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4-MANIFOLDS WITH INEQUIVALENT SYMPLECTIC FORMS AND 3-MANIFOLDS WITH INEQUIVALENT FIBRATIONS

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Abstract. We exhibit a closed, simply connected 4-manifold X carrying two symplectic structures whose first Chern classes in $H^2(X,\mathbb{Z})$ lie in disjoint orbits of the diffeomorphism group of X. Consequently, the moduli space of symplectic forms on X is disconnected.

The example X is in turn based on a 3-manifold M. The symplectic structures on X come from a pair of fibrations $\pi_0, \pi_1 : M \to S^1$ whose Euler classes lie in disjoint orbits for the action of $\mathrm{Diff}(M)$ on $H_1(M,\mathbb{R})$.

1. Introduction

Symplectic 4-manifolds. A *symplectic form* ω on a smooth manifold X^{2n} is a closed 2-form such that $\omega^n \neq 0$ pointwise. Given a pair of symplectic forms ω_0 and ω_1 on X, we say:

(i) ω_0 and ω_1 are *homotopic* if there is a smooth family of symplectic forms ω_t, $t \in [0,1]$, interpolating between them;

(ii) ω_0 is a *pullback* of ω_1 if $\omega_0 = f^*\omega_1$ for some diffeomorphism $f : X \to X$; and

(iii) ω_0 and ω_1 are *equivalent* if they are related by a combination of (i) and (ii).

Any symplectic form ω admits a compatible almost complex structure $J : TX \to TX$ (satisfying $\omega(v, Jv) > 0$ for $v \neq 0$). Let $c_1(\omega) \in H^2(X,\mathbb{Z})$ denote the first Chern class of the (canonical) complex line bundle $\wedge^n_{\mathbb{C}} TX$ determined by J. It is easy to see that the first Chern class is a deformation invariant of the symplectic structure; that is, $c_1(\omega_0) = c_1(\omega_1)$ if ω_0 and ω_1 are homotopic.

The purpose of this note is to show:

Theorem 1.1. *There exists a closed, simply-connected 4-manifold X which carries a pair of inequivalent symplectic forms. In fact, ω_0 and ω_1 can be chosen such that $c_1(\omega_0)$ and $c_1(\omega_1)$ lie in disjoint orbits for the action of* $\mathrm{Diff}(X)$ *on* $H^2(X,\mathbb{Z})$.

One can also formulate this result by saying that the moduli space $\mathcal{M} =$ (symplectic forms on X)$/\,\mathrm{Diff}(X)$ is disconnected.

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Fibered 3-manifolds. To construct the 4-dimensional example X, we first produce a compact 3-dimensional manifold M^3 that fibers over the circle in two unrelated ways.

To describe this example, we recall the correspondence between closed 1-forms and measured foliations. Let α be a closed 1-form on M, such that α and its pullback to ∂M are pointwise nonzero. Then α defines a *measured foliation* $\mathcal{F}$ of M^3, transverse to ∂M, with $T\mathcal{F} = \operatorname{Ker} \alpha$ and with transverse measure $\mu(T) = \int_T |\alpha|$. Conversely, a (transversally oriented) measured foliation $\mathcal{F}$ determines such a 1-form α. If α happens to have integral periods, then we can write $\alpha = d\pi$ for a *fibration* $\pi : M \to S^1 = \mathbb{R}/\mathbb{Z}$, and the leaves of $\mathcal{F}$ are then simply the fibers of π.

The *Euler class* of a measured foliation,

$$e(\mathcal{F}) = e(\alpha) \in H_1(M, \mathbb{Z})/(\text{torsion}),$$

is represented geometrically by the zero set of a section $s : M \to T\mathcal{F}$, such that the vector field $s|\partial M$ is inward pointing and nowhere vanishing.

Just as for symplectic forms, we say:

(i) α_0 and α_1 are *homotopic* if they are connected by a smooth family of closed 1-forms α_t, nonvanishing on M and ∂M;
(ii) α_0 is a *pullback* of α_1 if $\alpha_0 = f^*\alpha_1$ for some $f \in \operatorname{Diff}(M)$; and
(iii) α_0 and α_1 are *equivalent* if they are related by a combination of (i) and (ii).

In the 3-dimensional arena we will show:

Theorem 1.2. *There exists a compact link complement $M = S^3 - \mathcal{N}(K)$ which carries a pair of inequivalent measured foliations α_0 and α_1. In fact α_0 and α_1 can be chosen to be fibrations, with $e(\alpha_0)$ and $e(\alpha_1)$ in disjoint orbits for the action of $\operatorname{Diff}(M)$ on $H_1(M, \mathbb{Z})$.*

(Here and below, $\mathcal{N}(K)$ denotes an open regular neighborhood of a link K in a 3-manifold.)

Description of the manifolds. For the specific examples we will present, the link K is obtained from the Borromean rings $K_1 \cup K_2 \cup K_3$ by adding a fourth component K_4; see Figure 1. The fourth component is the *axis* of a rotation of S^3 cyclically permuting $\{K_1, K_2, K_3\}$; it can be regarded as a vertical line in $\mathbb{R}^3$, normal to a plane nearly containing the rings.

Alternatively, we can also write $M = T^3 - \mathcal{N}(L)$, where

- $T^3 = \mathbb{R}^3/\mathbb{Z}$ is the flat Euclidean 3-torus,
- $L \subset T^3$ is a union of 4 disjoint, oriented, closed geodesics,
- (L_1, L_2, L_3) gives a basis for $H_1(T^3, \mathbb{Z})$, and
- $L_4 = L_1 + L_2 + L_3$ in $H_1(T^3, \mathbb{Z})$.

The 4-manifold X of Theorem 1.1 is the fiber-sum of $T^3 \times S^1$ with 4 copies of the elliptic surface $E(1) \to \mathbb{CP}^1$, with the elliptic fiber $F \subset E(1)$ glued along

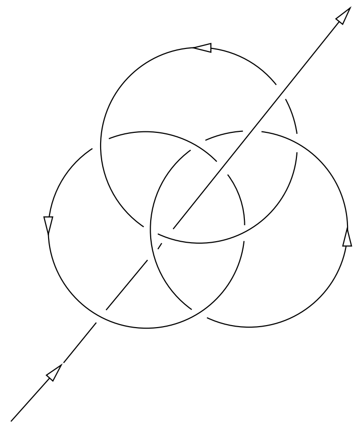

Figure 1. An axis added to the Borromean rings.

$L_i \times S^1$. The key to the example is that Diff(X) preserves the *Seiberg–Witten norm*

$$\|s\|_{\mathrm{SW}} = \sup\{|s \cdot t| \ : \ \mathrm{SW}(t) \neq 0\}$$

on $H^2(X, \mathbb{R})$, just as Diff(M) preserves the Alexander norm on $H^1(M, \mathbb{R})$. The Seiberg–Witten norm manifests the rigidity of the smooth structure on X, allowing us to check that the Chern classes $c_1(\omega_1), c_1(\omega_2)$ lie in different orbits of Diff(X).

On the other hand, using Freedman's work one can see that these two Chern classes *are* related by a homeomorphism of X. In fact, using the 3-torus we can write $H^2(X, \mathbb{Z})$ with its intersection form as a direct sum

$$(H^2(X,\mathbb{Z}), \wedge) \quad = \quad \left(\mathbb{Z}^6, \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)\right) \oplus (V, q),$$

where the Chern classes $c_1(\omega_1), c_1(\omega_2)$ lie in the first factor and are related by an integral automorphism preserving the hyperbolic form. By Freedman's result [FQ, §10.1], this automorphism of $H^2(X, \mathbb{Z})$ is realized by a homeomorphism of X.

Many more examples can be constructed along similar lines. For a simple variation, one can replace L_4 with a geodesic homologous to $L_1 + L_2 + (2m+1) \cdot L_3$, $m \in \mathbb{Z}$, and replace the elliptic surface $E(1)$ with its n-fold fiber sum, $E(n)$. The manifolds M and X resulting from these variations also satisfy the Theorems above.

Notes and references. Our examples exploit a dictionary between 3 and 4-dimensions, some of whose entries are summarized in Table 2.

The connection between the Thurston norm and the Seiberg–Witten invariant was developed by Kronheimer and Mrowka in [KM], [Kr2], [Kr1], while the work of Meng–Taubes and Fintushel–Stern brought the Alexander polynomial into play [MT], [FS1], [FS2], [FS3]. Inasmuch as the Alexander polynomial is tied to

3-manifolds	**4-manifolds**
Measured foliations $\mathcal{F}$ of M	Symplectic forms ω on X
Fibrations $M \to S^1$	Integral symplectic forms
Fibers minimize genus	Pseudo-holomorphic curves minimize genus
Euler class $e(T\mathcal{F})$	First Chern class $c_1(\wedge^2_{\mathbb{C}} TX)$
Alexander polynomial $\Delta_M \in \mathbb{Z}[H_1]$	Seiberg–Witten polynomial $\sum \mathrm{SW}(t) \cdot t \in \mathbb{Z}[H^2]$
Alexander norm on $H^1(M, \mathbb{R})$	Seiberg–Witten norm on $H^2(X, \mathbb{R})$

Table 2.

the Thurston norm in [Mc1], [Mc2], (see also [Vi]), there is an intriguing circle of ideas here which might be better understood.

2. The Alexander and Thurston norms

In this section we recall the Alexander and Thurston norms for a 3-manifold, and prove that Theorem 1.2 holds for the link complement pictured in the Introduction.

The Thurston norm. Let M be a compact, connected, oriented 3-manifold, whose boundary (if any) is a union of tori. For any compact oriented n-component surface $S = S_1 \sqcup \cdots \sqcup S_n$, let

$$\chi_-(S) = \sum_{\chi(S_i)<0} |\chi(S_i)|.$$

The *Thurston norm* on $H^1(M, \mathbb{Z})$ measures the minimum complexity of a properly embedded surface $(S, \partial S) \subset (M, \partial M)$ dual to a given cohomology class; it is given by

$$\|\phi\|_T = \inf\{\chi_-(S) \ : \ [S] = \phi\}.$$

The Thurston norm extends by linearity to $H^1(M, \mathbb{R})$.

Let $B_T = \{\phi \ : \ \|\phi\|_T \leq 1\}$ denote the unit ball in the Thurston norm; it is a finite polyhedron in $H^1(M, \mathbb{R})$. A basic result is:

Theorem 2.1. *Suppose $\phi_0 \in H^1(M, \mathbb{Z})$ is represented by a fibration $M \to S^1$ with fiber S. Then:*

- $\|\phi_0\|_T = \chi_-(S)$;
- *ϕ_0 is contained in the open cone $\mathbb{R}_+ \cdot F$ over a top-dimensional face F of the Thurston norm ball B_T;*
- *every cohomology class in $H^1(M, \mathbb{Z}) \cap \mathbb{R}_+ \cdot F$ is represented by a fibration;*
- *the classes in $H^1(M, \mathbb{R}) \cap \mathbb{R}_+ \cdot F$ are represented by measured foliations; and*
- *the Euler class $e = e(\phi_0) \in H_1(M, \mathbb{Z})$ is dual to the supporting hyperplane to F. More precisely, $\phi(e) = -1$ for all $\phi \in F$.*

In this case we say F is a *fibered face* of the Thurston norm ball. For more details, see [Th2] and [Fr].

The Alexander norm. Next we discuss the Alexander polynomial and its associated norm. Let $G = H_1(M, \mathbb{Z})/(\text{torsion}) \cong \mathbb{Z}^{b_1(M)}$. The *Alexander polynomial* Δ_M is an element of the group ring $\mathbb{Z}[G]$, well-defined up to a unit and canonically determined by $\pi_1(M)$. It can be effectively computed from a presentation for $\pi_1(M)$ (see e.g. [CF]). Writing

$$\Delta_M = \sum_G a_g \cdot g,$$

the *Newton polygon* $N(\Delta_M) \subset H_1(M, \mathbb{R})$ is the convex hull of the set of g such that $a_g \neq 0$. The *Alexander norm* on $H^1(M, \mathbb{R})$ measures the length of the image of the Newton polygon under a cohomology class $\phi : H_1(M, \mathbb{R}) \to \mathbb{R}$; that is,

$$\|\phi\|_A = |\phi(N(\Delta_M))|.$$

From [Mc1] we have:

Theorem 2.2. *If M is a 3-manifold with $b_1(M) \geq 2$, then we have*

$$\|\phi\|_A \leq \|\phi\|_T$$

for all $\phi \in H^1(M, \mathbb{R})$; and equality holds if ϕ is represented by a fibration $M \to S^1$.

Links in the 3-torus. We now turn to the Thurston norm for link-complements in the 3-torus. Let $T^3 = \mathbb{R}^3/\mathbb{Z}^3$ denote the flat Euclidean 3-torus. Every nonzero cohomology class $\phi \in H^1(T^3, \mathbb{Z})$ is represented by a fibration (indeed, a group homomorphism) $\Phi : T^3 \to S^1$.

Consider an n-component link $L \subset T^3$, consisting of disjoint, oriented, closed geodesics $L_1 \cup \cdots \cup L_n$. Define a norm on $H^1(T^3, \mathbb{R})$ by

$$\|\phi\|_L = \sum |\phi(L_i)|, \tag{1}$$

where the L_i are considered as elements of $H_1(M, \mathbb{Z})$. Let M be the link complement $T^3 - \mathcal{N}(L)$, equipped with the natural inclusion $M \subset T^3$.

Theorem 2.3. *Given $\phi \in H^1(T^3, \mathbb{Z})$, let ψ denote its pullback to $M = T^3 - \mathcal{N}(L)$. Then we have:*

$$\|\phi\|_L = \|\psi\|_T = \|\psi\|_A. \tag{2}$$

Moreover:

(a) *ψ is represented by a fibration $\Psi : M \to S^1$ $\iff$*
(b) *$\phi(L_i) \neq 0$ for all i $\iff$*
(c) *ϕ belongs to the open cone over a top-dimensional face of the norm ball $B_L = \{\phi : \|\phi\|_L \leq 1\} \subset H^1(T^3, \mathbb{R})$.*

Proof. We begin by showing (a-c) are equivalent. If ψ is represented by a fibration $\Psi : M \to S^1$, then the fibers are transverse to ∂M and thus $\phi(L_i) \neq 0$ for all i. On the other hand, the latter condition insures that the linear fibration $\Phi : T^3 \to S^1$ associated to ϕ restricts to a fibration of M representing ψ, so we have (a) $\iff$ (b). Finally $\|\phi\|_L$ behaves linearly on $H^1(T^3, \mathbb{R})$ unless one of the terms $\phi_i(L)$ changes sign, and thus the cone on the top dimensional faces is exactly the locus where $\phi(L_i) \neq 0$ for all i, showing (b) $\iff$ (c).

To establish equation (2), first suppose ψ is represented by a fibration $\Psi : M \to S^1$ with fiber S. Since we may take $\Psi = \Phi | M$, we see S is a union of tori with $\sum |\phi(L_i)|$ punctures, and thus

$$\chi_-(S) = \|\psi\|_T = \sum |\phi(L_i)| = \|\phi\|_L.$$

Equality with the Alexander norm holds by Theorem 2.2.

Thus (2) holds on the cone over the top-dimensional faces of B_L. Since this cone is dense, (2) holds throughout $H^1(T^3, \mathbb{Z})$ by continuity. □

The Borromean rings plus axis. We now turn to the study of the 4-component link $K \subset S^3$ pictured in Figure 1. Let $M = S^3 - \mathcal{N}(K)$, and let m_i denote the meridian linking K_i positively. Then (m_1, m_2, m_3, m_4) forms a basis for $H_1(M, \mathbb{Z}) \cong \mathbb{Z}^4$, and the Alexander polynomial Δ_M can be written as a Laurent polynomial in these variables.

Lemma 2.4. *The Alexander polynomial of $M = S^3 - \mathcal{N}(K)$ is given by*

$$\Delta_M(x,y,z,t) = -4 + \left(t + \frac{1}{t}\right) - \left(xy + \frac{1}{xy} + yz + \frac{1}{yz} + xz + \frac{1}{xz}\right) + \left(xyz + \frac{1}{xyz}\right) + \left(x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z}\right),$$

where $(x,y,z,t) = (m_1, m_2, m_3, m_4)$.

Proof. The projection in Figure 1 yields the Wirtinger presentation

$$\begin{aligned}\pi_1(M) \;=\; & \langle a,b,c,d,e,f,g,h,i,j,k,l \;: \\ & aj = jb, bi = ic, gc = ag, dc = ce, ae = fa, fj = jd, \\ & ge = eh, hj = ji, di = gd, jg = gk, kc = cl, le = ej\rangle.\end{aligned}$$

Here (a,b,c), (d,e,f), (g,h,i) and (j,k,l) are the edges of K_1, K_2, K_3 and K_4 respectively. Given this presentation, the calculation of Δ_M is a straightforward application of the Fox calculus [Fox]. □

Figure 3 shows the intersection of the Newton polygon $N(\Delta_M)$ with the (x,y,z)-hyperplane.

To bring the 3-torus into play, recall that 0-surgery along the Borromean rings determines a diffeomorphism

$$S^3 - \mathcal{N}(K_1 \cup K_2 \cup K_3) \cong T^3 - \mathcal{N}(L_1 \cup L_2 \cup L_3),$$

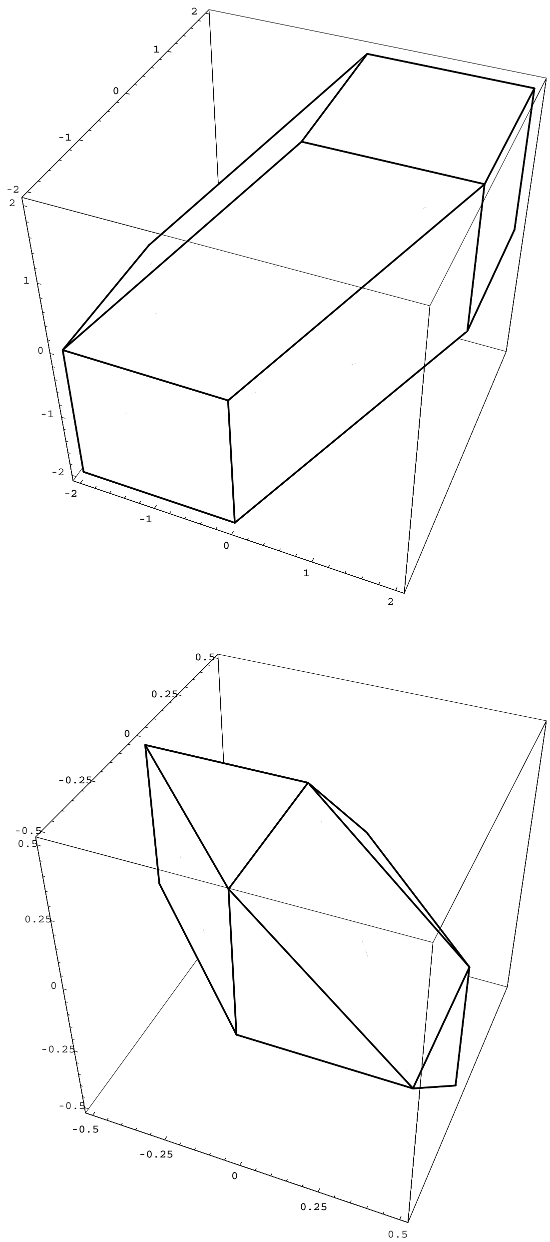

Figure 3. The Newton polygon of $\Delta_M(x, y, z, 1)$ (top), and its dual.

where (L_1, L_2, L_3) are disjoint closed geodesics forming a basis for $H_1(T^3, \mathbb{Z})$. Under this surgery, the meridians (m_1, m_2, m_3) go over to longitudes of

(L_1, L_2, L_3). On the other hand, K_4 goes over to the isotopy class of a geodesic $L_4 \subset T^3$, with

$$L_4 = L_1 + L_2 + L_3 \quad \text{in } H_1(T^3, \mathbb{Z}).$$

(To check the homology class of L_4, note that in S^3 we have $\operatorname{lk}(K_i, K_4) = 1$ for $i = 1, 2, 3$.)

The meridian m_4 goes over to a meridian of L_4, so unlike (m_1, m_2, m_3) it becomes trivial in $H_1(T^3, \mathbb{Z})$. Thus we have:

$$H^1(M, \mathbb{R}) \supset H^1(T^3, \mathbb{R}) = (\mathbb{R} \cdot m_4)^\perp.$$

Lemma 2.5. *The action of* $\operatorname{Diff}(M)$ *on* $H^1(M, \mathbb{R})$ *preserves the subspace* $H^1(T^3, \mathbb{R})$.

Proof. Consider the Newton polygon

$$N = N(\Delta_M) \subset H_1(M, \mathbb{R}),$$

where Δ_M is given by Proposition 2.4. Since $(t + 1/t)$ is the only expression in Δ_M involving t, we have $N = N_0 + [-1, 1] \cdot t$ where

$$N_0 = N(\Delta_M(x, y, z, 1))$$

is the polyhedron in (x, y, z)-space shown in Figure 3. The vertices $\pm t$ of N are thus combinatorially distinguished: they are the endpoints of 14 edges of N (coming from the 14 vertices of N_0), whereas all other vertices of N have degree 5. Since $\operatorname{Diff}(X)$ preserves N, it also stabilizes the special vertices $\{\pm t\}$, and thus $\operatorname{Diff}(X)$ stabilizes $H^1(T^3, \mathbb{R}) = (\mathbb{R} \cdot t)^\perp = (\mathbb{R} \cdot m_4)^\perp$. □

Proof of Theorem 1.2. For our chosen link $L \subset T^3$, we have

$$\|\phi\|_L = |\phi(m_1)| + |\phi(m_2)| + |\phi(m_3)| + |\phi(m_1 + m_2 + m_3)|.$$

The unit ball $B_L \subset H^1(T^3, \mathbb{R})$ of this norm is shown in Figure 3 (bottom); it is dual to the convex body N_0.

Note that B_L has both triangular and quadrilateral faces. Pick integral classes $\phi_0, \phi_1 \in H^1(T^3, \mathbb{Z})$ lying inside the cones over faces F_0 and F_1 of different types, and let $\alpha_0, \alpha_1 \in H^1(M, \mathbb{Z})$ denote their pullbacks to M.

By Theorem 2.3, the classes α_0 and α_1 correspond to fibrations $M \to S^1$. On the other hand, $\operatorname{Diff}(M)$ preserves the subspace $H^1(T^3, \mathbb{R}) \subset H^1(M, \mathbb{R})$ as well as the norm $\|\phi\|_L = \|\alpha\|_T$ on this subspace. Thus $\operatorname{Diff}(M)$ preserves B_L, so it cannot send the face F_0 to F_1. The supporting hyperplanes for α_0 and α_1 in B_T thus lie in different orbits of $\operatorname{Diff}(M)$. But these supporting hyperplanes are represented by $e(\alpha_0)$ and $e(\alpha_1)$, so their Euler classes are in different orbits as well. □

The Thurston norm. As was shown in [Mc1], the Alexander and Thurston norms agree for many simple links. The norms agree for the Borromean rings plus axis as well. To see this, just note that every vertex of the Alexander norm ball B_A is adjacent to a fibered face. (Indeed, $B_A = B_L \times [-2, 2]$, where $B_L \subset H^1(T^3, \mathbb{R}) \subset H^1(M, \mathbb{R})$ is shown in Figure 3 (bottom). Thus every vertex of B_A meets a face of the form $F \times [-2, 2]$, where F is a face of B_L. Since the slice $F \times \{0\}$ carries fibrations pulled back from T^3, the entire face $F \times [-2, 2]$ is fibered.)

The Alexander and Thurston norms agree on fibrations, so we have $B_A \subset B_T$ by convexity. The reverse inclusion comes from the general inequality $\|\phi\|_A \leq \|\phi\|_T$.

Further example: a closed 3-manifold. To conclude, we describe a *closed* 3-manifold N which fibers over the circle in two inequivalent ways.

Let $M = T^3 - \mathcal{N}(L) = S^3 - \mathcal{N}(K)$ be the link complement considered above. Note that the longitudes of K_1, K_2 and K_3 are all homologous to the meridian m_4 of K_4, since the components of the Borromean rings are unlinked, while each component links K_4 once. Since T^3 is obtained by 0-surgery on K, all the meridians of L are homologous to m_4.

Now let $N \to T^3$ be the 2-fold covering, branched over L, determined by the homomorphism

$$\xi : H_1(M, \mathbb{Z}) \to \{-1, 1\}$$

satisfying $\xi(m_1) = \xi(m_2) = \xi(m_3) = 1$ and $\xi(m_4) = -1$.

The pullback map $H^1(T^3, \mathbb{R}) \to H^1(N, \mathbb{R})$ is easily seen to be injective. We claim it is an isomorphism. To see surjectivity, let $N' \subset N$ be the preimage of $M \subset T^3$. Decomposing $H^1(N', \mathbb{R})$ into eigenspaces for the action of the $\mathbb{Z}/2$ deck group for $N' \to M$, we obtain an isomorphism

$$H^1(N', \mathbb{R}) \cong H^1(M, \mathbb{R}) \oplus H^1(M, \mathbb{R}_\xi),$$

where the last term represents cohomology coefficients twisted by the character ξ of $\pi_1(M)$. Since $\Delta_M(\xi) = \Delta_M(1, 1, 1, -1) = 4 \neq 0$, we have $H^1(M, \mathbb{R}_\xi) = 0$ (cf. [Mc1, §3]). Thus any cohomology class in $H^1(N, \mathbb{R})$ restricts to a $\mathbb{Z}/2$-invariant class on N', so it is the pullback of a class on T^3.

Moreover, every fibration of T^3 transverse to L lifts to a fibration of N, so we find:

Theorem 2.6. *The Thurston norm ball $B_T \subset H^1(N, \mathbb{R})$ agrees with the norm ball $B_L \subset H^1(T^3, \mathbb{R})$, and every face is fibered.*

Picking fibrations in combinatorially inequivalent faces of B_T as before, we have:

Corollary 2.7. *The closed 3-manifold N admits a pair of fibrations α_0, α_1 such that $e(\alpha_0), e(\alpha_1)$ lie in disjoint orbits for the action of $\mathrm{Diff}(N)$ on $H^2(N, \mathbb{Z})$.*

3. Fiber sum and symplectic 4-manifolds

In this section we recall the fiber sum construction, which can be used to canonically associate a 4-manifold $X = X(P, L)$ to a link L in a 3-manifold P. Under this construction, suitable fibrations of P give symplectic forms on $X(P, L)$, and the Alexander polynomial Δ_M of $M = P - \mathcal{N}(L)$ determines Seiberg–Witten invariants of X. It is then straightforward to prove Theorem 1.1 by taking $X = X(T^3, L)$, where $L \subset T^3$ is the 4-component link discussed in previous sections.

Fiber sum. Let $f_i : T^2 \times D^2 \to X_i$, $i = 1, 2$ be smooth embeddings of the torus cross a disk into a pair of smooth closed 4-manifolds. Let

$$X_i' = X_i - f(T^2 \times \operatorname{int} D^2);$$

it is a smooth manifold whose boundary is marked by $T^2 \times S^1$. The *fiber sum* Z of X_1 and X_2 is the closed smooth manifold obtained by gluing together X_1' and X_2' along their boundaries, such that $(x, t) \in \partial X_1'$ is identified with $(x, -t) \in \partial X_2'$. We denote the fiber sum by

$$Z = X_1 \underset{T_1 = T_2}{\#} X_2,$$

where $T_i = f(T^2 \times \{0\}) \subset X_i$; note that there is an implicit identification between the normal bundles of the tori T_i.

The fiber sum of symplectic manifolds along symplectic tori is also symplectic. More precisely, if ω_i are symplectic forms on X_i with $\omega_i > 0$ on T_i and $\int_{T_1} \omega_1 = \int_{T_2} \omega_2$, then Z carries a natural symplectic form ω with $\omega = \omega_i$ on X_i'.

For more details, see [Go], [MW], [FS1], [FS2], [FS3].

The elliptic surface $E(1)$. A convenient 4-manifold for use in the fiber-sum construction is the *rational elliptic surface* $E(1)$. The complex manifold $E(1)$ is obtained by blowing up the base-locus for a generic pencil of elliptic curves on $\mathbb{CP}^2$. Thus $E(1)$ is isomorphic to $\mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}$; it is simply-connected and unique up to diffeomorphism. The pencil provides a holomorphic map $E(1) \to \mathbb{CP}^1$ with generic fiber F an elliptic curve, and the canonical bundle of $E(1)$ is represented by the divisor $-F$.

The projection $E(1) \to \mathbb{CP}^1$ gives a natural trivialization of the normal bundle of the fiber torus F. Since $F \subset E(1)$ is a holomorphic curve in a projective variety, there is a symplectic (Kähler) form on $E(1)$ with $\omega | F > 0$.

Each of the nine exceptional divisors gives a holomorphic section

$$s : \mathbb{P}^1 \to E(1).$$

In particular, a meridian for the fiber F is contractible in $E(1) - \mathcal{N}(F)$, since it bounds the image of a disk under s. Since $E(1)$ is simply-connected, any loop in the complement of F is homotopic to a product of conjugates of meridians, so $E(1) - \mathcal{N}(F)$ is also simply-connected,

For a detailed discussion of the topology of elliptic surfaces, see [HKK, §1] or [GS].

From links to 4-manifolds. Now let $L \subset P^3$ be a framed n-component link in a closed, oriented 3-manifold. Such a link determines:

- a 3-dimensional *link complement* $M = P - \mathcal{N}(L)$, and
- a 4-dimensional *fiber-sum* $X = X(P, L) = (P \times S^1) \underset{L \times S^1 = nF}{\#} nE(1)$.

To describe the fiber-sum in more detail, note that each component L_i of L determines a torus

$$T_i = L_i \times S^1 \subset P \times S^1,$$

and the framing of L_i provides a trivialization of the normal bundle of T_i. Take n copies of the elliptic surface $E(1)$ with fiber F; as remarked above, the projection $E(1) \to \mathbb{CP}^1$ provides a natural trivialization of the normal bundle of F. Finally, choose an orientation-preserving identification between $L \times S^1$ and nF. The fiber-sum $X(P, L)$ is then defined using these identifications.

It turns out that every orientation-preserving diffeomorphism of F extends to a diffeomorphism of $E(1)$, preserving the normal data; indeed, the monodromy of the fibration $E(1) \to \mathbb{CP}^1$ is the full group $SL_2(\mathbb{Z})$. Thus the diffeomorphism type of $X(P, L)$ is the same for any choice of identification between $L \times S^1$ and nF.

Proposition 3.1. *The fiber-sum X is simply-connected if $\pi_1(M)$ is normally generated by $\pi_1(\partial M)$ (e.g. if M is homeomorphic to a link complement in S^3).*

Proof. When the simply-connected manifolds $n(E(1) - \mathcal{N}(F))$ are attached to $M \times S^1$ along $\partial M \times S^1$, they kill $\pi_1(\partial M \times S^1)$ by van Kampen's theorem. Since the latter groups normally generate $\pi_1(M \times S^1)$, the resulting manifold X is simply-connected. □

Promotion of cycles. The fiber-sum construction furnishes us with an inclusion $M \times S^1 = (P \times S^1)' \subset X$.

Proposition 3.2. *The map*

$$i : H_1(M, \mathbb{R}) \to H^2(X, \mathbb{R}),$$

sending a 1-cycle $\gamma \subset M$ to the Poincaré dual of $\gamma \times S^1 \subset X$, is injective.

Proof. The map i is a composition of three maps:

$$H_1(M) \to H_2(M \times S^1) \to H_2(X) \to H^2(X).$$

The first arrow is part of the Künneth isomorphism, and the last comes from Poincaré duality, so they are both injective. As for the middle arrow

$$H_2(M \times S^1) \to H_2(X),$$

we can use the exact sequence of the pair $(X, M \times S^1)$ to identify its kernel with

$$H_3(X, M \times S^1) \cong H_3(nE(1), nF) \cong H^1(nE(1) - nF) = 0.$$

Here we have used excision, Poincaré duality and the simple-connectivity of $E(1) - F$. Thus all three arrows are injective, and so i is injective. □

Corollary 3.3. *For an n-component link, we have*

$$b_2^+(X(P,L)) \geq b_1(M) \geq n.$$

Here $b_2^+(X)$ denotes the rank of the maximal subspace of $H_2(X,\mathbb{R})$ on which the intersection form is positive-definite.

Proof. Since 1-cycles in general position on M are disjoint, the intersection form on $H^2(X,\mathbb{R})$ restricts to zero on $i(H_1(M,\mathbb{R}))$. But the intersection form is non-degenerate, so it must admit a positive (and negative) subspace of dimension at least $b_1(M) = \dim i(H_1(M,\mathbb{R}))$.

For the second inequality, just note that we have $b_1(M) \geq b_1(\partial M)/2 = n$. Indeed, by Lefschetz duality, the kernel of $H_1(\partial M) \to H_1(M)$ is Lagrangian, so the image has dimension n. □

From fibrations to symplectic forms. A central point for us is that suitable fibrations α of P give rise to symplectic structures ω on $X(P,L)$.

Theorem 3.4. *For any fibration $\alpha \in H^1(P,\mathbb{Z})$ transverse to L, there is a symplectic form ω on $X(P,L)$ with*

$$c_1(\omega) \;=\; i(e(\alpha|M)).$$

Proof. Let $\alpha = d\pi$ be the closed 1-form representing a fibration $\pi : P \to S^1$ transverse to L.

Pick a closed 2-form β on M such that β restricts to an area form on each leaf of $\mathcal{F}$. (One can construct such a form by representing the monodromy of the fibration by an area-preserving map.) As observed by Thurston, for $\epsilon > 0$ sufficiently small, the closed 2-form

$$\omega_0 = \alpha \wedge dt + \epsilon\beta$$

is a symplectic form on $P \times S^1$, nowhere vanishing on $L \times S^1$ [Th1]. (Here $[dt]$ is the standard 1-form on $S^1 = \mathbb{R}/\mathbb{Z}$, and α and β have been pulled back to the product).

By scaling the Kähler form, we can provide the ith copy of $E(1)$ with a symplectic form ω_i such that $\int_F \omega_i = \int_{L_i \times S^1} \omega$. Then as mentioned above, ω_0 and (ω_i) joined together under fiber-sum to yield a symplectic form ω on X.

Let $K \to X$ denote the canonical bundle of (X,ω). We will compute $c_1(K)$ be constructing a section $\sigma : X \to K$.

Let $M = P - \mathcal{N}(L)$. As an oriented $\mathbb{R}^2$-bundle, $K|(M \times S^1)$ is isomorphic to the pullback of $T\mathcal{F}$ from M. Let $s : M \to T\mathcal{F}$ be a section such that $s|\partial M$ is inward pointing and nowhere vanishing. Then the zero set of s is a 1-cycle γ representing the Euler class $e(\alpha|M) \in H_1(M,\mathbb{R})$. Pulling back s, we obtain a section $\sigma_0 : M \times S^1 \to K$ with zero set $\gamma \times S^1$.

Now consider the 4-manifold $E(1)' = E(1) - \mathcal{N}(F)$ attached to $M \times S^1$ along $T_i \times S^1$. If we have $\omega_i(F) > 0$, then $K|E(1)'$ is just the pullback of the canonical bundle of $E(1)$. Since $-F$ is a canonical divisor on $E(1)$, there is a nowhere vanishing section $\sigma_i : E(1)' \to K$, namely the restriction of a meromorphic 2-form on $E(1)$ with divisor $-F$.

We claim σ_0 and σ_i fit together under the gluing identification between $T_i \times S^1$ and $F \times S^1$. To check this, we use the framings to identify $K|T_i \times S^1$ and $K|F \times S^1$ with the trivial bundle over $T^2 \times S^1$. Under this identification,

$$\sigma_0 : T^2 \times S^1 \to \mathbb{C}^*$$

is homotopic to the projection $T^2 \times S^1 \to S^1 \subset \mathbb{C}^*$, since the vector field $s|T_i$ runs along the meridians of ∂M. Similarly,

$$\sigma_i : T^2 \times S^1 \to \mathbb{C}^*$$

is homotopic to $1/\sigma_0$, because of the simple pole along F. Since $T_i \times S^1$ is identified with $F \times S^1$ using the involution $(x, t) \sim (x, -t)$ on $T^2 \times S^1$, the two sections correspond under gluing.

In the case where we have $\omega_i(F) < 0$, both homotopy classes are reversed, so σ_0 and σ_i still agree under gluing. Thus σ_0 and (σ_i) join together to form a global section $\sigma : X \to K$ with no zeros outside $M \times S^1$. It follows that $c_1(X, \omega)$ is Poincaré dual to $\gamma \times S^1$; equivalently, that $c_1(\omega) = i(\alpha|M)$. □

The Seiberg–Witten polynomial. A central feature of the fiber-sum $X = X(P, L)$ is that its Seiberg–Witten polynomial is directly computable.

Assume that X is simply-connected and $b_2^+(X) > 1$. Then the Seiberg–Witten invariant of X can be regarded as a map

$$\mathrm{SW} : H^2(X, \mathbb{Z}) \to \mathbb{Z},$$

well-defined up to a sign and vanishing outside a finite set. This information is conveniently packaged as a Laurent polynomial

$$\mathcal{SW}_X \;=\; \sum_t \mathrm{SW}(t) \cdot t \in \mathbb{Z}[H^2(X, \mathbb{Z})].$$

Theorem 3.5. *Suppose M is the complement of an n-component link $L \subset P$, and $\pi_1(\partial M)$ normally generates $\pi_1(M)$. Then $X = X(P, L)$ is simply-connected, we have $b_2^+(X) \geq n$, and*

$$\mathcal{SW}_X = \pm \sum a_t \cdot i(2t),$$

where $\Delta_M = \sum a_t \cdot t$ is the symmetrized Alexander polynomial of M.

Remarks. This Theorem was established by Fintushel and Stern in the special case where (P, L) is obtained by a certain surgery on a link in S^3 [FS2, Thm. 1.9].[1] To obtain the symmetrized Alexander polynomial, one multiplies $\Delta_K(t)$ by a monomial to arrange that its Newton polygon is centered at the origin. The exponents in the symmetrized polynomial may be half-integral.

[1]Note: contrary to [FS2, p. 371]: the cohomology classes $[T_j]$ in their formula for $\mathcal{SW}_X$ are always linearly independent in $H^2(X, \mathbb{R})$, by Proposition 3.2 above.

Proof. To compute $\mathcal{SW}_X$, we regard X as the union of manifolds $X_0 = M \times S^1$ and $X_i = E(1) - \mathcal{N}(F)$, $i = 1, \ldots, n$, glued together along their boundary. For such manifolds one can define a *relative* Seiberg–Witten polynomial $\mathcal{SW}_{X_i} \in \mathbb{Z}[H^2(X_i, \partial X_i; \mathbb{Z})$, such that

$$\mathcal{SW}_X = \mathcal{SW}_{X_0} \cdot \mathcal{SW}_{X_1} \cdots \mathcal{SW}_{X_n},$$

using the natural map $H^2(X_i, \partial X_i) \to H^2(X)$ to compute the product. For this gluing formula, developed by Morgan, Mrowka, Szabo and Taubes, see [FS2, Thm. 2.2] and [Ta].

Now for each $X_i = E(1) - \mathcal{N}(F)$, the relative polynomial is simply 1. To see this, just apply the product formula above to the K3 surface $Z = E(1) \#_F E(1)$, which satisfies $\mathrm{SW}_Z = 1$. (This well-known property of K3 surfaces follows, for example, from equations (4.17) and (4.20) in Witten's original paper [Wit].)

Thus we have $\mathcal{SW}_X = \mathcal{SW}_{X_0} = \mathcal{SW}_{M \times S^1}$. Finally the Seiberg-Witten polynomial for $M \times S^1$ is given in terms of Δ_M by the main result of [MT], yielding the formula for $\mathcal{SW}_X$ above.

To see $\pi_1(X) = \{1\}$ and $b_2^+(X) \geq n$, apply Proposition 3.1 and Corollary 3.3 above. □

Proof of Theorem 1.1. Using the Seiberg–Witten invariants to control the action of $\mathrm{Diff}(X)$, it is now easy to give an example of a simply-connected 4-manifold X with inequivalent symplectic forms.

For a concrete example, let $X = X(T^3, L)$ for the 4-component link $L \subset T^3$ studied in the preceding section, and choose any framing of L. As we have seen, the link-complement $M = T^3 - \mathcal{N}(L)$ is homeomorphic to the exterior $S^3 - \mathcal{N}(K)$ of the Borromean rings plus axis. In particular, $\pi_1(M)$ is the normal closure of $\pi_1(\partial M)$, so X is simply-connected and we have $b_2^+(X) \geq 4$.

Let m_i, $i = 1, \ldots, 4$ be the basis for $H_1(M, \mathbb{Z})$ coming from the meridians of $K \subset S^3$. Then the classes $t_i = i(m_i)$ form a basis for $i(H_1(M, \mathbb{Z})) \subset H^2(X, \mathbb{Z})$. By Theorem 3.5, we have:

The Seiberg–Witten polynomial of X is given by

$$\mathcal{SW}_X = \Delta_M(t_1^2, t_2^2, t_3^3, t_4^2),$$

where $\Delta_M(x, y, z, t)$ is given by Lemma 2.4.

In particular, the Newton polygons satisfy $N(\mathcal{SW}_X) = 2i(N(\Delta_M))$.

Now identify $H_1(T^3, \mathbb{R})$ with the subspace of $H_1(M, \mathbb{R})$ spanned by (m_1, m_2, m_3), and let

$$N_0 = N(\Delta_M) \cap H_1(T^3, \mathbb{R}).$$

As we have seen before, any vertex v of N_0 is dual to a fibered face F of the Thurston norm on $H^1(M, \mathbb{R})$; indeed, v is dual to a fibration pulled by from T^3. All fibrations ϕ in the cone over F have the same Euler class e, which satisfies

$$\|\phi\|_T = 2\phi(v) = -\phi(e);$$

thus $e = -2v$.

By Theorem 3.4, the vertex

$$i(e) = i(-2v) \in 2i(N_0)$$

is the first Chern class of a symplectic structure on X. Since $v \in N_0$ was an arbitrary vertex, we have:

Every vertex of $2i(N_0) \subset N(\mathcal{SW}_X)$ is the first Chern class of a symplectic structure on X.

Now pick a pair combinatorially distinct vertices

$$v_0, v_1 \in 2i(N_0) \subset N(\mathcal{SW}_X).$$

More precisely, referring to Figure 3 (top), we see $2i(N_0)$ has vertices of degrees 3 and 4; choose one of each type. Then v_0 and v_1 have degrees 5 and 6 as vertices of $N(\mathcal{SW}_X)$, since

$$N(\mathcal{SW}_X) \;\; = \;\; 2i(N_0) + [-2, 2] \cdot t_4$$

is simply the suspension of $2i(N_0)$. As a consequence, no automorphism of $H^2(X, \mathbb{R})$ stabilizing $N(\mathcal{SW}_X)$ can transport v_0 to v_1.

To complete the proof, choose symplectic forms on X with $c_1(\omega_0) = v_0$ and $c_1(\omega_1) = v_1$. Then the Chern classes of ω_0 and ω_1 lie in distinct orbits for the action of $\mathrm{Diff}(X)$ on $H^2(X, \mathbb{R})$, since diffeomorphisms preserve the Newton polygon of the Seiberg-Witten polynomial. In particular, ω_0 and ω_1 are inequivalent symplectic forms on X. □

Question. Could it be that $\mathrm{Diff}(X)$ actually preserves the submanifold $M \times S^1 \subset X$ up to isotopy?

Further example: skirting gauge theory. To conclude, we sketch an *elementary* example of a 4-manifold X carrying a pair of inequivalent symplectic forms — but with $\pi_1(X) \neq 1$. By elementary, we mean the proof does not use the Seiberg–Witten invariants; instead, it uses the fundamental group.

To construct the example, simply let $X = N \times S^1$, where N is the closed 3-manifold discussed at the end of §2.

By considering N as a covering of T^3 with a $\mathbb{Z}/2$-orbifold locus along L, one can show that $\pi_1(N)$ has trivial center. It follows that $\pi_1(S^1)$ is the center of $\pi_1(X)$, and thus the projection

$$\pi_1(X) \to \pi_1(N)$$

is canonical. In particular, every diffeomorphism of X induces an automorphism of $\pi_1(N)$.

Now let α_0, α_1 be fibrations of N whose Euler classes are in different orbits for the action of $\mathrm{Aut}(\pi_1(N))$ on $H_1(N, \mathbb{Z})$. (These classes exist as before, because the Alexander polynomial is functorially determined by $\pi_1(N)$, and hence preserved by automorphisms.) Then the Euler classes $e(\alpha_0), e(\alpha_1)$ lie in disjoint orbits for the action of $\mathrm{Diff}(X)$ on $H_1(N) = H_1(X)/H_1(S^1)$.

Now as we have seen above, each α_i gives a symplectic form ω_i on X with $c_1(\omega_i)$ dual to $e(\alpha_i) \times S^1$. Since the Euler classes lie in disjoint orbits for the action of Diff(X), so do these Chern classes. In particular, ω_0 and ω_1 are inequivalent symplectic forms on X. □

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