = \omega(b) = 0$, and we are done again. \qed

**Corollary 3.3.** Suppose that $-1 \in T$ and that for every $x \in K^\times \setminus T$ one has $1-x \in \bigcup_{r=0}^{p-1} x^r T$. Then $(H : T)|p$.

**Proof.** By Proposition 3.2, $(B : T)|p$. Now if $O^{-}(B)O^{-}(B) \subseteq 1-T$ then $H = B$, so $(H : T)|p$. If $O^{-}(B)O^{-}(B) \not\subseteq 1-T$ then $A = T$, by Proposition 2.2(b); hence $B = T$, so $(H : T) = (H : B) = p$. \qed

**4. The main result**

By combining the previous results we now obtain:

**Theorem 4.1.** Let $K$ be a field and let $(K^\times)^p \leq T \leq K^\times$ be an intermediate group. Suppose that:

(i) if $x \in K^\times \setminus T$ and $y \in T \setminus K^p$ then $\{x,y\} \neq 0$;

(ii) if $-1 \in T$ and if the cosets of $x, y \in K^\times$ in $K^\times/T$ are $\mathbb{F}_p$-linearly independent then $\{x,y\} \neq 0$.

Then $O$ above is a valuation ring. Furthermore, let $m, \tilde{K},$ and $\Gamma$, be its maximal ideal, residue field, and value group, respectively. Then:

(a) $1-m \subseteq (K^\times)^p$;

(b) if char $K \neq p$ then also char $\tilde{K}$, $\neq p$;

(c) if $-1 \in T$ then $(O^\times T : T) \leq p$;

(d) if $-1 \not\in T$ then $(O^\times B : B) \leq 2$;

(e) if $-1 \in T$ then $(\Gamma : p\Gamma) \geq (K^\times : T)/p$;

(f) if $-1 \not\in T$ then $(\Gamma : 2\Gamma) \geq (K^\times : B)/2$;

(g) if $\tilde{K} = \tilde{K}^p$ and $-1 \in T$ then $(\Gamma : p\Gamma) \geq (K^\times : T)$;

(h) if $\tilde{K} = \tilde{K}^p$ and $-1 \not\in T$ then $(\Gamma : 2\Gamma) \geq (K^\times : B)$.

**Proof.** By Proposition 2.1, $O$ is a valuation ring. Assumption (i) and Corollary 2.5 prove (a). Corollary 2.7 proves (b). By Proposition 2.2, $O^\times \subseteq H$.

Suppose that $-1 \in T$. For every $x \in K^\times \setminus T$ one has $\{x,1-x\} = 0$, so by (ii), $1-x \in \bigcup_{r=0}^{p-1} x^r T$. Corollary 3.3 now gives $(H : T)|p$, whence (c). Furthermore,

$$\left(\Gamma : p\Gamma\right) = (K^\times : O^\times (K^\times)^p) \geq (K^\times : H) \geq (K^\times : T)/p,$$

proving (e).

To prove (g), suppose that $\tilde{K} = \tilde{K}^p$. By Lemma 2.3, $O^\times = (1-m)(O^\times)^p \leq T$. If $H = O^\times (K^\times)^p$ then $H \leq T$; hence $H = T$, so $(\Gamma : p\Gamma) = (K^\times : T)$, and we
are done in this case. On the other hand, if $H > O^x(K^x)p$ then the inequalities above show that $(\Gamma : p\Gamma) > (K^x : T)/p$. Thus (g) holds in this case as well.

When $-1 \not\in T$ we have $p = 2$ and $(H : B) \leq 2$. Assertions (d),(f), and (h) are then proven similarly to (c), (e), and (g).

\textbf{Remark 4.2.} If $p = 2$ and $-1 \in T$ then assumption (ii) of Theorem 4.1 implies that for every $x \in K^x \setminus T$ one has $1 - x \in T \cup xT$. Hence $T = A = B$. This shows that the Main Theorem as stated in the introduction is a special case of Theorem 4.1.

\textbf{Example 4.3.} Let $p$ be a prime number and let $K$ be a field. Suppose that the canonical symbolic map induces an isomorphism $\wedge^2 (K^x/p) \cong K^3_2(K)/p$. Then (i) and (ii) of Theorem 4.1 hold with $T = (K^x)p$. Hence $K$ possesses a valuation satisfying (a)–(g) above.

In particular, this happens for $K = \mathbb{F}_l((t_1)) \cdots ((t_n))$, where $l$ is a prime number such that $p|l - 1$ and such that $4|l - 1$ if $p = 2$ [Wd, \S 2]. Then $\mathbb{F}_l$ contains a primitive $p$th root of unity, and $(K^x : (K^x)p) = p^{n+1}$ [Wd, Lemma 1.4]. Moreover, the value group $\Gamma$ of every valuation on $K$ satisfies $(\Gamma : p\Gamma) \leq p^n$. This shows that condition (e) of Theorem 4.1 cannot be strengthened to $(\Gamma : p\Gamma) \geq (K^x : T)$.

We conclude by proving a criterion for the existence of valuations having arbitrary residue characteristic:

\textbf{Theorem 4.4.} Let $p$ be an odd prime and let $K$ be a field. The following conditions are equivalent:

(a) There exists a valuation $v$ on $K$ with non-$p$-divisible value group;

(b) There exists an intermediate group $(K^x)^p \leq T < K^x$ such that for every $x \in K^x \setminus T$ one has $1 - x \in T \cup xT$.

\textbf{Proof.} (a)⇒(b): Let $T = v^{-1}(p\Gamma)$ and take $x \in K^x$. When $v(x) = 0$ (resp., $v(x) > 0$, $v(x) < 0$) we have $x \in T$ (resp., $1 - x \in T$, $1 - x \in xT$).

(b)⇒(a): We take $T$ as in (b). Since $p \neq 2$ we have $-1 \in T$, so $B = A = T$. Moreover, if $a \not\in T$ then $a^2 \not\in T$, so $T - a^2T \subseteq T \cup a^2T$. By Proposition 1.3(c), $O^{-}(T)O^{-}(T) \subseteq 1 - T$, whence $H = T$. Propositions 2.1 and 2.2 give rise to a valuation ring $O$ such that $O^x \leq T$. Its value group $\Gamma$ satisfies $(\Gamma : p\Gamma) = (K^x : O^x (K^x)p) \geq (K^x : T) > 1$. □

\textbf{References}


