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# NEW EXAMPLES OF COMPACT 8-MANIFOLDS OF HOLONOMY SPIN(7)

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A Spin(7)-structure $\Phi$ on a manifold $M$ can be encoded in a 4-form given by

$$(1)\qquad \begin{aligned}\Phi = \; & y_1\wedge y_2\wedge y_3\wedge y_4 + y_5\wedge y_6\wedge y_7\wedge y_8 + y_1\wedge y_3\wedge y_5\wedge y_7\\ & + y_2\wedge y_4\wedge y_6\wedge y_8 - y_1\wedge y_2\wedge y_5\wedge y_6 - y_3\wedge y_4\wedge y_7\wedge y_8\\ & - y_1\wedge y_2\wedge y_7\wedge y_8 - y_3\wedge y_4\wedge y_5\wedge y_6 - y_1\wedge y_3\wedge y_6\wedge y_8\\ & - y_2\wedge y_4\wedge y_5\wedge y_7 - y_1\wedge y_4\wedge y_5\wedge y_8 - y_2\wedge y_3\wedge y_6\wedge y_7\\ & - y_1\wedge y_4\wedge y_6\wedge y_7 - y_2\wedge y_3\wedge y_5\wedge y_8,\end{aligned}$$

where $\{y_i\}_{i=1}^8$ is an orthonormal basis of the tangent space $T_mM$. The subgroup of $GL(8,\mathbb{R})$ preserving $\Phi$ is Spin(7). Much analogous to the Kähler form $\omega$ on Calabi-Yau manifolds which has holonomy SU($n$), the form $\Phi$ is parallel if and only $M$ has holonomy in Spin(7).

The Joyce construction of compact manifolds of holonomy Spin(7) [1] generalizes the Kummer construction of $K3$ surfaces. Starting from $T^4$, let $\mathbb{Z}_2$ act on $T^4$ by negating each of the four coordinates. The resulting orbifold has 16 singular points. Each of these singular points can by resolved by gluing the so-called Eguchi-Hanson space $U$. Eguchi-Hanson space [1] is the blow-up of $\mathbb{C}^2/\{\pm 1\}$ biholomorphic to $T^*\mathbb{CP}^1$. Give $\mathbb{C}^2$ coordinates $(z_1, z_2)$, where $dz_1 = dx_1 + idx_2$, and $dz_2 = dx_3 + idx_4$, and let $u = |z_1|^2 + |z_2|^2$ and $t > 0$. If we define

$$f_t = \sqrt{u^2+t^2} + t^2 \ln u - t^2\, \ln\Big(\sqrt{u^2+t^2} + t^2\Big),$$

then $\omega_t = \frac{1}{2} i\partial\bar{\partial} f_t$ is the Kähler form of a Kähler metric on $U$. $\omega_t$ together with $\omega_2 = dx_1\wedge dx_3 - dx_2\wedge dx_4$ and $\omega_3 = dx_1\wedge dx_4 + dx_2\wedge dx_3$ form the triplet of smooth closed 2-forms on $U$ determining the Hyperkähler structure on $U$ and the Eguchi-Hanson metric with holonomy SU(2). Eguchi-Hanson metrics are asymptotic to the flat metric on $\mathbb{C}^2/\{\pm 1\}$ at infinity. The orbifold resolved is $K3$.

Moving from $T^4$ to $T^8$, and changing the group of action from $\mathbb{Z}_2$ to $G$, a power of $\mathbb{Z}_2$'s, and resolving $T^8$ by gluing Eguchi-Hanson spaces to singular points, one can achieve manifolds of exceptional holonomy Spin(7).

Having obtained the resolution M of $T^8/G$ using Eguchi-Hanson spaces, it remains to be shown using analysis that there exists Spin(7)-structure $\Phi$ on M with small torsion, further these structures can be deformed to one that is

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torsion-free. The condition for the metric $g$ associated to $\Phi$ to have holonomy in Spin(7) is exactly that $\Phi$ should be torsion-free.

If $\mathrm{Hol}(g) \subset \mathrm{Spin}(7)$, one can determine $\mathrm{Hol}(g)$ by the $\hat{A}$-genus of $M$, given that $M$ is simply-connected. $\mathrm{Hol}(g)$ is Spin(7) iff $\hat{A} = 1$; SU(4) iff $\hat{A} = 2$; Sp(2) iff $\hat{A} = 3$; and $\mathrm{SU}(2) \times \mathrm{SU}(2)$ iff $\hat{A} = 4$.

In [1], Joyce constructed altogether 95 topologically distinct compact manifolds of Spin(7) holonomy; they were obtained by resolving $T^8/G$ where $G = \mathbb{Z}_2^4$ or $\mathbb{Z}_2^5$ and the action on $T^8$ preserves the Spin(7)-structure, namely the associated form $\Phi$. In his examples, $b^3 = 16, 8$, and 4. Here we expand the collection of Spin(7) compact manifolds by considering $G = \mathbb{Z}_2^6$, and produce new examples of Spin(7) manifolds with $b^3 = 0$.

Let $(x_1, \ldots, x_8)$ be coordinates on $T^8 = \mathbb{R}^8/\mathbb{Z}^8$, where $x_i \in \mathbb{R}/\mathbb{Z}$. Let $\alpha, \beta, \gamma, \delta, \epsilon$ be the involutions on $T^8$ defined by

$$
\begin{aligned}
(2)\ \alpha((x_1, \ldots, x_8)) &= (-x_1, -x_2, -x_3, -x_4, x_5, x_6, x_7, x_8),\\
\beta((x_1, \ldots, x_8)) &= (x_1, x_2, x_3, x_4, -x_5, -x_6, -x_7, -x_8),\\
\gamma((x_1, \ldots, x_8)) &= (c_1 - x_1, c_2 - x_2, x_3, x_4, c_5 - x_5, c_6 - x_6, x_7, x_8),\\
\delta((x_1, \ldots, x_8)) &= (d_1 - x_1, x_2, d_3 - x_3, x_4, d_5 - x_5, x_6, d_7 - x_7, x_8),\\
\epsilon((x_1, \ldots, x_8)) &= (c_1 + x_1, c_2 + x_2, e_3 + x_3, e_4 + x_4, c_5 + x_5, c_6 + x_6,\\
&\qquad e_7 + x_7, e_8 + x_8),
\end{aligned}
$$

where $c_i$, $d_i$ and $e_i$ take values in $\{0, \frac{1}{2}\}$. These five elements all preserve the 4-form $\Phi$ which defines the Spin(7)-structure, and they all commute with each other. The 95 examples found by Joyce come from the groups $G = \langle \alpha, \beta, \gamma, \delta \rangle$ and $G = \langle \alpha, \beta, \gamma, \delta, \epsilon \rangle$, where $e_3 = e_7 = 0$, and $e_4 = e_8 = \frac{1}{2}$. The examples all have the property that $\sigma = b^4_+ - b^4_- = 64$ and $\chi = 144$, where $b^4_+ = 103 - b^2 + b^3$ and $b^3 = 4, 8$ or 16. The singularities involved are of the five types specified by [1] as follows.

- *(i)* $T^4 \times (\mathbb{R}^4/\{\pm 1\})$, its resolution is $T^4 \times U$, which adjusts the Betti numbers of $T^8/G$ by increasing $b^2$ by 1, $b^3$ by 4, $b^4_+$ and $b^4_-$ by 3.
- *(ii)* $(T^4/\{\pm 1\}) \times (\mathbb{R}^4/\{\pm 1\})$, its resolution is $(T^4/\{\pm 1\} \times U$, which adjusts the Betti numbers of $T^8/G$ by increasing $b^2$ by 1, $b^4_+$ and $b^4_-$ by 3.
- *(iii)* $(\mathbb{R}^4/\{\pm 1\} \times (\mathbb{R}^4/\{\pm 1\}$, its resolution is $U \times U$, which adjusts the Betti numbers of $T^8/G$ by increasing $b^4_-$ by 1.
- *(iv)* $T^4 \times (\mathbb{R}^4/\{\pm 1\})/\sigma$, where $\sigma$ is an isometric involution defined by

$$\sigma((x_1, \cdots, x_8)) = (\tfrac{1}{2} + x_1, x_2, -x_3, -x_4, x_5, x_6, -x_7, -x_8),$$

  $\sigma$ acts freely on the $T^4$ part. There are two topologically distinct resolutions of $(T^4 \times U)/\sigma$ due to the action of $\sigma$ on the $U$. The holomorphic resolution, where $\sigma(z_1, z_2) = (z_1, -z_2)$, increases $b^2$ by 1, $b^3$ by 2, $b^4_+$ and $b^4_-$ by 1; the antiholomorphic resolution, where $\sigma(z_1, z_2) = (\bar{z_1}, \bar{z_2})$, increases $b^3, b^4_+$, and $b^4_-$ all by 2.

(*v*) $(T^4/\{\pm 1\}) \times (\mathbb{R}^4/\{\pm 1\})/\sigma$, where $\sigma$ is as defined above. Again, we have two topologically distinct resolutions. The holomorphic one increases $b^2, b^4_+$, and $b^4_-$ all by 1; the antiholomorphic one increases $b^4_+$ and $b^4_-$ by 2.

Here, we will construct examples of Spin(7) manifolds with $b^3 = 0$.

Let $S_g$ denote the fixed point set of $T^8$ under $g \in G$. Looking at Joyce's examples closely, we see that in the examples arising from $G = \langle \alpha, \beta, \gamma, \delta \rangle$, $b^3 = 16$, and $S_{\alpha\beta} = S_\alpha \cap S_\beta$ is a set of 256 singular points which yield 64 singularities of type (*iii*), $S_\gamma$ and $S_\delta$ yield singularities of type (*i*) or (*iv*) which increase $b^3$. In the examples arising from $G = \langle \alpha, \beta, \gamma, \delta, \epsilon \rangle$, $b^3 = 4$ or 8; $S_{\alpha\beta} = S_\alpha \cap S_\beta$ and $S_{\alpha\beta\epsilon} = S_\gamma \cap S_{\alpha\beta\gamma\epsilon}$ each yield 32 singularities of type (*iii*), $S_\delta$ yield singularities of type (*i*) or (*iv*) which increase $b^3$. Furthermore, in order for the singular points to come from $S_\alpha$, $S_\beta$, $S_\gamma$, $S_\delta$, $S_{\alpha\beta}$, $S_{\alpha\beta\epsilon}$, and $S_{\alpha\beta\gamma\epsilon}$ only, and for them to be one of the specified 5 types and one has to require that

$$(3) \qquad \begin{aligned} &(c_1, c_2) \neq (0,0), \quad (c_5, c_6) \neq (0,0), \quad (d_1, d_3) \neq (0,0), \\ &(d_5, d_7) \neq (0,0), \quad (c_1, c_5) \neq (d_1, d_5), \quad (d_3, d_7) \neq (0,0). \end{aligned}$$

When we add another generator $\eta$ to form group $G = \mathbb{Z}_2^6 = \langle \alpha, \beta, \gamma, \delta, \epsilon, \eta \rangle$, so that we have 4 singular sets $S_{\alpha\beta}$, $S_{\alpha\beta\epsilon}$, $S_{\alpha\beta\eta}$, and $S_{\alpha\beta\epsilon\eta}$, each yielding 16 singularities of type (*iii*). Also, none of the singular sets arising from $G$ produces singularities of type (*i*) or (*iv*), so $b^3$ stays 0.

To define $\eta$, we need to see how the singular sets $S_{\alpha\beta\eta}$ and $S_{\alpha\beta\epsilon\eta}$ can arise from singular sets which contain type (*ii*) and (*v*) singularities. There are three distinct possibilities(— other possibilities are similar to one of the three).

(*i*) $S_{\alpha\beta\eta} = S_\beta \cap S_{\alpha\eta}$ and $S_{\alpha\beta\epsilon\eta} = S_\delta \cap S_{\alpha\beta\delta\epsilon\eta}$.
(*ii*) $S_{\alpha\beta\eta} = S_\delta \cap S_{\alpha\beta\delta\eta}$ and $S_{\alpha\beta\epsilon\eta} = S_\beta \cap S_{\alpha\epsilon\eta}$.
(*iii*) $S_{\alpha\beta\eta} = S_\gamma \cap S_{\alpha\beta\gamma\eta}$ and $S_{\alpha\beta\epsilon\eta} = S_\delta \cap S_{\alpha\beta\delta\epsilon\eta}$.

Let

$$(4) \quad \eta((x_1, \ldots, x_8)) = (f_1 + x_1, f_2 + x_2, f_3 + x_3, f_4 + x_4, f_5 + x_5, f_6 + x_6, f_7 + x_7, f_8 + x_8),$$

where $f_i \in \{0, \frac{1}{2}\}$.

In the first case, $S_{\alpha\beta\eta} = S_\beta \cap S_{\alpha\eta}$ implies that

$$f_5 = f_6 = f_7 = f_8 = 0,$$

and $S_{\alpha\beta\epsilon\eta} = S_\delta \cap S_{\alpha\beta\delta\epsilon\eta}$ implies that

$$c_1 + d_1 = f_1, e_3 + d_3 = f_3, c_5 + d_5 = f_5 = 0, d_7 + e_7 = f_7 = 0.$$

The fact that the only singularities are $S_\alpha$, $S_\beta$, $S_\gamma$, $S_\delta$, $S_{\alpha\eta}$, $S_{\alpha\beta\gamma\epsilon}$, $S_{\alpha\beta\delta\epsilon\eta}$, $S_{\alpha\beta}$, $S_{\alpha\beta\epsilon}$, $S_{\alpha\beta\eta}$, and $S_{\alpha\beta\epsilon\eta}$ imply that

$$f_1 = f_2 = f_3 = f_4 = \tfrac{1}{2},$$

and the other constants must take the following values:

| $c_1$ | $c_2$ | $c_5$ | $c_6$ | $d_1$ | $d_3$ | $d_5$ | $d_7$ | $e_3$ | $e_4$ | $e_7$ | $e_8$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\frac{1}{2}$ | 0 | $a$ | $b$ | 0 | $\frac{1}{2}$ | $a$ | $c$ | 0 | $\frac{1}{2}$ | $c$ | $d$ |
| 0 | $\frac{1}{2}$ | $a$ | $b$ | $\frac{1}{2}$ | 0 | $a$ | $c$ | $\frac{1}{2}$ | 0 | $c$ | $d$ |

where $a, b, c, d \in \{0, \frac{1}{2}\}$ and $(a, b) \neq (0, 0)$ and $(a, c) \neq (0, 0)$.

In the second case, we can deduce from the nature of singularities that

$$d_1 = f_1, \quad d_3 = f_3, \quad d_5 = f_5 = c_5, \quad d_7 = f_7 = e_7, \quad f_6 = c_6, \quad f_8 = e_8,$$
$$c_1 + d_1 = f_4 + e_4 = e_3 + d_3 = c_2 + f_2 = \tfrac{1}{2}$$

and the other constants must take the following values:

| $c_1$ | $c_2$ | $c_5$ | $c_6$ | $d_1$ | $d_3$ | $d_5$ | $d_7$ | $e_3$ | $e_4$ | $f_2$ | $f_4$ | $f_6$ | $f_8$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | $\frac{1}{2}$ | $a$ | $b$ | $\frac{1}{2}$ | 0 | $a$ | $c$ | $\frac{1}{2}$ | 0 | 0 | $\frac{1}{2}$ | $b$ | $d$ |
| $\frac{1}{2}$ | 0 | $a$ | $b$ | 0 | $\frac{1}{2}$ | $a$ | $c$ | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $b$ | $d$ |

where $(a, b) \neq (0, 0)$, $(a, c) \neq (0, 0)$, and $(b, d) \neq (0, 0)$.

In the last case, we deduce that

$$c_1 = f_2, \quad c_2 = f_2, \quad c_5 = f_5, \quad c_6 = f_6, \quad d_1 = d_5 = 0, \quad d_3 = d_7 = \tfrac{1}{2},$$
$$d_3 + e_3 = f_3, \quad d_7 + e_7 = f_7, \quad f_4 + e_4 = \tfrac{1}{2}, \quad f_8 + e_8 = \tfrac{1}{2},$$

and the other constants must take the following values:

| $c_1$ | $c_2$ | $c_5$ | $c_6$ | $e_3$ | $e_4$ | $e_7$ | $e_8$ | $f_3$ | $f_4$ | $f_7$ | $f_8$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $a$ | $b$ | $c$ | $d$ | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | 0 | 0 | $\frac{1}{2}$ |
| $a$ | $b$ | $c$ | $d$ | $\frac{1}{2}$ | 0 | 0 | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 |

where $(a, b) \neq (0, 0)$, $(c, d) \neq (0, 0)$, $(a, c) \neq (0, 0)$ and $(b, d) \neq (0, 0)$.

*Example* 1. Let $\alpha, \beta, \gamma, \delta, \epsilon$, and $\eta$ be defined as above, and let

$$(c_1, c_2, c_5, c_6) = (\tfrac{1}{2}, 0, 0, \tfrac{1}{2}), \quad (d_1, d_3, d_5, d_7) = (0, \tfrac{1}{2}, 0, \tfrac{1}{2}),$$
$$(e_3, e_4, e_7, e_8) = (0, \tfrac{1}{2}, \tfrac{1}{2}, 0), \quad (f_1, f_2, f_3, f_4) = (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}),$$
$$(f_5, f_6, f_7, f_8) = (0, 0, 0, 0).$$

We have

(*i*) 2 singularities of type (*ii*): $S_\alpha$ and $S_{\alpha\eta}$ each contributing 1.

(*ii*) 64 singularities of type (*iii*): $S_{\alpha\beta}, S_{\alpha\beta\epsilon}, S_{\alpha\beta\eta}, S_{\alpha\beta\epsilon\eta}$ each contributing 16.

(*iii*) 12 singularities of type (*v*):$S_\gamma, S_\delta, S_{\alpha\beta\delta\epsilon\eta}$, and $S_{\alpha\beta\gamma\epsilon}$ each contributing 2, and $S_\beta$ contributing 4.

Computing the Betti numbers of the resolution of $T^8/G$, we get

$$b^0 = 1, \quad b^1 = 0, \quad b^2 = 2 + k, \quad b^3 = 0, \quad b^4_+ = 101 - k, \quad b^4_- = 37 - k,$$

where $0 \leq k \leq 12$.

In the 13 examples we find here, $\sigma = 64, \chi = 144$. Further,

$$\hat{A} = \frac{1}{24}(-1 + b^1 - b^2 + b^4_+ - 2b^4_-) = 1,$$

so the resolved manifold has holonomy exactly Spin(7).

*Example* 2. Let

$$(c_1, c_2, c_5, c_6) = (\tfrac{1}{2}, 0, \tfrac{1}{2}, 0), \quad (d_1, d_3, d_5, d_7) = (0, \tfrac{1}{2}, \tfrac{1}{2}, 0),$$
$$(e_3, e_4, e_7, e_8) = (0, \tfrac{1}{2}, 0, \tfrac{1}{2}), \quad (f_1, f_2, f_3, f_4) = (0, \tfrac{1}{2}, \tfrac{1}{2}, 0),$$
$$(f_5, f_6, f_7, f_8) = (\tfrac{1}{2}, 0, 0, \tfrac{1}{2}).$$

We have

($i$) 1 singularity of type ($ii$) coming from $S_\alpha$.

($ii$) 64 singularities of type ($iii$): $S_{\alpha\beta}, S_{\alpha\beta\epsilon}, S_{\alpha\beta\eta}, S_{\alpha\beta\epsilon\eta}$ each contributing 16.

($iii$) 14 singularities of type ($v$):$S_\gamma, S_\delta, S_{\alpha\beta\delta\eta}, S_{\alpha\epsilon\eta}$, and $S_{\alpha\beta\gamma\epsilon}$ each contributing 2, and $S_\beta$ contributing 4.

We compute that

$$b^0 = 1, \quad b^1 = 0, \quad b^2 = 1 + j, \quad b^3 = 0, \quad b^4_+ = 102 - j, \quad b^4_- = 38 - j,$$

where $0 \le j \le 14$.

Again $\sigma = 64, \chi = 144$, and $\hat{A} = 1$ in these 15 new examples.

As $\chi(M) = \frac{1}{8}(4p_2 - p_1^2)$, $\sigma(M) = \frac{1}{45}(7p_2 - p_1^2)$. In the case of holonomy Spin(7), $\hat{A}(M) = 1 = \frac{1}{45 \cdot 2^7}(7p_1^2 - 4p_2) = \frac{1}{24}(-1 + b^3 + (2b^2 + b^4) - 3(b^2 + b^4_-))$. According to [2], the data that can be deduced physically are $b^3$ and $2b^2 + b^4$ up to exchange, thus also $b^2 + b^4_-$.

Here is a table of all the important topological data associated to all the compact manifolds of holonomy Spin(7) found in [1] and the examples above.

| | | | | |
|---|---|---|---|---|
| $2b^2 + b^4$ | 174 | 158 | 150 | 142 |
| $b^3$ | 16 | 8 | 4 | 0 |
| $b^2 + b^4_-$ | 55 | 47 | 43 | 39 |
| $\chi$ | 144 | 144 | 144 | 144 |
| $\sigma$ | 64 | 64 | 64 | 64 |

If one expands the possibilities of singularities beyond the five types specified by [1], one may obtain more examples of holonomy Spin(7) manifolds.

## References

[1] D.D. Joyce, *Compact Riemannian 8-manifolds with holonomy Spin(7)*, Invent. Math. **123** (1996), 507–552.

[2] S.L. Shatashvili and C. Vafa, *Superstrings and Manifolds of Exceptional Holonomy*, Selecta Math. (N.S.) **1** (1995), 347–381.

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