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THE LCM-LATTICE IN MONOMIAL RESOLUTIONS

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1. Introduction

Describing the properties of the minimal free resolution of a monomial ideal I is a difficult problem posed in the early 1960's. The main directions of progress on this problem were:

- constructing the minimal free resolutions of special monomial ideals, cf., [AHH, BPS]
- constructing non-minimal free resolutions; for example, Taylor's resolution (cf., [Ei, p. 439]) and the cellular resolutions
- the Stanley-Reisner theory for computing the Betti numbers of I by simplicial complexes; it has a long tradition and has led to important results in combinatorics and commutative algebra [St].

Working with special monomial ideals or with non-minimal resolutions simplifies the problem significantly because it removes the main difficulty (finding minimal generators of the homology in general). In this paper we obtain results on the minimal free resolution of an arbitrary monomial ideal. We introduce a new approach inspired by the topological theory of subspace arrangements. Some of the best results in this theory show that the cohomology algebra of the complement of a complex subspace arrangement is independent of the geometric position of the subspaces and is determined by the structure of a certain lattice. Inspired by this, we introduce the lcm-lattice of a monomial ideal and show how its structure relates to the Betti numbers, the maps in the minimal free resolution, and the structure of the Tor-algebra for the ideal. This is outlined more precisely below:

The intersection lattice $\mathcal{L}$ of a complex subspace arrangement plays a significant role in describing the cohomology of the complement $\mathcal{M}$ of the arrangement:

- the Goresky-MacPherson Formula [GM, III.1.5. Theorem A] expresses the dimensions of the cohomology groups of $\mathcal{M}$ in terms of the dimensions of the homology groups of $\mathcal{L}$.
- $\mathcal{L}$ together with the additional data of the codimensions of the intersection spaces determine the algebra structure of the rational cohomology of $\mathcal{M}$ [DP, Yu]. (Throughout we use the expression "the lattice of A determines the structure of $X(A)$" in the sense that objects A and B with isomorphic lattices

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yield isomorphic structures of $X(A)$ and $X(B)$, and we do not mean that one can obtain explicit formulas/descriptions of the structures.)

In this paper we introduce the lcm-lattice of a monomial ideal and show that it plays the same role as above in describing the homology of the ideal. Let $S = k[x_1, \dots , x_n]$ be the polynomial ring over a field k, and I a monomial ideal minimally generated by monomials $m_1, \dots , m_d$. We denote by L_I the lattice with elements labeled by the least common multiples of $m_1, \dots , m_d$ ordered by divisibility; in particular the atoms in L_I are $m_1, \dots , m_d$, the maximal element is $\mathrm{lcm}(m_1, \dots , m_d)$, and the minimal element is 1 regarded as the lcm of the empty set. The least common multiple of elements in L_I is their join (i.e., their least common upper bound in the poset L_I). We call L_I the *lcm-lattice* of I. Let $\mathcal{L}_I$ be the lattice L_I without the labeling. We obtain the following results:

- Theorem 2.1 expresses the Betti numbers of S/I in terms of the dimensions of the homology groups of $\mathcal{L}_I$.
- Theorem 3.3 shows that $\mathcal{L}_I$ determines the minimal free resolution of S/I up to relabeling (see 3.2 for the definition of relabeling).
- Theorem 3.5 shows that $\mathcal{L}_I$ together with the additional data, which pairs of minimal monomial generators are relatively prime, determine the algebra structure on $\mathrm{Tor}_*^S(S/I, k)$ and Golodness.

An application of our approach is given in [GPW]. There we relate the (co)homological properties of two objects: square-free monomial ideals and real coordinate subspace arrangements. In particular, we discover the equivalence of results, which on the one hand were proved for subspace arrangements by Björner and on the other hand were recently proved for monomial ideals by Eagon-Reiner and Terai.

2. Multigraded Betti numbers

In this section we study the multigraded Betti numbers of a monomial ideal using the lcm-lattice.

Consider the polynomial ring $S = k[x_1, \dots , x_n]$ over a field k as $\mathbf{N}^n$-graded by letting $\deg(x_i)$ be the i^{th} standard basis vector in $\mathbf{R}^n$. Let I be a monomial ideal minimally generated by monomials $m_1, \dots , m_d$. The ideal I and the minimal free resolution of S/I over S are $\mathbf{N}^n$-graded. Therefore we have $\mathbf{N}^n$-graded Betti numbers

$$b_{i,\mathbf{x}^\alpha}(S/I) \,=\, \dim_k \mathrm{Tor}_{i,\alpha}^S(S/I, k),$$

for $i \geq 0$, $\alpha = (\alpha_1, \dots , \alpha_n) \in \mathbf{N}^n$, and $\mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

Let L be a lattice with minimal element $\hat{0}$ and $p \in L$. We write $(\hat{0}, p)_L$ for the *open interval* $\{q \in L \mid \hat{0} < q < p\}$ in L. In particular, for $m \in L_I$ we denote by $(\hat{0}, m)_{L_I}$ the open lower interval in L_I below m. We implicitly think of a poset P (such as $(\hat{0}, m)_{L_I}$) as a topological space by considering its *order complex* $\Delta(P)$, that is the abstract simplicial complex whose faces are the chains in the poset.

Taylor's resolution (cf., [Ei, p. 439]) shows that $b_{i,m}(S/I) = 0$ if $m \notin L_I$.

Theorem 2.1. *For $i \geq 1$ and $m \in L_I$ we have*

$$b_{i,m}(S/I) \;=\; \dim \widetilde{\mathrm{H}}_{i-2}\big((\hat{0}, m)_{L_I}; k\big)\,.$$

Forgetting about the multigrading in Theorem 2.1 we obtain for $i \geq 1$ the formula

$$b_i(S/I) \;=\; \sum_{\substack{m \in L_I \\ m \neq \hat{0}}} \dim \widetilde{\mathrm{H}}_{i-2}\big((\hat{0}, m)_{L_I}; k\big)\,.$$

This formula is an analogue of the Goresky-MacPherson Formula [GM, III.1.5. Theorem A], which expresses the dimensions of the cohomology groups of the complement of a subspace arrangement in terms of the dimensions of the homology groups of the lower intervals in the intersection lattice.

In order to compute the Betti numbers one can use the lcm-lattice built on any set of monomial generators of I; this follows from the following result:

Lemma 2.2. *Let L' be a lattice of the least common multiples of monomials generating I. For $i \geq 0, m \in L'$ we have*

$$\dim \widetilde{\mathrm{H}}_i\big((\hat{0}, m)_{L'}; k\big) \;=\; \begin{cases} \dim \widetilde{\mathrm{H}}_i\big((\hat{0}, m)_{L_I}; k\big)\,, & \textit{if } m \in L_I \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. The atoms in L' are $m_1, \ldots, m_d$. Thus L_I is the join sublattice of L' generated by its atoms. By [Bj, Corollary 10.12], for any $m \in L_I$ the open interval $(\hat{0}, m)_{L'}$ in L' is homotopy equivalent to the open interval $(\hat{0}, m)_{L_I}$ in L_I.

Suppose that $m \in L' \setminus L_I$. Consider the map $f : (\hat{0}, m)_{L'} \longrightarrow (\hat{0}, m)_{L'}$ which maps a monomial to the join of all atoms dividing it. As f is order preserving and $f(l) \leq l$ for each $l \in (\hat{0}, m)_{L'}$, we have by [Bj, Corollary 10.12] that the image $(\hat{0}, f(m)]_{L_I} = (\hat{0}, f(m)) \cup \{f(m)\}$ is homotopic to $(\hat{0}, m)_{L'}$. The assertion follows as the order complex of $(\hat{0}, f(m)]_{L_I}$ is a cone over $f(m)$ and hence contractible. $\square$

Proof of Theorem 2.1. Let Γ_m be the simplicial complex with faces

$$\Big\{ J \subseteq [n] \;\Big|\; \frac{m}{\prod_{i \in J} x_i} \in I \Big\}\,.$$

By [BH, Proposition 1.1] we have $b_{i,m}(S/I) = \dim \widetilde{\mathrm{H}}_{i-2}\big(\Gamma_m; k\big)$. Denote by $X_{\preceq m}$ the full simplex with vertices the minimal monomial generators of I which divide m. Let $X_{\prec m}$ be the subcomplex of $X_{\preceq m}$ obtained by deleting each face whose vertices have least common multiple equal to m. It is easy to see (cf., [BS, proof of 1.13]) that $X_{\preceq m}$ and Γ_m are homotopic. Thus, $b_{i,m}(S/I) \;=\; \dim \widetilde{\mathrm{H}}_{i-2}\big(X_{\prec m}; k\big)\,.$

On the other hand, the set C_m of the minimal monomial generators of I which divide m forms a crosscut of the poset $(\hat{0}, m)_{L_I}$. Its crosscut complex has faces the subsets of C_m whose lcm is strictly smaller than m, so it coincides with the complex $X_{\prec m}$. By [Bj, Theorem 10.8] the crosscut complex $X_{\prec m}$ is homotopic to the order complex of $(\hat{0}, m)_{L_I}$. □

Below we relate the lcm-lattice to the Stanley-Reisner theory [St]. For a simplicial complex Δ let I_Δ be the *Stanley-Reisner* monomial ideal associated to Δ, that is

$$I_\Delta \;=\; \Big(\{\, x_{j_1} \cdots x_{j_p} \,|\, \{j_1, \dots, j_p\} \notin \Delta\}\Big)\,.$$

The *Alexander dual complex* $\Delta^\vee$ of Δ is

$$\Delta^\vee \;=\; \{\, [n] \setminus \{j_1, \dots, j_p\} \,|\, \{j_1, \dots, j_p\} \notin \Delta\,\}\,.$$

Recall, that any monomial ideal I which is generated by square-free monomials is a Stanley-Reisner ideal $I = I_\Delta$ for some simplicial complex Δ.

The *support* $\operatorname{supp}(m)$ of a monomial m is $\{i \mid x_i \text{ divides } m\}$. The *proper part* L° of a lattice L, with minimal element $\hat{0}$ and maximal element $\hat{1}$, is $L \setminus \{\hat{0}, \hat{1}\}$.

Proposition 2.3.

(a) *For an ideal I in S let I_{pol} be its polarization. The lattices L_I and $L_{I_{pol}}$ are isomorphic.*

(b) *Let Δ be a simplicial complex. Let $L_{\Delta^\vee}$ be the lattice of all non-empty intersections of the facets of $\Delta^\vee$ ordered by reverse inclusion, and enlarged by an additional minimal element $\hat{0}$ and an additional maximal element $\hat{1}$. The lattices L_{I_Δ} and $L_{\Delta^\vee}$ are isomorphic.*

(c) *Under the assumptions in* (b), *we have that $\Delta^\vee$ is homotopy equivalent to the proper part $L^\circ_{I_\Delta}$ of L_{I_Δ}.*

Proof. It is easy to see that (a) holds. Part (c) follows from (b) because $L^\circ_{I_\Delta} = (\hat{0}, x_1 \cdots x_n)_{L_{I_\Delta}}$. We will prove (b). For $J \subseteq [n]$ we denote by $\mathbf{x}_J$ the monomial $\prod_{i \in J} x_i$. Let $\sigma_1, \dots, \sigma_d$ be the facets (i.e., maximal faces) of $\Delta^\vee$. They correspond to the minimal monomial generators of I_Δ via the correspondence

$$\sigma_i \in \Delta^\vee \quad \longleftrightarrow \quad \frac{x_1 \cdots x_n}{\mathbf{x}_{\sigma_i}} \in I_\Delta.$$

Furthermore, for any $\emptyset \neq A \subseteq [d]$ we consider the correspondence

$$\bigcap_{i \in A} \sigma_i \quad \longleftrightarrow \quad \frac{x_1 \cdots x_n}{\bigcap_{i \in A} \operatorname{supp}(\mathbf{x}_{\sigma_i})} \;=\; \operatorname{lcm}\left(\frac{x_1 \cdots x_n}{\mathbf{x}_{\sigma_i}} \,\middle|\, i \in A\right).$$

The lattices $L_{\Delta^\vee}$ and L_{I_Δ} are isomorphic via the above correspondence. □

3. Homological properties of a monomial ideal determined by the poset structure of its lcm-lattice

In this section we prove that if I and I' are two monomial ideals in polynomial rings S and S' respectively, and L_I is isomorphic to $L_{I'}$, then S/I and S'/I' have the same homological behavior.

We start with the observation that the poset structure of the lcm-lattice determines the total Betti numbers of S/I and its projective dimension; this is an immediate consequence of Theorem 2.1.

Observation 3.1. *If I and I' are two monomial ideals in polynomial rings S and S' respectively, and L_I is isomorphic to $L_{I'}$, then S/I and S'/I' have the same total Betti numbers.*

Proof. The following is a very quick proof that does not use Theorem 2.1. Denote by $\mathbf{T}$ and $\mathbf{T}'$ the Taylor's resolutions of S/I and S'/I' respectively; note that $L_I \cong L_{I'}$ implies $\mathbf{T} \otimes k \cong \mathbf{T}' \otimes k$. □

Next we show that the poset structure of the lcm-lattice determines the minimal free resolution up to relabeling. The procedure of relabeling is as follows:

Construction 3.2. *(Relabeling)*
Let I and I' be two monomial ideals in polynomial rings S and S' respectively. Let $f : L_I \to L_{I'}$ be a map which is a bijection on the atoms and preserves joins. Note that this implies f is a surjection.

Relabeling an Lcm-lattice: If f is an isomorphism, then we can think of $L_{I'}$ as obtained from L_I by relabeling the lattice elements with new monomials.

Suppose that $\mathbf{P}$ is a bounded complex of free S-modules that is homologically and $\mathbf{N}^n$-graded, with homogeneous differential ∂, and such that the free modules in $\mathbf{P}$ have generators with multidegrees in L_I. Using the map f we will construct a complex $f(\mathbf{P})$. We will do this by using f to "relabel" the $\mathbf{N}^n$-graded modules and the homogeneous maps between them.

Relabeling Free Modules: We denote by $S(\frac{1}{m})$ the free S module with a generator whose degree is the exponent of the monomial m; we say that the generator of this module has degree m. For any sequence $(a_m)_{m\in L_I}$ of natural numbers set $f\Big(\oplus_{m\in L_I} S(\frac{1}{m})^{a_m}\Big) = \oplus_{m\in L_I} S\big(\frac{1}{f(m)}\big)^{a_m}$. If $T = \bigoplus_{i\in\mathbf{N}} T_i$ is an $\mathbf{N}$-graded free S-module such that all T_i's are $\mathbf{N}^n$-graded free S-modules with generators in multidegrees lying in L_I, then define $f(T)$ to be the $\mathbf{N}$-graded free S-module $\bigoplus_{i\in\mathbf{N}} f(T_i)$.

Relabeling Homogeneous Maps: Let $m, \ell \in L_I$ and $\alpha : S(\frac{1}{m}) \to S(\frac{1}{\ell})$ be a homogeneous map. Then α acts as multiplication by $c \cdot \frac{m}{\ell}$ for some $c \in k$. We set $f(\alpha)$ to be the map $f(\alpha) : S'(\frac{1}{f(m)}) \to S'(\frac{1}{f(\ell)})$ which acts as multiplication by $c \cdot \frac{f(m)}{f(\ell)}$. If $\alpha_i : S(\frac{1}{m}) \to S(\frac{1}{l_i})$ are homogeneous maps, all $l_i \in L_I$, and $\beta = \oplus_i \alpha_i : S(\frac{1}{m}) \to \oplus_i S(\frac{1}{l_i})$, then we set $f(\beta) = \oplus_i f(\alpha_i)$. Finally, if $\gamma : R = \oplus_j S(\frac{1}{m_j}) \to \bar{R} = \oplus_i S(\frac{1}{l_i})$ is a homogeneous map and all $m_j, l_i \in L_I$, then we can write $\gamma = \sum_j \beta_j$ with $\beta_j : S(\frac{1}{m_j}) \to \oplus_i S(\frac{1}{l_i})$ homogeneous; we set $f(\gamma) = \sum_j f(\beta_j)$. Thus $f(\gamma)$ is a homogeneous map from $\oplus_j S'(\frac{1}{f(m_j)})$ to $\oplus_i S'(\frac{1}{f(l_i)})$. In the definition of $f(\gamma)$ we used fixed bases of R and $\bar{R}$. It is important that defining the map $f(\gamma)$ is invariant under change of bases in R and $\bar{R}$ that keep the matrix of γ homogeneous.

Now the complex $f(\mathbf{P})$ is constructed as follows:

- $f(\mathbf{P})$ as an S'-module is obtained by applying f to the homologically and $\mathbf{N}^n$ graded S-module $\mathbf{P}$;
- The differential of $f(\mathbf{P})$ is $f(\partial)$ defined using any basis of the free modules in which the matrices of ∂ are homogeneous.

Example 3.4(d) illustrates how the procedure of relabeling works. □

Theorem 3.3. *Let I and I' be two monomial ideals in polynomial rings S and S' respectively. Let $f : L_I \to L_{I'}$ be a map which is a bijection on the atoms and preserves joins. Denote by $\mathbf{F}_I$ the minimal free resolution of S/I. Then $f(\mathbf{F}_I)$ is defined and is a free resolution of S'/I'. If f is an isomorphism of lattices then $f(\mathbf{F}_I)$ is the minimal free resolution of S'/I'.*

Proof. Assume I is minimally generated by the monomials $m_1, \ldots, m_d$. Let $\mathbf{E}$ be the exterior algebra on d generators $e_1, \ldots, e_d$, and $\mathbf{T}_I = S \otimes \mathbf{E}$. For a subset J of $\{1, \ldots, d\}$ we set $m_J = \operatorname{lcm}(m_i \mid i \in J)$. Denote by "$\vee$" the join operation in a lattice, and note that $m_J = \bigvee_{i \in J} m_i$ in the lattice L_I. The *Taylor resolution* of S/I is the module $\mathbf{T}_I$ equipped with the differential

$$\partial(e_{j_1} \wedge \cdots \wedge e_{j_r}) \ = \ \sum\nolimits_{1 \le i \le r} (-1)^{i+1} \cdot \frac{m_J}{m_{J \setminus i}} \cdot e_{j_1} \wedge \cdots \wedge \widehat{e}_{j_i} \wedge \cdots \wedge e_{j_r} \ ,$$

where $J = \{j_1, \ldots, j_r\}$ and $\widehat{e}_{j_i}$ means that e_{j_i} is omitted in the product. This resolution is $\mathbf{N}^n$-graded. Since f preserves joins, we have that

$$f(m_J) = f\big(\bigvee_{i \in J} m_i\big) = \bigvee_{i \in J} f(m_i) = \operatorname{lcm}\big(f(m_i) \mid i \in J\big) .$$

Our crucial observation is that $f(\mathbf{T}_I) = \mathbf{T}_{I'}$.

By [Ei, Theorem 20.2], $\mathbf{T}_I = \mathbf{F}_I \oplus \mathbf{P}_I$ as complexes, where $\mathbf{P}_I$ is a direct sum of exact complexes of the form $0 \to S \to S \to 0$. Therefore, $f(\mathbf{T}_I) = f(\mathbf{F}_I) \oplus f(\mathbf{P}_I)$ and $f(\mathbf{P}_I)$ is a direct sum of exact complexes of the form $0 \to S' \to S' \to 0$. Since $f(\mathbf{P}_I)$ is exact and $f(\mathbf{T}_I) = \mathbf{T}_{I'}$ is exact except at homological degree zero, it follows that $f(\mathbf{F}_I)$ is exact except at homological degree zero. As f is a bijection on the atoms, we see that the homology of $f(\mathbf{F}_I)$ is S'/I'. Therefore, $f(\mathbf{F}_I)$ is a free resolution of S'/I'.

Let m be a monomial entry in the matrices of the differential in Taylor's resolution $\mathbf{T}_I$ of S/I. Denote by l the corresponding monomial entry in the matrices of the differential in $f(\mathbf{T}_I) = \mathbf{T}_{I'}$. Although 0 is not a monomial, we allow here $m = l = 0$. There exists a set $J \subseteq [d]$ and a $j \in J$ such that $m = \frac{m_J}{m_{J \setminus j}}$. Suppose that f is an isomorphism of lattices. Then $m_J = m_{J \setminus j}$ if and only if $f(m_J) = f(m_{J \setminus j})$. Hence we have that $m \notin k^*$ implies $l \notin k^*$. Since $\mathbf{F}_I$ is minimal, it follows that $\partial\big(f(\mathbf{F}_I)\big) = f\big(\partial(\mathbf{F}_I)\big) \subseteq \mathfrak{m}' f(\mathbf{F}_I)$, where $\mathfrak{m}'$ is the irrelevant maximal ideal in S'. Thus $f(\mathbf{F}_I)$ is minimal. □

Examples 3.4.

(a) The method of degenerating of exponents introduced in [BPS, Section 4] is a particular case of relabeling and Theorem 3.3. This method provides a non-minimal (but usually small) resolution.

(b) Let I be a monomial ideal and I' be its radical. Define a map f on L_I by mapping each monomial to its radical. This map preserves joins. Theorem 3.3 (with a minor modification) implies that relabeling the minimal free resolution of S/I provides a non-minimal free resolution of S/I'; this was shown using Gröbner basis methods in [Ra]. In particular, the Betti numbers of S/I are bigger than those of $S/\mathrm{rad}(I)$.

(c) Let I' be the polarization of I. It is known (see [Fr]), that the minimal free resolution of S/I is obtained from the minimal free resolution of S'/I' by depolarizing (the proof consists of showing that depolarizing is equivalent to factoring out a regular sequence). The above fact immediately follows from Theorem 3.3 as depolarizing defines an isomorphism $f:\ L_{I'} \to L_I$.

(d) This is a short explicit example demonstrating how relabeling works. Let $S = k[x, y]$, $I = (x^2, xy, y^2)$ and $S' = k[u, v, w]$, $I' = (uv, vw, uw)$. Define $f:\ L_I \to L_{I'}$ by $f(x^2) = uv$, $f(xy) = vw$, $f(y^2) = uw$, $f(x^2y) = uvw$, $f(xy^2) = uvw$, $f(x^2y^2) = uvw$. Then f is a bijection on the atoms and preserves joins; however f is not an isomorphism. The minimal free resolution $\mathbf{F}_I$ of S/I is:

$$\mathbf{F}_I:\quad 0 \to S^2 \xrightarrow{\begin{pmatrix} y & 0 \\ -x & y \\ 0 & -x \end{pmatrix}} S^3 \xrightarrow{(x^2 \quad xy \quad y^2)} S \to 0\,.$$

The free resolution $f(\mathbf{F}_I)$ obtained by applying Theorem 3.3 is:

$$\mathbf{F}_{I'}:\quad 0 \to S'^2 \xrightarrow{\begin{pmatrix} w & 0 \\ -u & u \\ 0 & -v \end{pmatrix}} S'^3 \xrightarrow{(uv \quad vw \quad uw)} S' \to 0\,.$$

□

Next we show that the poset structure of the lcm-lattice, together with the data which pairs of minimal monomial generators are relatively prime, determine the structure of the Tor algebra.

Theorem 3.5. *Let I and I' be monomial ideals over the polynomial rings $S = k[x_1, \dots, x_n]$ and $S' = k[x'_1, \dots, x'_{n'}]$ respectively. Suppose that $f:\ L_I \to L_{I'}$ is an isomorphism of lattices. Assume that for all $m, \ell \in L_I$ we have*

$$\gcd(m, \ell) = 1 \quad \Longleftrightarrow \quad \gcd(f(m), f(\ell)) = 1\ . \tag{3.6}$$

Then the following properties hold:

(a) *$Tor^S_*(S/I, k)$ and $Tor^{S'}_*(S'/I', k)$ are isomorphic as graded algebras.*
(b) *S/I is Golod if and only if S'/I' is Golod.*

Before we present the long proof of Theorem 3.5, we make a few remarks. Condition (3.6) is necessary because if $m, \ell \in L_I$ and $\gcd(f(m), f(\ell)) \neq 1$, then one can have $\mathrm{Tor}^S_{*,m}(S/I, k) \cdot \mathrm{Tor}^S_{*,\ell}(S/I, k) \neq 0$ while the corresponding product $\mathrm{Tor}^{S'}_{*,f(m)}(S'/I', k) \cdot \mathrm{Tor}^{S'}_{*,f(\ell)}(S'/I', k)$ vanishes for degree reasons. For example let $I = (x, y)$, $S = k[x, y]$, $I' = (uv, uw)$, and $S' = k[u, v, w]$. Then any bijection f between the minimal sets of generators of I and I' extends to a lattice isomorphism of L_I and $L_{I'}$. But clearly the Tor-algebras are not isomorphic for exactly the reasons mentioned above.

In the case when I and I' are square-free and $\mathrm{char}(k) \neq 2$, Babson and Chan can provide a different proof of Theorem 3.5(a) [Babson-Chan, personal communication]. Also, Theorem 3.5(a) could be proved quickly using the proof of 3.1. In order to prove Theorem 3.5(b) we need to analyze the Koszul homology (and Massey operations); this analysis proves Theorem 3.5 (a) and (b) simultaneously.

Recall that S/I is called *Golod* if

$$\mathrm{Poin}^k_{S/I}(t, x_1, \ldots, x_n) = \frac{(1 + tx_1) \cdots (1 + tx_n)}{1 - t^2\, \mathrm{Poin}^I_S(t, x_1, \ldots, x_n)},$$

where $\mathrm{Poin}^I_S(t, x_1, \ldots, x_n)$ is the multigraded Poincaré series of the minimal free resolution of I over S and $\mathrm{Poin}^k_{S/I}(t, x_1, \ldots, x_n)$ is the multigraded Poincaré series of the minimal free resolution of k over S/I. Golod rings are important because the minimal free resolution of k over S/I is nicely structured if S/I is Golod. Golodness can be detected using finite data: the ring S/I is Golod if and only if its Koszul complex admits a trivial Massey operation.

Proof. First we will make several reductions which show that it suffices to prove the theorem in a special case and after that we will deal with this special case in Lemma 3.7.

Let I_{pol} be the polarization of I and S_{pol} the polynomial ring in which I_{pol} lives. By [Fr], depolarizing S_{pol}/I_{pol} to S/I consists of factoring a regular sequence of linear forms. Therefore, $\mathrm{Tor}^S_*(S/I, k) \cong \mathrm{Tor}^{S_{pol}}_*(S_{pol}/I_{pol}, k)$ as graded algebras, and S/I is Golod if and only if S_{pol}/I_{pol} is. Similarly, we can polarize I' and get that $\mathrm{Tor}^{S'}_*(S'/I', k) \cong \mathrm{Tor}^{S'_{pol}}_*(S'_{pol}/I'_{pol}, k)$ as graded algebras, and S'/I' is Golod if and only if S'_{pol}/I'_{pol} is. Also note that depolarization defines an isomorphism of the lcm-lattices of the original ideal and the polarized ideal, and that this isomorphism satisfies (3.6).

So in order to prove the theorem, we can assume that both I and I' are square-free. (We remark that our proof can be extended to the case of arbitrary monomial ideals; we work in the square-free case because assuming in Lemma 3.7 that either $f(m) = m$ or $f(m) = ym$ simplifies the proof.)

Consider the ring $R = k[x_1, \ldots, x_n, x'_1, \ldots, x'_{n'}]$. Let N be the ideal in R generated by the monomials $\{\, mf(m) \mid m \text{ is a minimal generator of } I \,\}$. Define $g : L_I \to L_N$ by $g(m) = mf(m)$. Then g is an isomorphism of lattices satisfying condition (3.6). We will show that g induces an isomorphism

$\mathrm{Tor}_*^S(S/I,k) \cong \mathrm{Tor}_*^R(R/N,k)$ of graded algebras, and that S/I is Golod if and only if R/N is. Similarly, one can prove that $\mathrm{Tor}_*^{S'}(S'/I',k) \cong \mathrm{Tor}_*^R(R/N,k)$ and that S'/I' is Golod if and only if R/N is. This will prove both (a) and (b).

Consider a tower

$$R_1 = S \subset R_2 = R_1[x'_1] \subset \cdots \subset R_{i+1} = R_i[x'_i] \subset \cdots \subset R_{n'+1} = R.$$

For $1 \le i \le n'+1$ let N_i be the ideal in R_i generated by the monomials obtained by setting $x'_i = x'_{i+1} = \ldots = x'_{n'} = 1$ in the minimal monomial generators of N; note that $N_1 = I$ and $N_{n'+1} = N$. Let $g_i : L_{N_{i+1}} \to L_{N_i}$ be the map which sets $x'_i = 1$ in each monomial. Then g_i^{-1} is an isomorphism of lattices satisfying the condition (3.6). Lemma 3.7 below applied to g_i^{-1} shows that

$$\mathrm{Tor}_*^{R_i}(R_i/N_i,k) \cong \mathrm{Tor}_*^{R_{i+1}}(R_{i+1}/N_{i+1},k)$$

as graded algebras, and that R_i/N_i is Golod if and only if R_{i+1}/N_{i+1} is. This implies the desired isomorphism of graded algebras $\mathrm{Tor}_*^S(S/I,k) \cong \mathrm{Tor}_*^R(R/N,k)$, and shows that S/I is Golod if and only if R/N is. □

The main work in proving Theorem 3.5 is in the proof of the following lemma:

Lemma 3.7. *Let I and I' be square-free monomial ideals over the polynomial rings $S = k[x_1,\ldots,x_n]$ and $S' = S[y]$ respectively. Suppose that $f : L_I \to L_{I'}$ is an isomorphism of lattices which satisfies condition* (3.6) *and that for every $m \in L_I$ either $f(m) = m$ or $f(m) = ym$. Then*

(a) *$Tor_*^S(S/I,k)$ and $Tor_*^{S'}(S'/I',k)$ are isomorphic as graded algebras;*
(b) *S/I is Golod if and only if S'/I' is Golod.*

Proof. Let

$$E = \bigwedge(kX_1 \oplus \cdots \oplus kX_n), \quad E' = \bigwedge(kY \oplus kX_1 \oplus \cdots \oplus kX_n),$$

be the exterior algebras on the bases $X_1,\ldots,X_n$ and $Y,X_1,\ldots,X_n$ respectively. Then $\mathbf{K} = E \otimes S/I$ equipped with the differential defined by $\partial(X_i) = x_i$ is the Koszul complex of k over S/I, and $\mathbf{K}' = E' \otimes S'/I'$ equipped with the differential defined by $\partial'(X_i) = x_i$, $\partial'(Y) = y$ is the Koszul complex of k over S'/I'. Both $\mathbf{K}$ and $\mathbf{K}'$ are multigraded by $\deg(Y) = \deg(y)$, $\deg(X_i) = \deg(x_i)$. We have the graded algebra isomorphisms $\mathrm{Tor}_{*,*}^S(S/I,k) \cong \mathrm{H}_{*,*}(\mathbf{K})$ and $\mathrm{Tor}_{*,*}^{S'}(S'/I',k) \cong \mathrm{H}_{*,*}(\mathbf{K}')$. Golodness is determined by whether each of the complexes $\mathbf{K}$ and $\mathbf{K}'$ admits a trivial Massey operation.

Define $\mathbf{K}_I = \oplus_{m \in L_I} K_m$. We give $\mathbf{K}_I$ an algebra structure by setting $K_m K_\ell = 0$ if $\gcd(m,\ell) \neq 1$ and when $\gcd(m,\ell) = 1$ we keep the product $K_m K_\ell \subseteq K_{m\ell}$ the same as in $\mathbf{K}$. Consider the ideal $\mathbf{P} = \oplus\{K_m \,|\, m \text{ is not square-free}\}$ in the Koszul complex $\mathbf{K}$ (here $\mathbf{K}$ is considered as a differential graded algebra). Denote by $\mathbf{W}$ the quotient algebra $\mathbf{K}/\mathbf{P}$. Then the algebra $\mathbf{K}_I$ is isomorphic to the subalgebra $\oplus_{m\in L_I} W_m$ of $\mathbf{W}$. Note that $\mathbf{K} = \mathbf{P} \oplus \big(\oplus\{K_m \,|\, m \text{ is square-free}\}\big)$ as complexes. Taylor's resolution shows that all Betti numbers of S/I are concentrated in square-free multidegrees, hence $\mathbf{P}$ is exact and therefore $\mathrm{H}_{*,*}(\mathbf{K}) \cong \mathrm{H}_{*,*}(\mathbf{W})$. Furthermore, Taylor's resolution shows that all Betti numbers of S/I are concentrated in multidegrees in L_I, hence $\mathrm{H}_{*,*}(\mathbf{W}) \cong \mathrm{H}_{*,*}(\mathbf{K}_I)$. Thus, $\mathrm{Tor}^S_{*,*}(S/I,k) \cong \mathrm{H}_{*,*}(\mathbf{K}_I)$ as graded algebras. Also, all Massey products in nonsquare-free multidegrees vanish, so S/I is Golod if and only if all Massey products in $\mathbf{K}_I$ vanish.

Similarly, we give $\mathbf{K'}_{I'} = \oplus_{\substack{m\in L_{I'}\\ m\neq 1}} K'_m$ an algebra structure as above. By the same argument as above we conclude that $\mathrm{H}_{*,*}(\mathbf{K}'_{I'}) \cong \mathrm{Tor}^{S'}_{*,*}(S'/I',k)$ as graded algebras and that S'/I' is Golod if and only if all Massey products in $\mathbf{K}'_{I'}$ vanish.

To prove the lemma we will compare $\mathbf{K}_I$ and $\mathbf{K}'_{I'}$, and their homologies.

For each $1 \neq m \in L_I$ let $a_m \in \{0,1\}$ be such that $y^{a_m} = \frac{f(m)}{m}$ and let $\alpha_m : \ K_m \longrightarrow K'_{f(m)}$ be the map which acts as multiplying by y^{a_m}. Consider the map

$$\alpha: \ \mathbf{K}_I = \oplus_{m\in L_I} K_m \ \longrightarrow \ \mathbf{K}'_{I'} = \oplus_{m\in L_{I'}} K'_m,$$

defined by $\alpha|_{K_m} = \alpha_m$ for $m \in L_I$ and extended by linearity. If s is a monomial in S, then $sX_{i_1} \wedge \cdots \wedge X_{i_r} \notin I(E\otimes S)$ implies $y^p sX_{i_1} \wedge \cdots \wedge X_{i_r} \notin I'(E'\otimes S')$ for any $p \geq 0$. As I and I' are monomial ideals, we conclude that α_m is a monomorphism of k-vector spaces for each $m \in L_I$, and hence so is α.

Claim. *α is a homomorphism of algebras and commutes with differentiation.*

Proof of the claim. Since f satisfies condition (3.6), it follows that exactly one of the following two cases holds:

- If $\gcd(m,\ell) \neq 1$, then $\gcd(f(m), f(\ell)) \neq 1$, $K_m K_\ell = 0$, $K'_{f(m)}K'_{f(\ell)} = 0$.
- If $\gcd(m,\ell) = 1$, then $\gcd(f(m), f(\ell)) = 1, f(m\ell) = f(\mathrm{lcm}(m,\ell)) = \mathrm{lcm}(f(m), f(\ell)) = f(m)f(\ell)$, $a_m + a_\ell = a_{m\ell}$.

Using this, it is a straightforward computation to verify the claim.

First, we show that α is a homomorphism. Let $\mu \in K_m$ and $\nu \in K_l$ for some $m,l \in L_I$. We consider two cases:

Suppose that $\gcd(m,\ell) \neq 1$. Then $\mu\nu = 0$, so $\alpha(\mu\nu) = 0$. On the other hand, $\gcd(f(m), f(\ell)) \neq 1$ in this case, so $K'_{f(m)}K'_{f(\ell)} = 0$, hence $\alpha(\mu)\alpha(\nu) = 0$.

Now suppose that $\gcd(m,\ell) = 1$. We have that $\alpha(\mu\nu) = y^{a_{ml}}\mu\nu$. Since $a_m + a_\ell = a_{m\ell}$ in this case, we can write $\alpha(\mu\nu) = y^{a_m + a_l}\mu\nu$. As $\gcd(f(m), f(\ell)) = 1$, we conclude that in $\mathbf{K}'_{I'}$ we have the equality $y^{a_m+a_l}\mu\nu = (y^{a_m}\mu)(y^{a_l}\nu)$. By the definition of the map α we get $(y^{a_m}\mu)(y^{a_l}\nu) = \alpha(\mu)\alpha(\nu)$.

It remains to show that α commutes with the differential. Let $\mu \in K_m$ for some $m \in L_I$. We have

$$\partial'(\alpha(\mu)) = \partial'(y^{a_m}\mu) = y^{a_m}\partial'(\mu) = y^{a_m}\partial(\mu) = \alpha(\partial(\mu)).$$

The proof of the claim is finished.

Denote by $\widetilde{\mathbf{K}}_I$ the image of α. We have shown that α is an isomorphism of the differential graded algebras $\mathbf{K}_I$ and $\widetilde{\mathbf{K}}_I$. But $\widetilde{\mathbf{K}}_I$ is a subalgebra of $\mathbf{K}'_{I'}$. We will finish the proof of the lemma by establishing the following:

Claim. *If a set of elements in $\widetilde{\mathbf{K}}_I$ is a basis for the homology $\mathrm{H}_{*,*}(\widetilde{\mathbf{K}}_I)$ then this set is also a basis for the homology $\mathrm{H}_{*,*}(\mathbf{K}'_{I'})$.*

Proof of the claim. For an element z in a complex $\mathbf{T}$ we denote by $\mathrm{c}l(z)_{\mathbf{T}}$ the class of z in the homology of $\mathbf{T}$. By Theorem 2.1 applied to I and to I', we see that the k-vector spaces $\mathrm{H}_{*,m}(\mathbf{K}_I)$ and $\mathrm{H}_{*,f(m)}(\mathbf{K}'_{I'})$ have the same dimension for each $m \in L_I$. But since $\widetilde{\mathbf{K}}_I \cong \mathbf{K}_I$, this implies that the k-vector spaces $\mathrm{H}_{*,*}(\widetilde{\mathbf{K}}_I)$ and $\mathrm{H}_{*,*}(\mathbf{K}'_{I'})$ also have the same dimension. Therefore, in order to prove the desired property it suffices to show that:

$$z \in \widetilde{\mathbf{K}}_I \text{ and } \mathrm{c}l(z)_{\widetilde{\mathbf{K}}_I} \neq 0 \qquad \Longrightarrow \qquad \mathrm{c}l(z)_{\mathbf{K}'_{I'}} \neq 0\,. \tag{3.8}$$

Since both $\widetilde{\mathbf{K}}_I$ and $\mathbf{K}'_{I'}$ are multigraded and their homology is concentrated in multidegrees in $L_{I'}$, it is enough to show for any $m \in L_I$ that (3.8) holds for $z = y^{a_m} Z \in (\widetilde{\mathbf{K}}_I)_{my^{a_m}}$.

If $a_m = 0$ then $(\widetilde{\mathbf{K}}_I)_m = (\mathbf{K'}_{I'})_m$ and (3.8) clearly holds. Suppose that $a_m = 1$. Then $(\mathbf{K}'_{I'})_{ym} = yK'_m \oplus \big(Y \wedge K'_m\big)$. Note that $(\widetilde{\mathbf{K}}_I)_{ym} = yK'_m$. Let $z = yZ$, $Z \in K'_m$ and $\mathrm{c}l(z)_{\mathbf{K}'_{I'}} = 0$. We have $z = \partial(yF + Y \wedge G)$ for some $F, G \in K'_m$. Set $u = z - \partial(yF)$. Then $\mathrm{c}l(u)_{\widetilde{\mathbf{K}}_I} = \mathrm{c}l(z)_{\widetilde{\mathbf{K}}_I}$ and $u = \partial(Y \wedge G) = yG - Y \wedge \partial(G)$. Since $u = z - \partial(yF) \in yK'_m$ we conclude that $\partial(G) = 0$ in $\mathbf{K}'_{I'}$. Hence $u = yG$. By assumption f sends an $l \in L_I$ either to $l \in L_{I'}$ or $yl \in L_{I'}$. Hence, $f(m) = ym$ implies that m does not appear as an image under f (note that y does not divide m). Thus $m \notin L_{I'}$ and by Theorem 2.1 it follows that $\mathbf{K}'_{I'}$ does not have homology in multidegree m. Since G is of multidegree m and since $\partial(G) = 0$ we conclude that $G = \partial(H)$ for some $H \in K'_m$. So $u = y\partial(H)$, which means that $\mathrm{c}l(u)_{\widetilde{\mathbf{K}}_I} = 0$. Thus (3.8) holds. This finishes the proof of the claim. $\square$

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