A REMARK ON LOCAL RIGIDITY OF CONFORMAL ACTIONS ON THE SPHERE

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Introduction

Let M be a closed orientable C^∞ riemannian manifold of dimension $n+1 \geq 3$ with constant sectional curvature -1 . It is covered by the hyperbolic space H^{n+1} , and a deck transformation is naturally extended to a C^∞ diffeomorphism of the sphere at infinity, which is conformal with respect to the standard metric of the sphere. In consequence, we are provided with a canonical C^∞ action of the fundamental group $\Gamma = \pi_1(M)$ on the n -sphere S^n , or, alternatively speaking, a homomorphism ρ_0 of Γ into the diffeomorphism group $\text{Diff}(S^n)$. Another action or homomorphism $\rho : \Gamma \rightarrow \text{Diff}(S^n)$ is said to be C^k -close to ρ_0 , if $\rho(\gamma)$ is close to $\rho_0(\gamma)$ relative to the C^k -topology for finitely many $\gamma \in \Gamma$ that altogether generate Γ (recall that Γ is finitely generated). One of the main results in the present note is

Theorem A (Rigidity of the action at infinity). *A C^∞ action ρ of Γ which is sufficiently C^2 -close to the standard action ρ_0 is C^∞ -conjugate to ρ_0 (i.e., there exists a C^∞ diffeomorphism h of S^n such that $\rho(\gamma) = h \circ \rho_0(\gamma) \circ h^{-1}$ for all $\gamma \in \Gamma$).*

Let us denote by ϕ_0^t the geodesic flow of M which is defined on the unit tangent bundle V of M . It is an Anosov flow with C^∞ weak and strong (un)stable foliations. Denote by $\mathcal{F}_0^+$ the weak unstable foliation. We then have

Theorem B (Rigidity of the weak unstable foliation). *Any C^∞ foliation $\mathcal{F}^+$ of V of dimension $n+1$ ($= \dim \mathcal{F}_0^+$) that is sufficiently close to $\mathcal{F}_0^+$ with respect to the C^2 -topology has to be C^∞ -conjugate to $\mathcal{F}_0^+$: Namely, for such a foliation $\mathcal{F}^+$, there always exists a C^∞ diffeomorphism of V onto itself that sends $\mathcal{F}^+$ to $\mathcal{F}_0^+$.*

The theorems are false in dimension $n+1 = 2$, as the hyperbolic structure of M is deformable. There is however a 2-dimensional counterpart due to E. Ghys [G2] (see also [KY] for an alternative proof). Incidentally, in the succeeding paper [G3], Ghys amazingly established a kind of *global* rigidity in dimension $n+1 = 2$.

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One of the breakthroughs brought by Ghys [G2] is the observation that one can translate a rigidity question for actions of discrete groups into that for foliations. In particular, his method enables us to derive Theorem A from Theorem B in the following manner (cf. Yue [Y]). Given an C^∞ action $\rho : \Gamma \rightarrow \text{Diff}(S^n)$, we can form the foliated bundle $(M \times_\rho S^n, \mathcal{H}_\rho)$ over M : The total space $M \times_\rho S^n$ is the quotient of $H^{n+1} \times S^n$ by the diagonal action of Γ (the action on the first component H^{n+1} is by the deck transformations, and on the second via ρ), while the foliation $\mathcal{H}_\rho$ comes from that of $H^{n+1} \times S^n$ by the ‘horizontal’ submanifolds $H^{n+1} \times \{y\}$. This procedure being applied especially to the standard action ρ_0 , the unit tangent bundle $V \rightarrow M$ endowed with the weak unstable foliation $\mathcal{F}_0^+$ is rebuilt. It can be shown that, if ρ is sufficiently C^2 -close to the standard one ρ_0 , then there exists a C^∞ diffeomorphism of $M \times_\rho S^n$ onto V that maps $\mathcal{H}_\rho$ to a foliation which is C^2 -close to $\mathcal{F}_0^+$. Thus Theorem A follows from Theorem B.

Rigidity theory for actions of lattices of higher-rank semisimple Lie groups has been progressing rapidly: For instance, the ‘projective’ actions of those groups on some ‘boundaries’ have recently been shown to be rigid by Katok and Spatzier [KS1], [KS2] (see also [K2] for some partial result). On the contrary, there is no rigidity theorem for actions of rank-one lattices except those by Ghys in dimension $n + 1 = 2$ mentioned earlier. There is however a remarkable attempt by C. Yue, who proved in [Y] that (a) *the standard action ρ_0 appearing in Theorem A (or, more generally, the actions arising in the same manner from locally symmetric riemannian manifolds of strictly negative curvature) is structurally stable*, and that, (b) *if the topological conjugacy between the perturbed action and the original one is absolutely continuous, then they are actually conjugate through a C^∞ diffeomorphism*. One would like to bridge over the gap between these two statements of Yue. This is actually not hard to carry out in the case of real hyperbolic manifolds, as we show here.

Proof of Theorem B

The first step of the proof of Theorem B is to construct a new Anosov flow ϕ^t tangent to the perturbed foliation $\mathcal{F}^+$, which can be thought of as a perturbation of the geodesic flow ϕ_0^t . The new flow ϕ^t will eventually be shown to be a ‘contact’ Anosov flow with C^∞ strong stable and unstable foliations. Then, the theorem of Benoist, Foulon and Labourie [BFL] would imply Theorem B. We should mention that the stream of our proof is quite similar to that of Yue [Y]. Actually, the present article might be thought of as a small rearrangement of his paper. It should be also pointed out that quite a few of ideas employed in our proof go back to Ghys [G2].

We fix some notation to start with. To denote the strong and weak (un)stable foliations of an Anosov flow, we are going to use the following symbols, attaching “0” to them to refer to those that come from the geodesic flow ϕ_0^t :

	stable foliation	unstable foliation
strong	$\mathcal{E}^-$	$\mathcal{E}^+$
weak	$\mathcal{F}^-$	$\mathcal{F}^+$

Due to the assumption $\mathcal{F}^+ \approx \mathcal{F}_0^+$ (in C^2 -topology) in Theorem B, we may assume that the tangent bundle $T\mathcal{F}^+$ of $\mathcal{F}^+$ is transverse to $T\mathcal{E}_0^-$, the tangent bundle of the strong stable foliation of the geodesic flow ϕ_0^t . There is thus a unique C^∞ section g of $T^*\mathcal{F}_0^+ \otimes T\mathcal{E}_0^-$ such that

$$T\mathcal{F}^+ = \{X_0^+ + g(X_0^+) : X_0^+ \in T\mathcal{F}_0^+\};$$

that is, to say, $T\mathcal{F}^+$ is realized as the ‘graph’ of g . Let Z_0 be the vector field on V that generates the geodesic flow ϕ_0^t , and introduce a new C^∞ vector field Z of V which is tangent to $\mathcal{F}^+$ by

$$Z = Z_0 + g(Z_0).$$

The flow generated by Z will be denoted by ϕ^t . It is not hard to see that the diffeomorphisms ϕ^t ($0 \leq t \leq 1$) are C^2 -close to ϕ_0^t ’s uniformly in $0 \leq t \leq 1$.

Claim 1. *ϕ^t is an Anosov flow which is topologically conjugate to the original one ϕ_0^t . Moreover, the strong and weak stable foliations $\mathcal{E}^-$, $\mathcal{F}^-$ of ϕ^t coincide with those $\mathcal{E}_0^-$, $\mathcal{F}_0^-$ attached to ϕ_0^t .*

Proof. The first statement is just a standard fact in the theory of Anosov systems. To show that $\mathcal{E}^- = \mathcal{E}_0^-$, recall the following fact: Other than $\mathcal{E}^-$, there is no ϕ^t -invariant codimension-one subfoliation of $\mathcal{F}^-$ that is transverse to the orbits of ϕ^t . Meanwhile, by the definition of the vector field Z that generates ϕ^t , $\mathcal{E}_0^-$ has to be ϕ^t -invariant. Thus, we are led to $\mathcal{E}^- = \mathcal{E}_0^-$. Finally, to see $\mathcal{F}^- = \mathcal{F}_0^-$, notice that $T\mathcal{F}^-$ is spanned by $T\mathcal{E}^- = T\mathcal{E}_0^-$ and $Z = Z_0 + g(Z_0)$ with $g(Z_0) \in T\mathcal{E}_0^-$, while $T\mathcal{F}_0^-$ by $T\mathcal{E}_0^-$ and Z_0 . $\square$

Since ϕ^t is an Anosov flow, there must be constants $\Lambda \geq \lambda > 0$, $c > 0$ satisfying the following conditions for all $t \geq 0$:

$$\begin{aligned} c^{-1} \cdot e^{-\Lambda t} |X^-| &\leq |d\phi^t X^-| \leq c \cdot e^{-\lambda t} |X^-|, & X^- \in T\mathcal{E}^-; \\ c^{-1} \cdot e^{\lambda t} |X^+| &\leq |d\phi^t X^+| \leq c \cdot e^{\Lambda t} |X^+|, & X^+ \in T\mathcal{E}^+. \end{aligned}$$

Recall that the original flow ϕ_0^t satisfies the same inequalities (in which $T\mathcal{E}^\pm$ are to be replaced by $T\mathcal{E}_0^\pm$) with $\Lambda = \lambda = c = 1$. Since $\phi^t \approx \phi_0^t$ (in C^1 -topology), it follows that the constants Λ , λ and c for ϕ^t are all sufficiently close to 1. In particular, we may assume

$$1 \leq \Lambda/\lambda < 2:$$

Namely, the ‘2-pinching condition’ on the hyperbolicity of ϕ^t is fulfilled. The theorem of Hirsch–Pugh [HP] therefore guarantees that the tangent bundles $T\mathcal{E}^+$

of $\mathcal{E}^+$ are C^1 -differentiable. Meanwhile, we already know that $T\mathcal{E}^- = T\mathcal{E}_0^-$ is C^∞ .

Let θ be the 1-form on V defined by

$$\theta(Z) = 1, \quad \theta(X) = 0 \quad (X \in T\mathcal{E}^\pm).$$

It is obvious that θ is C^1 and ϕ^t -invariant. Note further that θ is completely determined by Z and $T\mathcal{E}^+$, for $T\mathcal{E}^- = T\mathcal{E}_0^-$ is kept invariant under the perturbation of $\mathcal{F}^+$ (or ϕ^t). Due to the assumption, the vector field Z is C^1 -close to Z_0 . In addition, ϕ^t ($0 \leq t \leq 1$) is C^2 -close to ϕ_0^t . Meanwhile, going back to the proof of the theorem of Hirsch–Pugh [HP], we can easily see that $T\mathcal{E}^+$ changes C^1 -differentiably when the time-one map ϕ^1 is perturbed in C^2 -topology. It therefore follows that θ is C^1 -close to θ_0 , which is by definition the 1-form defined by $\theta_0(Z_0) = 1$, $\theta(X) = 0$ ($X \in T\mathcal{E}_0^\pm$). Since θ_0 is the canonical contact form of the unit tangent bundle V , we are led to

Claim 2. θ is a contact form of V ; that is, $\theta \wedge (d\theta)^n$ is a volume form of V .

According to Claim 2, $\Omega = \theta \wedge (d\theta)^n$ is a continuous volume form of V , which is simultaneously ϕ^t -invariant. The regularity theorem of de la Llave–Marco–Moriyón [dLMM] (cf. Hurder–Katok [HK]) guarantees that Ω is indeed C^∞ . We are now able to improve the differentiability of $\mathcal{E}^+$, appealing to the ϕ^t -invariant smooth volume form Ω . (Incidentally, Yue [Y] proved the following assertion assuming the existence of an invariant volume form Ω , and then derived Claim 2. We go in the opposite way.)

Claim 3. The strong unstable foliation $\mathcal{E}^+$ of ϕ^t is also C^∞ .

Proof. (cf. Ghys [G2], Yue [Y]). Recall that the strong stable foliation $\mathcal{E}^- = \mathcal{E}_0^-$ is uniquely ergodic: It has a holonomy-invariant transverse measure that is unique up to constant. Moreover, since $\mathcal{E}^- = \mathcal{E}_0^-$ comes from a metric of constant curvature, the invariant transverse measure is represented by a smooth ‘transverse volume form’ τ^- , a nonvanishing C^∞ differential form of degree $n+1$ ($= \text{codim } \mathcal{E}^-$) that annihilates the vectors tangent to $\mathcal{E}^-$, and is invariant under the holonomy. Since $\mathcal{E}_0^- = \mathcal{E}^-$ is invariant not only under ϕ_0^t but also under ϕ^t , the uniqueness of τ^- (up to constant) implies $\mathcal{L}_Z \tau^- = -\text{const} \cdot \tau^-$ with $\text{const} > 0$, where $\mathcal{L}$ denotes the Lie derivative. Define another differential form ω^- by $\omega^- = i_Z \tau^-$, with i being the inner product. It also satisfies $\mathcal{L}_Z \omega^- = -\text{const} \cdot \omega^-$.

Recall that V carries a ϕ^t -invariant C^∞ volume form $\Omega = \theta \wedge (d\theta)^n$. We can find a unique differential form ω^+ of degree n ($= \text{codim } \mathcal{F}^+$) that annihilates $T\mathcal{F}^+$ and satisfies $\omega^- \wedge \omega^+ = i_Z \Omega$. Note that ω^+ is C^∞ . Take a C^∞ 1-form α^+ of V so that it satisfies $d\omega^+ = \alpha^+ \wedge \omega^+$. We can easily show that the tangent bundle of $\mathcal{E}^+$ is characterized by $T\mathcal{E}^+ = \{X^+ \in T\mathcal{F}^+ : i_X \alpha^+ = 0\}$ (see [G2], [Y] for a detailed proof). In consequence, $T\mathcal{E}^+$ turns out to be C^∞ . $\square$

Notice that the claim implies that the contact form θ is actually C^∞ .

In the sequel, we have observed that ϕ^t is an Anosov flow satisfying the following conditions:

- (i) The strong stable and unstable foliations of ϕ^t are both C^∞ ;
- (ii) there is a ϕ^t -invariant C^∞ contact form θ such that $\theta(Z) = 1$, where Z is the vector field generating ϕ^t .

We can now apply the theorem of Benoist–Foulon–Labourie [BFL] (see also [G1], [K1], [FK], [F] for partial results prior to [BFL]) to conclude that ϕ^t is C^∞ -conjugate to the original flow ϕ_0^t up to time change. Although the conjugacy between the flows ϕ^t and ϕ_0^t may not preserve the parameterizations of the flows or the strong (un)stable foliations, it does preserve the *weak* stable and unstable foliations. It therefore turns out that $\mathcal{F}^+$ is C^∞ -conjugate to $\mathcal{F}_0^+$. This proves Theorem B.

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