Mathematical Research Letters **7**, 177–194 (2000)

ON CERTAIN EXTERIOR PRODUCT MAPS OF CHOW GROUPS

Chad Schoen

0. Introduction

Let $k \subset K$ be a field extension, W a k-variety, and E a K-variety. In this paper we consider an exterior product homomorphism of Chow groups,

$$(0.1) \qquad CH(W) \otimes CH(E) \xrightarrow{\times} CH(W \times_k E).$$

This map is most familiar in the case $k = K$. However, it is worth investigating and is sometimes more easily understood when $k \subset K$ has positive transcendence degree. This is illustrated by our main result:

(0.2) Theorem. *Suppose that W is smooth and proper and that E is a genus 1 curve whose J-invariant is transcendental over k. If both k and K are algebraically closed, then the exterior product,*

$$(0.3) \qquad CH(W) \otimes CH(E)_{\text{tors}} \xrightarrow{\times} CH(W \times_k E),$$

is injective.

This result gives insight into the nature of the torsion in the Chow groups of certain product varieties. Indeed, if l is a prime, $CH_0(E)_{\text{tors}}$ contains a direct summand isomorphic to $\mathbb{Q}_l/\mathbb{Z}_l$. Thus (0.2) yields

(0.4) Corollary. *Suppose that $CH_s(W) \otimes \mathbb{Q}_l/\mathbb{Z}_l$ has infinite corank. Then the l-primary component of the torsion subgroup of $CH_s(W \times_k E)$ has infinite corank.*

In §3 we give examples of d dimensional varieties, W, and primes, l, for which $CH_s(W) \otimes \mathbb{Q}_l/\mathbb{Z}_l$ has infinite corank for all s in the range $0 < s < d - 1$.

To put (0.4) in perspective recall that Bloch has defined a map,

$$\lambda^r : CH^r(V)[l^\infty] \to H^{2r-1}(V, \mathbb{Q}_l/\mathbb{Z}_l(r)),$$

Received January 7, 2000.

1991 *Mathematics Subject Classification* 14C25.

Key words and phrases. Algebraic cycles, torsion in Chow groups, exterior product, zero cycles

Partial support by NSF and NSA gratefully acknowledged.

for V smooth and projective over an algebraically closed field of characteristic prime to l [Bl]. When $r = 1$ this map may be identified with the map coming from the Kummer sequence and is thus an isomorphism [Bl, 3.6]. When V has dimension r Roitman proved that λ^r is an isomorphism [Ro], [Bl, 4.2]. Merkuriev and Suslin showed that λ^2 is injective [Me-Su, §18]. Since $H^{2r-1}(V, \mathbb{Q}_l/\mathbb{Z}_l(r))$ has finite corank [Mi, VI.2.1], it follows that $CH^r(V)[l^\infty]$ has finite corank for $r = 1$, 2, or $\dim(V)$. Bloch and Srinivas show that this continues to hold for $r = 3$ and $r = \dim(V) - 1$ when $CH_0(V)$ is not too far from being representable [Bl-Sri, §3]. A consequence of (0.3) is the existence of smooth, projective varieties, V, such that the corank of $CH^r(V)[l^\infty]$ is not finite for all r in the range $2 < r < \dim(V)$.

The first example of non-injectivity of λ^3 was given by Totaro [To], who used complex cobordism to detect an element in the kernel. Motivated by thoughts concerning unramified cohomology and [Sch] he suggested that there might exist varieties defined over algebraically closed fields for which $CH^3(V)[l]$ could fail to be finite [To2]. This paper owes its existence to Totaro's suggestion.

Here is an overview of the contents of the individual sections: After covering preliminaries in §1, Theorem (0.2) is proved in §2. In §3 results from [Sch] (or [Sch3]) are used to give examples of varieties, W, and primes, l, for which $CH(W) \otimes \mathbb{Q}_l/\mathbb{Z}_l$ has infinite corank. This leads to examples of product varieties where the l-primary component of the torsion in the Chow group does not have finite corank. Certain abelian fourfolds are shown to have this property.

It is natural to ask to what extent the methods used to prove injectivity of (0.3) may be generalized to give injectivity results for (0.1). Section 4 contains an injectivity result for (0.1) in the case that both W and E are curves and certain additional hypotheses hold. A weaker result, (4.18), is shown to hold when W has arbitrary dimension. A situation in which (0.1) fails to be injective is discussed in §5. Combining these results yields

(0.5) Theorem. *Keep the hypotheses of* (0.2). *Assume in addition that W is a curve and that k is not the algebraic closure of a finite field. Then* (0.1) *is injective if and only if W has no non-constant map to a genus* 1 *curve.*

As an interesting contrast to (0.5) we recall the following intriguing speculation about the behaviour of the Chow group of degree zero zero cycles, denoted $CH_0(\ )_0$, under the ordinary exterior product:

(0.6) Recurring Fantasy. (*Bloch and Beilinson.*) *Let W and E be smooth, irreducible, proper varieties over $\bar{\mathbb{Q}}$. Then the exterior product map,*

$$CH_0(W)_0 \otimes CH_0(E)_0 \xrightarrow{\times} CH_0(W \times E),$$

is zero.

I thank B. Totaro for his stimulating suggestion and for comments which led to an improvement in the exposition.

Notations.

- For W a separated scheme of finite type over a field k and $k \subset K$ a field extension, $W \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K)$ will be denoted by W_K or $W \times_k K$.
- $Z_s(W) :=$ the group of cycles of dimension s on W.
- $Z_s(W)_{\mathrm{rat}} :=$ the subgroup of cycles of dimension s which are rationally equivalent to 0 [Fu, 1.3].
- $CH_s(W) := Z_s(W)/Z_s(W)_{\mathrm{rat}}$.
- $Z_s(W)_{\mathrm{alg}} :=$ the subgroup of cycles of dimension s which are algebraically equivalent to 0 [Fu, 10.3].
- $Z(W) := \oplus_{s\in\mathbb{Z}} Z_s(W), \qquad CH(W) := \oplus_{s\in\mathbb{Z}} CH_s(W)$.
- $CH^r(W) := CH_{d-r}(W)$ if W has pure dimension d.
- $CH^r(W)_0 := \mathrm{Ker} : \ CH^r(W) \to \prod_{l\neq \mathrm{char}(k)} H^{2r}(W, \mathbb{Z}_l(r))$, when W/k is smooth and proper and k is algebraically closed.
- $A[m] := \mathrm{Ker} : A \xrightarrow{\cdot m} A$, where A is an abelian group and $\cdot m$ is multiplication by an integer, m.
- $A[l^\infty] := \cup_{n\in\mathbb{N}} A[l^n]$.
- $A_{\mathrm{tors}} :=$ the torsion subgroup of A.
- $\mathbb{N} = \{n \in \mathbb{Z} : n > 0\}$.

1. Preliminaries on Chow groups and exterior products

Let $k \subset K$ be a field extension. Suppose that W is a finite type, separated k-scheme and E is a finite type, separated K-scheme. If $V \subset W$ is a closed k-subvariety and $U \subset E$ is a closed K-subvariety, then $V \times_k U$ is a closed subscheme of $W \times_k E \simeq W \times_k K \times_K E$. Each closed subscheme of $W \times_k E$ gives rise to a cycle in $Z(W \times_k E)$. This proceedure defines an exterior product homomorphism on cycles,

$$Z_s(W) \otimes Z_t(E) \xrightarrow{\times} Z_{s+t}(W \times_k E). \tag{1.1}$$

Recall that the natural map, $W_K \to W$, induces a pullback map on cycles,

$$\upsilon_{K/k} : Z_s(W) \to Z_s(W_K). \tag{1.2}$$

Since $V \times_k U \simeq (V \times_k K) \times_K U$, (1.1) may be viewed as the composition,

$$Z_s(W) \otimes Z_t(E) \xrightarrow{\upsilon_{K/k}\otimes Id} Z_s(W_K) \otimes Z_t(E) \xrightarrow{\times} Z_{s+t}(W_K \times_K E), \tag{1.3}$$

where the second map is the exterior product in [Fu,1.10]. If L is an intermediate field in the extension, $k \subset K$, there is a commutative diagram,

$$\begin{array}{ccc} Z_s(W) \otimes Z_t(E) & \xrightarrow{\times} & Z_{s+t}(W \times_k E) \\ {\scriptstyle \upsilon_{L/k}\otimes Id}\downarrow & & \downarrow \simeq \\ Z_s(W_L) \otimes Z_t(E) & \xrightarrow{\times} & Z_{s+t}(W_L \times_L E). \end{array} \tag{1.4}$$

(1.5) Lemma.

(i) *If* $\alpha \in Z_s(W)_{\mathrm{rat}}$ *or* $\beta \in Z_t(E)_{\mathrm{rat}}$*, then* $\alpha \times \beta \in Z_{s+t}(W \times_k E)_{\mathrm{rat}}$.

(ii) *The exterior product induces a homomorphism,*

$$CH_s(W) \otimes CH_t(E) \xrightarrow{\times} CH_{s+t}(W \times_k E).$$

(iii) *If* $\alpha \in Z_s(W)_{\mathrm{alg}}$ *or* $\beta \in Z_t(E)_{\mathrm{alg}}$*, then* $\alpha \times \beta \in Z_{s+t}(W \times_k E)_{\mathrm{alg}}$.

Proof. (i) By (1.3) and the fact that $\upsilon_{K/k}$ respects rational equivalence, one is reduced to verifying the assertion when $k = K$. This is done in [Fu, 1.10].

(ii) is an immediate consequence of (i).

(iii) The proof of (iii) is analogous to the proof of (i) since algebraic equivalence satisfies the same formal properties as rational equivalence [Fu, 10.3]. □

(1.6) Remark. Classical examples show that the homomorphism in (1.5)(ii) need not be either injective or surjective. Perhaps the simplest illustration of this is the case that $k = K$ is algebraically closed (but not the algebraic closure of a finite field) and $W = E$ is a genus 1 curve. Then surjectivity fails because the class of the diagonal is not in the image. To show non-injectivity choose points $p, e \in E$ such that $p - e \in CH(E)$ has infinite order. Then $(p-e) \otimes (p-e)$ is a non-zero element of the kernel as one sees by choosing $q \in E$ such that $2(q-e) = p - e \in CH(E)$ and writing

$$(4.2) \quad (p,p) - (p,e) - (e,p) + (e,e) = [(p,p) + (e,e) - 2(q,q)] - [(p,e) + (e,p) - 2(q,q)].$$

Each of the bracketed expressions gives a cycle which is rationally equivalent to zero on a copy of E embedded in $E \times E$ (cf. [Scholl, 3.5]). □

The next three lemmas give formal statements of well known facts which will be used repeatedly in the sequel.

(1.7) Lemma. *Let* W *be a variety defined over an algebraically closed field,* k*. Let* K *be an algebraically closed extension field of* k*. The base change map* $W_K \to W$ *induces an injection,* $CH(W) \to CH(W_K)$.

Proof. According to [Bl2, Lemma 3 p.1.21] the kernel of this map is contained in $CH(W)_{\mathrm{tors}}$. By a theorem of Lecomte [Le] inspired by Suslin [Su], the map $CH(W)_{\mathrm{tors}} \to CH(W_K)_{\mathrm{tors}}$ is an isomorphism. □

(1.8) Lemma. *Let* $k \subset K$ *be an algebraic field extension. Let* $\mathcal{K}$ *be the directed system of intermediate fields which are finite extensions of* k *ordered by inclusion. If* W *is a* k*-variety, then*

$$CH(W_K) \simeq \varinjlim_{L \in \mathcal{K}} CH(W_L).$$

(1.9) Lemma. *Let $\pi : Y \to X$ be a morphism of varieties over a field with X irreducible. Write η for the generic point of X. Let $\mathcal{U}$ be a directed system of non-empty open subsets of X ordered by inclusion. If $\cap_{U\in\mathcal{U}} U = \eta$, then $CH(\pi^{-1}(\eta)) \simeq \varinjlim_{U\in\mathcal{U}} CH(\pi^{-1}(U))$.*

Proof. The proof is essentially the same as that of the special case treated in [Bl2, p. 1.20]. □

2. Proof of Theorem (0.2)

Let $k \subset K$ be an extension of algebraically closed fields. Let W be a smooth, proper k-variety and let E_K be a smooth, irreducible, proper K-curve of genus 1. (The reason we now write E_K instead of simply E will become apparent shortly.) Write $J \in K$ for the J-invariant of E_K [Ha, IV.4] or [Sil, IV.1.4(b)]. Fix a prime number l. Now Theorem (0.2) is an immediate consequence of

(2.1) Proposition. *Suppose that J is transcendental over k. Then the exterior product map,*

$$CH(W) \otimes CH^1(E_K)[l^\infty] \xrightarrow{\times} CH(W \times_k E_K), \tag{2.2}$$

is injective.

Proof. Let E be a genus 1 curve defined over $k(J)$ such that the base change of E to K becomes isomorphic to E_K. By choosing E with care, we may arrange that it has a degree one point, s, which gives E the structure of elliptic curve [Sil, III.1.4(c)]. Let $K_0 \subset K$ denote the algebraic closure of $k(J)$. Using the isomorphism, $CH^1(E_{K_0})[l^\infty] \simeq CH^1(E_K)[l^\infty]$ we may identify (2.2) with the composition,

$$CH(W) \otimes CH^1(E_{K_0})[l^\infty] \xrightarrow{\times} CH(W \times_k E_{K_0}) \to CH(W \times_k E_K). \tag{2.3}$$

As the second map is injective by (1.7), we need only show that the first map is injective.

For $z \in CH(W)$ and $\tau \in CH^1(E_{K_0})[l^\infty]$, $z \otimes \tau = 0$ if either z is torsion or z is infinitely divisible by l. Set $r(l) = 1$ if $l = \mathrm{char}(k)$ and $r(l) = 2$ otherwise. An arbitrary non-zero element of the left hand term in (2.3) may be written in the form $\mathfrak{c} = \sum_{i=1}^{r(l)} z_i \otimes \tau_i$, where $\{\tau_1, \tau_{r(l)}\}$ is a basis of $CH^1(E_{K_0})[l^n]$, $n \in \mathbb{N}$, $z_i \in CH(W)$ and $z_1 \notin lCH(W) + CH(W)_{\mathrm{tors}}$. To prove (2.1) we assume that $\mathfrak{c}$ is a non-zero element in the kernel of the map $\times$ in (2.3) and derive a contradiction.

Let $L_0 \subset K$ be a finite extension of $k(J)$ such that $E_{L_0}[l^n] \simeq (\mathbb{Z}/l^n)^{r(l)}$. For any finite extension, $L_0 \subset L$, we may view $\mathfrak{c}$ as an element of $CH(W) \otimes CH^1(E_L)[l^n]$. By (1.8)

$$CH(W \times_k E_{K_0}) \simeq \lim_{L \subset K_0} CH(W \times_k E_L),$$

where L varies over finite extensions of L_0. It follows that there is a finite extension, $L_0 \subset L_1$, contained in K_0 such that for any finite extension, $L_1 \subset L$, $\mathfrak{c}$ is in the kernel of

$$CH(W) \times CH^1(E_L)[l^n] \xrightarrow{\times} CH(W \times_k E_L). \tag{2.4}$$

Given such an L, let X denote a smooth, irreducible, projective k-curve whose generic point gives the field extension, $k \subset L$. There is a smooth, projective k-surface, Y, and a morphism, $\pi : Y \to X$, which is a relatively minimal model of the curve, E_L. We may choose L so that that π is semi-stable [Sil, VII.5.4] and every singular fiber has Kodaira type I_M for some M with $l^n|M$. The origin $s \in E_L$ extends to a section, s, of π, which we call the identity section. Choose a basis, $\varsigma_1, \varsigma_{r(l)}$ of $E_L[l^n]$ so that $\tau_i = \varsigma_i - s$. The points $\varsigma_1, \varsigma_{r(l)}$ also extend to sections of π, denoted again $\varsigma_1, \varsigma_{r(l)}$. By (1.9) there is a non-empty open subset, $X' \subset X$ such that $\sum_{i=1}^{r(l)} z_i \otimes (\varsigma_i - s)$ lies in the kernel of

$$CH(W) \times CH^1(Y') \xrightarrow{\times} CH(W \times_k Y'), \tag{2.5}$$

where $Y' := \pi^{-1}(X')$.

The next lemma describes how the sections ς_i meet a certain singular fiber of π. Recall that the set of reduced components of a fiber in a relatively minimal elliptic pencil with section inherit a group structure from the Néron model of the generic fiber. In the case of a fiber of Kodaira type I_M this component group is isomorphic to $\mathbb{Z}/M$. We label the components $F_j, 0 \leq j \leq M-1$, where F_0 meets the identity section, $F_1 \cdot F_0 = 1 = F_{M-1} \cdot F_0$ and $F_j \cdot F_{j'} = 0$ for $j, j' \in \{1, ..., M-1\}$ unless $|j - j'| \leq 1$. The intersection matrix, $(F_j \cdot F_{j'})$, $1 \leq j, j' \leq M_1 - 1$ is given by

$$F_j \cdot F_j = -2, \qquad F_j \cdot F_{j'} = 1 \text{ if } |j - j'| = 1, \qquad F_j \cdot F_{j'} = 0 \text{ if } |j - j'| > 1. \tag{2.6}$$

An isomorphism from the component group to $\mathbb{Z}/M$ is given by $F_j \mapsto j$.

(2.7) Lemma. *There is a point $x \in X$ such that*

(i) *The fiber $\pi^{-1}(x)$ has Kodaira type $I_{l^n N}$ for some $N \in \mathbb{N}$.*
(ii) *The unique component of $\pi^{-1}(x)$ meeting ς_1 has the form F_{mN}, with $gcd(m, l) = 1$.*
(iii) *If $r(l) = 2$, then the unique component of $\pi^{-1}(x)$ meeting ς_2 is F_0.*

Proof. (i) follows from the fact that π has at least one singular fiber, which holds because J is transcendental over k.

If $l \neq \text{char}(k)$ and if $l^n \neq 2$ there is a universal generalized elliptic curve with level l^n-structure [De-Ra], which we denote $\pi_{l^n} : Y(l^n) \to X(l^n)$. By collapsing the components of order not dividing l^n in all singular fibers of π we obtain a generalized elliptic curve, $\bar{\pi} : \bar{Y} \to X$, whose level l^n structure is given by the sections ς_2 and ς_1. Now $(\bar{Y}, \varsigma_2, \varsigma_1)$ is obtained from the universal family by pullback with respect to a morphism, $\xi : X \to X(l^n)$. Removing the singular points from a singular fiber of π_{l^n} gives a group isomorphic to $\mathbb{G}_m \times \mathbb{Z}/l^n$. Consider the level l^n structures of the form $(\zeta_2, 0), (\zeta_1, m)$ with $\zeta_1, \zeta_2 \in \mu_{l^n}$, $m \in \mathbb{Z}/l^n$ and m and ζ_2 generators. The canonical level l^n structure on $\pi_{l^n}^{-1}(x')$ has this form for some $x' \in \mathrm{im}(\xi)$. This follows from the universal property of π_{l^n} and the fact that the irreducible components of $X(l^n)$ are parametrized by the value of the Weil pairing on the tautological basis of the l^n-torsion. If $x \in \xi^{-1}(x')$, then (i), (ii) and (iii) hold. The same argument may be made to work when $l^n = 2$ by replacing l^n with $l^n p'$ for some auxilliary odd prime, p', distinct from the characteristic. In this case one must choose L_0 above so that $E_{L_0}[l^n p'] \simeq (\mathbb{Z}/l^n p')^2$.

It remains to prove (ii) in the case $l = \mathrm{char}(k)$. The only way (ii) could fail to hold is if the section of order l, $\kappa = l^{n-1}\varsigma_1$, were to meet every singular fiber in the identity component. This amounts to the divisor, $l(\kappa - s)$, meeting every fiber component in a cycle which is rationally equivalent to zero. Such a divisor is algebraically equivalent to $m\mathfrak{f}$, a multiple of a smooth fiber. To see that $m = 0$, intersect $l(\kappa - s)$ with a torsion multi-section of order prime to l and the characteristic of k. Since κ and s are contained in the Néron model, $\mathcal{Y}$, one can check fiberwise that such a multi-section is disjoint from κ and s. This implies that $l(\kappa - s)$ is algebraically equivalent to zero. To get a contradiction note

$$(\kappa - s)\cdot s \geq -s\cdot s = -\deg(R^1\pi_*\mathcal{O}_Y) = \chi(\mathcal{O}_X) - \chi(R^1\pi_*\mathcal{O}_Y) = \chi(\mathcal{O}_Y) = e(Y)/12,$$

where $e(Y)$ is the etale euler characteristic, the last equality is Noether's formula, the previous one comes from the Leray spectral sequence, the one prior to that from Riemann-Roch on X, and the first one from the formula for the normal sheaf to a section of an elliptic surface. We claim that the euler characteristic, $e(Y)$, is equal to the total number of singular points in all the fibers and is thus positive. This may be verified by removing the singular fibers from Y to form $\dot{Y}$, showing $e(\dot{Y}) = 0$, and applying the long exact cohomology sequence of the pair, $(Y, \dot{Y})$. Alternately, one may use [Mi,V.2.15(c)] and the fact that the representation in the torsion prime to $\mathrm{char}(k)$ of a semi-simple elliptic curve is tamely ramified. The positivity of $e(Y)$ contradicts the notion that $l(\kappa - s)$ could be algebraically equivalent to zero. □

(2.8) Lemma. *Let $x \in X$ and N be as in* (2.7). *Then there is a divisor, D, supported on $\pi^{-1}(x)$ such that*

(i) *If F is a divisor supported in a closed fiber of π, then $D\cdot F \equiv 0 \mod l^n N$.*
(ii) *$D \cdot (\varsigma_1 - s) = mN$ for some m with $\gcd(m, l) = 1$.*
(iii) *If $r(l) = 2$, then the divisors D and $\varsigma_2 - s$ have disjoint support.*

Proof. Set $M = Nl^n$ and define $D := \sum_{j=1}^{M-1} jF_j$. Now (i) is clear if F is not supported on $\pi^{-1}(x)$. If F is supported on $\pi^{-1}(x)$, (i) follows from an easy computation using (2.6):

$$D \cdot F_j \equiv 0 \mod M \quad \forall\ j, \quad 0 \leq j \leq M-1.$$

The remaining assertions are immediate from (2.7). □

Returning to the proof of (2.1) set $\mathbf{F} = Y - Y'$ and construct a commutative diagram whose middle column is the localization sequence,

$$\begin{array}{ccccc} CH(W)\otimes CH^1(Y') & \xrightarrow{\times} & CH(W\times_k Y') & & \\ {\scriptstyle v_0}\uparrow & & {\scriptstyle v_1}\uparrow & & \\ CH(W)^{r(l)} & \xrightarrow{h_1} & CH(W\times_k Y) & \xrightarrow{h_2} & CH(W), \\ & & {\scriptstyle v_2}\uparrow & & \\ & & CH(W\times \mathbf{F}) & & \end{array} \tag{2.9}$$

by defining

$$h_1((\mathfrak{z}_1, \mathfrak{z}_{r(l)})) = \sum_{i=1}^{r(l)} \mathfrak{z}_i \times (\varsigma_i - s), \qquad v_0((\mathfrak{z}_1, \mathfrak{z}_{r(l)})) = \sum_{i=1}^{r(l)} \mathfrak{z}_i \otimes (\varsigma_i - s),$$

and letting h_2 be the composition,

$$CH(W\times Y) \xrightarrow{\cdot(W\times D)} CH(W\times Y) \xrightarrow{pr_{W*}} CH(W).$$

Since the top row is (2.5), $(\times \circ v_0)((z_1, z_{r(l)})) = 0$. Thus $v_1 \circ h_1((z_1, z_{r(l)})) = 0$, so

$$h_1((z_1, z_{r(l)})) \in \mathrm{im}(v_2). \tag{2.10}$$

By (2.8) (ii) and (iii) $h_2 \circ h_1((z_1, z_{r(l)})) = mNz_1$ for some m with $\gcd(m, l) = 1$.

(2.11) Lemma. *The image of $h_2 \circ v_2$ is contained in $l^n NCH(W)$.*

Proof. Let $\mathbf{F}' \subset \mathbf{F}$ denote the subvariety consisting of irreducible components which do not meet the support of D. Clearly $h_2 \circ v_2(CH(W \times \mathbf{F}')) = 0$. Each component, F_j, of $\mathbf{F}$ which is not contained in $\mathbf{F}'$ is isomorphic to $\mathbb{P}^1_k$ (2.7)(i). By [Fu, 3.3(b)]

$$CH(W)\otimes CH(F_j) \xrightarrow{\times} CH(W\times F_j)$$

is an isomorphism. For $\mathfrak{z}, \mathfrak{z}' \in CH(W)$ and a point $y_0 \in F_j$,

$$h_2 \circ v_2(\mathfrak{z}\times y_0 + \mathfrak{z}'\times F_j) = pr_{W*}(\mathfrak{z}\times(y_0\cdot D) + \mathfrak{z}'\times(F_j\cdot D)) = (F_j\cdot D)\mathfrak{z}'.$$

By (2.8)(i) $F_j \cdot D \equiv 0 \mod l^n N$. □

We claim $h_2 \circ h_1((z_1, z_{r(l)})) \notin \mathrm{im}(h_2 \circ v_2)$. Otherwise $mNz_1 = l^n N\mathfrak{z}$ for some $\mathfrak{z} \in CH(W)$, which leads to $mz_1 = l^n\mathfrak{z}$ in $CH(W)/CH(W)_{\mathrm{tors}}$, contradicting $z_1 \notin lCH(W) + CH(W)_{\mathrm{tors}}$. The claim implies that $h_1((z_1, z_{r(l)})) \notin \mathrm{im}(v_2)$ which contradicts (2.10) and proves (2.1). □

3. Varieties, V, for which corank$(CH(V)[l^\infty]) = \infty$

In this section W denotes a smooth, proper irreducible variety over an algebraically closed field, k, and $l \neq \mathrm{char}(k)$ is a prime number.

(3.1) Proposition. *If $CH^2(W)/l$ is infinite, then $CH^2(W) \otimes \mathbb{Q}_l/\mathbb{Z}_l$ has infinite corank.*

Proof. By the result of Merkuriev and Suslin refered to in the introduction, the cycle class map,

$$\lambda^2 : CH^2(W)[l^\infty] \to H^3(W, \mathbb{Q}_l/\mathbb{Z}_l(2)),$$

is injective. It follows that $CH^2(W)_{\mathrm{tors}}/l$ is a finite group. The proposition now follows from a purely group theoretic lemma.

(3.2) Lemma. *Let A be an abelian group with $A \otimes \mathbb{Z}/l$ infinite and $A_{\mathrm{tors}} \otimes \mathbb{Z}/l$ finite. Then $A \otimes \mathbb{Q}_l/\mathbb{Z}_l$ does not have finite corank.*

Proof. Since $\bar{A} := A/A_{\mathrm{tors}}$ is torsion free, it is flat [Ha, III.9.1.3]. Thus

$$0 \longrightarrow \bar{A} \otimes \mathbb{Z}/l \longrightarrow \bar{A} \otimes \mathbb{Q}_l/\mathbb{Z}_l \xrightarrow{l} \bar{A} \otimes \mathbb{Q}_l/\mathbb{Z}_l \longrightarrow 0$$

is exact. As $\bar{A} \otimes \mathbb{Q}_l/\mathbb{Z}_l$ is a divisible, l-power torsion group, it is isomorphic to a direct sum of $\mathbb{Q}_l/\mathbb{Z}_l$'s [Rot, 10.13]. Since $\bar{A} \otimes \mathbb{Z}/l$ is not finite, there are infinitely many summands. Thus $A \otimes \mathbb{Q}_l/\mathbb{Z}_l \simeq \bar{A} \otimes \mathbb{Q}_l/\mathbb{Z}_l$ does not have finite corank.

□ □

(3.3) Corollary. *If $\dim(W) = 3$ and $CH^2(W)/l$ is not finite, then $W \times \mathbb{P}^{d-3}$ is a d-dimensional variety for which $CH_s(W \times \mathbb{P}^{d-3}) \otimes \mathbb{Q}_l/\mathbb{Z}_l$ has infinite corank for all s in the range $0 < s < d-1$.*

Proof. $CH_s(W \times \mathbb{P}^{d-3})$ contains a direct summand which is isomorphic to $CH^2(W)$ if $0 < s < d-1$ [Fu, 3.3(b)]. □

Examples of primes l and varieties, W, satisfying the hypotheses of (3.1) and (3.3) may be found in [Sch, §9 and §10] and [Sch3]. In particular, if k is any algebraically closed field of characteristic 0, $l \equiv 1 \mod 3$, $B \subset \mathbb{P}^2_k$ is defined by $x_0^3 + x_1^3 + x_2^3 = 0$, and $W = B^3$, then the hypotheses of (3.1) and (3.3) hold [Sch, (0.2)].

(3.4) Corollary. *There is an Abelian fourfold, V, defined over an algebraically closed extension, K, of $\mathbb{Q}$ of transcendence degree one so that $CH_1(V)[l^\infty]$ does not have finite corank for any prime $l \equiv 1 \mod 3$.*

Proof. Let $J \in \mathbb{C}$ be a transcendental number, $k = \bar{\mathbb{Q}}$, $K = \overline{\mathbb{Q}(J)}$, E/K an elliptic curve with J-invariant equal to J and $V = B^3 \times_k E$. Now V/K satisfies the desired properties by (0.4). □

(3.5) Open Problem. Does there exist a smooth, *projective* variety, V, over $\bar{\mathbb{Q}}$ such that $CH(V)[l^\infty]$ has infinite corank?

(3.6) Remark. There do exist smooth, *quasi-projective* varieties, V, defined over $\bar{\mathbb{Q}}$ with $CH(V)[l]$ infinite. An example is given by $V = W \times (\mathbb{P}^2 - C)$ where W is a threefold over $\bar{\mathbb{Q}}$ with $CH(W)/l$ infinite and C is an irreducible, rational curve of degree l. The proof is similar to that of the following simpler example pointed out by Totaro:

(3.7) Lemma. *Let P be a line bundle over $\mathbb{P}^1$ whose sheaf of sections is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(l)$. Let $C \subset P$ be the zero section. Then $CH(W \times (P - C)) \simeq CH(W)/l \oplus CH(W)$.*

Proof. This follows from the commutative diagram with exact rows,

$$\begin{array}{ccccccc}
CH(W) \otimes \mathbb{Z}^2 & \xrightarrow{Id\otimes(0,l)} & CH(W) \otimes \mathbb{Z}^2 & \longrightarrow & CH(W) \oplus CH(W)/l & \longrightarrow & 0 \\
\simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow & & \\
I \otimes CH(C) & \longrightarrow & I \otimes CH(P) & \longrightarrow & I \otimes CH(P - C) & \longrightarrow & 0 \\
\simeq\downarrow & & \simeq\downarrow & & \downarrow & & \\
CH(W \times C) & \longrightarrow & CH(W \times P) & \longrightarrow & CH(W \times (P - C)) & \longrightarrow & 0,
\end{array}$$

where $I = CH(W)$ and the lower vertical isomorphisms follow from [Fu, 3.3]. □

4. Injectivity results for the exterior product map

Assume that $k \subset K$ is an extension of algebraically closed fields. We begin with some elementary lemmas concerning injectivity of the exterior product map (0.1) restricted to various subgroups of the domain.

(4.1) Lemma. *Suppose that W is a k-variety and E is an irreducible, K-variety of dimension, r. Then the exterior product map,*

$$CH(W) \otimes CH_r(E) \xrightarrow{\times} CH(W \times_k E), \tag{4.2}$$

is injective. Furthermore, the intersection of the image of (4.2) *and the image of*

$$CH(W) \otimes (\oplus_{i<r} CH_i(E)) \xrightarrow{\times} CH(W \times_k E) \tag{4.3}$$

is 0.

Proof. Fix a closed point, s, in the non-singular locus of E. The map,

$$CH(W) \to CH(W_K), \qquad z \mapsto (z \times E) \cdot (W \times s),$$

agrees with the map of (1.7) and is hence injective. Thus (4.2) is injective.

It remains to show that each element, $\mathfrak{z}$, of the image of (4.3) is annihilated by $\cdot(W \times s)$ for a suitable choice of s. In fact we may choose s and a cycle Z representing $\mathfrak{z}$ such that $W \times s$ is disjoint from the support of Z. □

Suppose that W is a proper k-variety and E is an irreducible, proper K-variety. Let $s \in E$ be a closed point in the non-singular locus of E. Write $CH_0(E)_0 \subset CH_0(E)$ for the kernel of the degree map. There is a direct sum decomposition,

$$CH_0(E) \simeq CH_0(E)_0 \oplus \mathbb{Z}s.$$

Write $pr_W : W \times_k E \to W_K$ for the composition,

$$W \times_k E \simeq W \times_k K \times_K E \xrightarrow{pr_1} W \times_k K.$$

(4.4) Lemma. *The composition,*

$$CH(W) \otimes CH_0(E) \xrightarrow{\times} CH(W \times_k E) \xrightarrow{pr_{W*}} CH(W_K),$$

annihilates $CH(W) \otimes CH_0(E)_0$ and restricts to an injective map on $CH(W) \otimes \mathbb{Z}s$.

Proof. The first assertion is clear. The restriction of this map to $CH(W) \otimes \mathbb{Z}s$ may be identified with the map of (1.7) which is injective. □

Suppose now that E is a smooth, irreducible, projective curve over K. Write $CH^1(E)_0$ for the group of degree zero zero cycles. By (4.1) and (4.4) to show that the exterior product,

$$CH(W) \otimes CH(E) \xrightarrow{\times} CH(W \times_k E), \tag{4.5}$$

is injective, it suffices to show that

$$CH(W) \otimes CH^1(E)_0 \xrightarrow{\times} CH(W \times_k E), \tag{4.6}$$

is injective. Since $CH^1(E)_{\mathrm{tors}} \subset CH^1(E)_0$ is a divisible subgroup of a divisible group, there is a $\mathbb{Q}$-vector space, $CH^1(E)_{tf}$, and a (non-canonical) direct sum decomposition

$$CH(W) \otimes CH^1(E)_0 \simeq CH(W) \otimes CH^1(E)_{\mathrm{tors}} \oplus CH(W) \otimes CH^1(E)_{tf}. \tag{4.7}$$

To show that (4.6) is injective, it would suffice to show that (4.6) restricted to each of the summands on the right hand side of (4.7) is injective. We have already dealt with the first summand in §2 (in the case that E has genus 1).

(4.8) Proposition. *Let $k \subsetneq K$ be an extension of algebraically closed fields. Let E_K be a smooth, irreducible, projective curve of genus g over K. Let W be a smooth, projective curve over k. Suppose that no positive dimensional abelian subvariety of $\mathrm{Pic}^0(E_K)$ descends to an abelian variety defined over k. Suppose further that $\mathrm{Hom}(\mathrm{Jac}(W), \mathrm{Jac}(C)) = 0$ for every curve C of genus $\leq g$. Then the exterior product map,*

$$CH(W) \otimes CH^1(E_K)_{tf} \xrightarrow{\times} CH(W \times_k E_K), \tag{4.9}$$

is injective.

Proof. The proof is by contradiction. Suppose that divisors, $\tau_i \in Z^1(E_K)$, which map to elements of $CH^1(E_K)_{tf}$, and cycles, $z_i \in Z(W)$, existed such that $\sum_{i=1}^r z_i \otimes \tau_i$ represents a non-zero element of the kernel of (4.9). We may assume that $\{\tau_1, ..., \tau_r\} \subset CH^1(E_K)_{tf}$ is linearly independent. Let $k \subset L_0$ be a finitely generated extension with the property that E_K is obtained from a curve, E, defined over L_0 by extension of scalars. (We require that L_0 and all other fields mentioned in this proof be subfields of K.) By replacing L_0 with a finitely generated extension field we may assume that each divisor τ_i, $1 \leq i \leq r$ has the form $\tau_i = \sum_j n_{ij} a_{ij}$, where $n_{ij} \in \mathbb{N}$ and a_{ij} is a degree one point of E_{L_0}. Let $L_1 \subset L_0$ be a subfield, finitely generated over k and of transcendence degree one less than the transcendence degree of L_0/k. We may choose L_1 such that no positive dimensional abelian subvariety of $\mathrm{Pic}^0(E_K)$ descends to an abelian variety defined over the algebraic closure, L, of L_1.

Write $\mathbf{z}_i$ for the image of z_i under the injective map, $\upsilon_{L/k} : CH(W) \to CH(W_L)$. Since $CH^1(E_K)_{tf}$ is flat over $\mathbb{Z}$ it follows that

$$CH(W) \otimes CH^1(E_K)_{tf} \xrightarrow{\upsilon_{L/k} \otimes Id} CH(W_L) \otimes CH^1(E_K)_{tf},$$

is injective. Since $\sum_{i=1}^r z_i \otimes \tau_i \neq 0$ it follows that $\mathfrak{c} := \sum_{i=1}^r \mathbf{z}_i \otimes \tau_i \neq 0$. Let K_0 denote the algebraic closure of L_0 in K. We may view $\tau_1, ..., \tau_r$ as linearly independent elements of $CH^1(E_{K_0})_0$ and $\mathfrak{c}$ as an element of the domain of the exterior product map,

$$CH(W_L) \otimes CH^1(E_{K_0})_0 \xrightarrow{\times} CH(W_L \times_L E_{K_0}). \tag{4.10}$$

In fact $\mathfrak{c}$ is in the kernel of (4.10) since $CH(W_L \times_L E_{K_0}) \to CH(W_L \times_L E_K)$ is injective (1.7). Let $L_2 := LL_0$. For any finite extension, $L_2 \subset L_3$, we may view $\mathfrak{c}$ as an element of the domain of the exterior product map,

$$CH(W_L) \otimes CH^1(E_{L_3})_0 \xrightarrow{\times} CH(W_L \times_L E_{L_3}). \tag{4.11}$$

By (1.8) we may choose $L_3 \subset K_0$ such that $\mathfrak{c}$ is in the kernel of (4.11).

The field extension, $L \subset L_3$, is finitely generated of transcendence degree one. Write X for the corresponding smooth, projective, irreducible L-curve. By resolution of singularities for surfaces, there is a smooth, projective L-surface, Y, and a morphism, $\pi : Y \to X$, which is a relatively minimal model for the curve, E_{L_3}. Given a non-empty open subset, $X' \subset X$, define $Y' := \pi^{-1}(X')$ and consider the exterior product,

$$CH(W_L) \otimes CH^1(Y') \xrightarrow{\times} CH(W_L \times Y'). \tag{4.12}$$

Write a'_{ij} for the closure in Y' of the degree one point, $a_{ij} \in E_{L_3}$ and define $\tau'_i := \sum_j n_{ij} a'_{ij} \in Z^1(Y')$. By (1.9) we may choose X' so that $\mathfrak{c}' := \sum_{i=1}^r \mathbf{z}_i \otimes \tau'_i$ lies in the kernel of (4.12). Write $\widehat{a}_{ij}$ for the closure of a_{ij} in Y and define $\widehat{\tau}_i := \sum_j n_{ij} \widehat{a}_{ij} \in Z^1(Y)$.

(4.13) Lemma. *There are elements of the Néron-Severi group, $D_1, \dots, D_r \in N.S.(Y)$, whose intersection number with every component of every fiber of π is zero and whose intersection numbers with the $\widehat{\tau}_j$'s are given by*

$$D_i \cdot \widehat{\tau}_j = c_i \delta_{ij}, \qquad c_i \in \mathbb{N}, \qquad 1 \le i, j \le r.$$

Proof. Since no positive dimensional abelian subvariety of $Pic^0(E_{L_3})$ descends to an abelian variety defined over L, the quotient, $CH^1(Y)/\pi^* CH^1(X)_0$, is isomorphic to the Néron-Severi group, $N.S.(Y)$. Thus intersection with the generic fiber gives a well defined map,

$$N.S.(Y) \to CH^1(E_{L_3}).$$

Let s be a section of π. Write $N_0 \subset N.S.(Y)$ for the submodule generated by components of fibers of π and the section s. The intersection pairing restricted to N_0 has signature $(1, \mathrm{rank}(N_0) - 1)$ since the intersection form on the subspace generated by fiber components which don't meet s is negative definite [Des, p. 7]. Let $N \subset N.S.(Y)$ denote the span of $\widehat{\tau}_1, ..., \widehat{\tau}_r$. Now $N \cap N_0 = 0$, because the restriction of N to $CH^1(E_{L_3})$ is a rank r submodule of $CH^1(E_{L_3})_0$ while the restriction of N_0 is generated by the class of s. The orthogonal projection with respect to the intersection pairing maps N injectively to $N_0^\perp$. By the Hodge index theorem, the intersection pairing on $N_0^\perp$ is negative definite. Thus there are divisor classes $D_1, ..., D_r \in N_0^\perp$ such that the intersection numbers are given by

$$D_i \cdot \widehat{\tau}_j = c_i \delta_{ij}, \qquad c_i \in \mathbb{N}, \qquad 1 \le i, j \le r. \qquad \square$$

Set $\mathbf{F} = Y - Y'$. Construct a commutative diagram whose middle column is the localization sequence,

$$\begin{array}{ccccc} CH(W_L) \otimes CH^1(Y') & \xrightarrow{\times} & CH(W_L \times Y') & & \\ \uparrow{\scriptstyle v_0} & & \uparrow{\scriptstyle v_1} & & \\ CH(W_L)^r & \xrightarrow{h_1} & CH(W_L \times Y) & \xrightarrow{h_2} & CH(W_L)^r, \\ & & \uparrow{\scriptstyle v_2} & & \\ & & CH(W_L \times \mathbf{F}) & & \end{array} \tag{4.14}$$

by defining

$$h_1((\mathfrak{z}_1, ..., \mathfrak{z}_r)) = \sum_{i=1}^{r} \mathfrak{z}_i \times \widehat{\tau}_i, \qquad v_0((\mathfrak{z}_1, ..., \mathfrak{z}_r)) = \sum_{i=1}^{r} \mathfrak{z}_i \otimes \tau_i',$$
$$h_2(\mathfrak{Z}) = (q_1(\mathfrak{Z}), ..., q_r(\mathfrak{Z})),$$

where q_i is the composition,

$$CH(W_L \times Y) \xrightarrow{\cdot(W_L \times D_i)} CH(W_L \times Y) \xrightarrow{pr_{W*}} CH(W_L).$$

Since $\mathfrak{c}'$ lies in the kernel of (4.12), $(\times \circ v_0)((\mathbf{z}_1, ..., \mathbf{z}_r)) = 0$. Thus, $v_1 \circ h_1((\mathbf{z}_1, ..., \mathbf{z}_r)) = 0$, so

$$h_1((\mathbf{z}_1, ..., \mathbf{z}_r)) \in \mathrm{im}(v_2). \tag{4.15}$$

Now $CH(W_L)^r$ is a module over the ring $\mathbb{Z}^r$. By (4.13) the composition, $h_2 \circ h_1$, is multiplication by the non-zero divisor, $(c_1, ..., c_r) \in \mathbb{Z}^r$. Since $\sum_i \mathbf{z}_i \otimes \tau_i \in CH(W_L) \otimes CH^1(E_K)_{tf}$ is not zero and $CH^1(E_K)_{tf}$ is a $\mathbb{Q}$-vector space, some $\mathbf{z}_i \in CH(W_L)$ is not torsion. It follows that $h_2 \circ h_1((\mathbf{z}_1, ..., \mathbf{z}_r)) \neq 0$.

We claim $h_2 \circ v_2 = 0$. To verify this let F be the normalization of an irreducible component of $\mathbf{F}$. Write $g : F \to Y$ and $G : W_L \times F \to W_L \times Y$ for the natural maps. For $Z \in CH(W_L \times F)$

$$G_*(Z) \cdot (W_L \times D_i) = G_*(Z \cdot G^*(W_L \times D_i)) = G_*(Z \cdot (W_L \times g^*(D_i))). \tag{4.16}$$

Since F is a curve of genus $\leq g$ the hypotheses of (4.8) imply $\mathrm{Hom}(\mathrm{Jac}(W_L), \mathrm{Jac}(F)) = 0$. Thus

$$CH(W_L) \otimes CH(F) \xrightarrow{\times} CH(W_L \times F) \tag{4.17}$$

is surjective. Write $Z = Z_1 \times F + Z_2 \times \xi$ with $\xi \in CH_0(F)$. Now $(Z_2 \times \xi) \cdot (W_L \times g^*(D_i)) = 0$, and

$$pr_{W*}G_*((Z_1 \times F) \cdot (W_L \times g^*(D_i))) = pr_{W*}(Z_1 \times g_*g^*(D_i)) = 0,$$

since $g_*g^*(D_i)$ is a degree zero zero cycle. Thus $q_i \circ v_2(Z) = 0$ and the claim is verified.

It follows that the image of $h_1((\mathbf{z}_1, ..., \mathbf{z}_r)) \notin \mathrm{im}(v_2)$ contradicting (4.15). This contradiction proves the proposition. □

The hypothesis in Proposition (4.8) that W is a curve was used only to show that (4.17) is surjective. To describe a weak analog of (4.8) which is valid for varieties W of arbitrary dimension we recall the notation $CH(W)_{\mathrm{alg}} \subset CH(W)$ for the group of cycle classes which are algebraically equivalent to zero [Fu, 10.3].

(4.18) Proposition. *Let $k \subsetneq K$ be an extension of algebraically closed fields. Let E_K be a smooth, projective curve over K such that no positive dimensional abelian subvariety of $\mathrm{Pic}^0(E_K)$ descends to an abelian variety defined over k. If $A \subset CH(W)$ satisfies $A \cap CH(W)_{\mathrm{alg}} = 0$, then the exterior product map,*

$$A \otimes CH_0(E)_{tf} \xrightarrow{\times} CH(W \times_k E_K),$$

is injective.

Proof. The proof is the same as that of (4.8) with one modification. As (4.17) may fail to be surjective, $h_2 \circ v_2$ may fail to be zero. Nonetheless, $\mathrm{im}(h_2 \circ v_2) \subset CH(W_L)^r_{\mathrm{alg}}$ since, in the notation of (4.16), $g^*(D_i) \in CH(F)_{\mathrm{alg}}$, which implies that (4.16) lies in $CH(W_L \times Y)_{\mathrm{alg}}$. The image of A under the (injective) base change map, $v_{L/k} : CH(W) \to CH(W_L)$ satisfies $v_{L/k}(A) \cap CH(W_L)_{\mathrm{alg}} = 0$. Given $\mathbf{z}_1, ..., \mathbf{z}_r \in v_{L/k}(A)$ with at least one $\mathbf{z}_i$ not torsion, we have as before,

$$h_2 \circ h_1((\mathbf{z}_1, ..., \mathbf{z}_r)) = (c_1\mathbf{z}_1, ..., c_r\mathbf{z}_r) \notin \mathrm{im}(h_2 \circ v_2) \subset CH(W_L)^r_{\mathrm{alg}}.$$

Thus $h_1((\mathbf{z}_1, ..., \mathbf{z}_r)) \notin \mathrm{im}(v_2)$, contradicting (4.15). □

5. Non-injectivity for certain exterior product maps

The purpose of this section is to prove a partial converse of (4.8).

(5.1) Proposition. *Let k be an algebraically closed field which is not the algebraic closure of a finite field. Let $k \subset K$ be an extension of algebraically closed fields. Let W be a smooth, projective k-curve which admits a non-constant map to a smooth genus 1 curve. Let E_K be a genus 1 curve whose J-invariant, $J \in K$, is transcendental over k. Then the exterior product map,*

$$CH(W) \otimes CH(E_K) \xrightarrow{\times} CH(W \times_k E_K),$$

is not injective.

Proof. We postphone considerations of general genus 1 curves, E_K, to the end of the argument and begin with a geometric construction.

Let E_0 be a smooth, projective, genus 1 curve over k. Let $\gamma_0 : W \to E_0$ be a non-constant morphism. In the Grassmannian of pencils of degree 3 curves in $\mathbb{P}^2$ consider the open subset, U, consisting of pencils with 9 simple base points and only irreducible nodal curves as singular curves. Since the degree of the J-invariant of such a pencil is 12, we may choose two fibers isomorphic to E_0. Call them E_0 and E_1. Given two points, $\bar{s}_1, \bar{s}_2 \in E_0 \cap E_1$, the divisor class, $\bar{s}_1 - \bar{s}_2 \in \mathrm{Pic}^0(gE_1) \simeq \mathrm{Pic}^0(E_1)$ sweeps out an open subset as g varies in $\{g \in \mathrm{Aut}(\mathbb{P}^2_k, \bar{s}_1) : \bar{s}_2 \in g(E_1)$ and $\mathrm{Span}\{E_0, gE_1\} \in U\}$. Since k is not the algebraic closure of a finite field, we may choose g so that

$\kappa^*\mathcal{O}_{gE_1}(\bar{s}_1 - \bar{s}_2) \notin \mathcal{O}_{E_0}(\bar{s}_1 - \bar{s}_2)\mathbb{Q} \subset \mathrm{Pic}^0(E_0)\otimes\mathbb{Q}$, for any isomorphism, $\kappa : E_0 \to gE_1$. Write $E_\infty := gE_1$, $\sigma : Y \to \mathbb{P}^2$ for the blow up along $E_0 \cap E_\infty$, $s_i = \sigma^{-1}(\bar{s}_i)$, and $\pi : Y \to X := \mathbb{P}^1$ for the elliptic fibration. Also write η for the generic point of X and $\hat{X}$ for the set $X - \{\eta\}$.

Fix an isomorphism, $\kappa : E_0 \to E_\infty$, and define $\gamma_\infty := \kappa \circ \gamma_0 : W \to E_\infty$. For $i \in \{0, \infty\}$ the graph of γ_i will be denoted Γ_i. Consider the commutatitive diagram,

(5.2)

$$\begin{array}{ccccc}
CH(W)\otimes \oplus_{x\in\hat{X}} CH(\pi^{-1}(x)) & \xrightarrow[b_5]{} & \oplus_{x\in\hat{X}} CH(W\times\pi^{-1}(x)) & & \\
\downarrow b_1 & & \downarrow b_2 & & \\
CH(W)\otimes CH(Y) & \xrightarrow[b_6]{\sim} & CH(W\times Y) & \xrightarrow[b_8]{} & CH(W), \\
\downarrow b_3 & & \downarrow b_4 & & \\
CH(W)\otimes CH(\pi^{-1}(\eta)) & \xrightarrow{b_7} & CH(W\times\pi^{-1}(\eta)) & &
\end{array}$$

where b_5, b_6, b_7 are exterior product maps and $b_8(\mathfrak{z}) := pr_{W*}(pr_Y^*(s_2 - s_1)\cdot\mathfrak{z})$. The map b_6 may be seen to be an isomorphism by recalling that exterior product with $\mathbb{P}^i$ is an isomorphism and comparing the short exact sequence of Chow groups associated with the blowup of $W \times \mathbb{P}^2$ along $W \times (E_0 \cap E_\infty)$ with the short exact sequence associated with the blowup of $\mathbb{P}^2$ along $E_0 \cap E_\infty$ tensored with $CH(W)$ [Fu, 6.7(e)]. We may view the cycle $\Gamma_0 - \Gamma_\infty$ as an element of the upper right hand corner of the diagram. We claim, $b_8 \circ b_2 \circ b_5 = 0$. To see this let F denote the normalization of a fiber, $\pi^{-1}(x)$, $p_1 \in F$, $p_2 \in W$, and write $(s_2 - s_1)_x := (s_2 - s_1)\cdot F$. Then

$$pr_{W*}([W \times (s_2 - s_1)_x]\cdot[(W\times p_1) + (p_2 \times F)])_{W\times F} = 0,$$

which proves the claim. On the other hand $b_8 \circ b_2(\Gamma_0 - \Gamma_\infty)$ has infinite order. This follows from

$$b_8 \circ b_2(\Gamma_i) = \gamma_i^*(s_2 - s_1)|_{E_i}, \qquad i \in \{0, \infty\},$$

the fact that the kernel of $\gamma_0^* : CH^1(E_0)_0 \to CH^1(W)_0$ is finite, and the fact that $(s_2 - s_1)|_{E_0} \in CH^1(E_0)_0$ is not a rational multiple of $\kappa^*(s_2 - s_1)|_{E_\infty}$. By a diagram chase $Z := b_3 \circ b_6^{-1} \circ b_2(\Gamma_0 - \Gamma_\infty)$ lies in $\mathrm{Ker}(b_7)$ and has infinite order.

The generic point $\eta \in X$ corresponds to a field extension of k whose algebraic closure we denote by K_0. Define $E'_{K_0} : \pi^{-1}(\eta) \times_\eta \mathrm{Spec}(K_0)$ and recall that the kernel of the base change map, $CH(\pi^{-1}(\eta)) \to CH(E'_{K_0})$, is torsion [Bl2, Lemma 3 p.1.21]. It follows easily from the commutative diagram,

$$\begin{array}{ccc}
CH(W)\otimes CH(\pi^{-1}(\eta)) & \xrightarrow{\times} & CH(W\times\pi^{-1}(\eta)) \\
b_9\downarrow & & \downarrow \\
CH(W)\otimes CH(E'_{K_0}) & \xrightarrow{b_{10}} & CH(W\times_k E'_{K_0}),
\end{array}$$

that $b_9(Z)$ is an element of infinite order in the kernel of b_{10}. This completes the proof of (5.1) in the special case $E_K = E'_{K_0}$.

To treat the general case let $K_1 \subset K$ denote the algebraic closure of $k(J)$, where J is the J-invariant of the genus 1 curve E_K. Let $J_0 \in K_0$ denote the J-invariant of E'_{K_0}. The existence of a nodal fiber in the pencil defining Y implies that J_0 is transcendental over k. An isomorphism to k-algebras, $k(J_0) \to k(J)$ sending J_0 to J may be extended to an isomorphism, $K_0 \to K_1$ and even to an isomorphism to k-schemes, $E'_{K_0} \to E_{K_1}$. This gives rise to a commutative diagram whose vertical arrows are isomorphisms,

$$\begin{array}{ccc} CH(W) \otimes CH(E'_{K_0}) & \xrightarrow{\times} & CH(W \times_k E'_{K_0}) \\ \downarrow & & \downarrow \\ CH(W) \otimes CH(E_{K_1}) & \xrightarrow{\times} & CH(W \times_k E_{K_1}). \end{array}$$

To complete the proof of (5.1) it remains only to extend scalars from K_1 to K and to apply (1.7). □

Proof of Theorem (0.5). The only if statement is (5.1). By §4 the if statement follows from (2.1) (which is trivial when W is a curve) and (4.8).

(5.3) Remark. Zero cycles on products of curves were investigated from a somewhat different point of view in [Sch2].

References

[Bl] S. Bloch, *Torsion algebraic cycles and a theorem of Roitman*, Compositio Math. **39** (1979), 107–127.

[Bl2] ———, *Lectures on algebraic cycles*, Duke University Mathematics Series, IV, Duke University, Mathematics Department, Durham, N.C., 1980.

[Bl-Es] S. Bloch and H. Esnault, *The coniveau filtration and non-divisibility for algebraic cycles*, Math. Ann. **304** (1996), 303–314.

[Bl-Sri] S. Bloch and V. Srinivas, *Remarks on correspondences and algebraic cycles*, Amer. J. Math. **105** (1983), 1235–1253.

[De-Ra] P. Deligne and M. Rapoport, *Les schémas de modules de courbes elliptiques*, Modular Functions of One Variable II, Lecture Notes in Mathematics, 349, Springer-Verlag, New York, 1973, pp. 143–316.

[Des] M. Deschamps, *Réduction semi-stable*, Séminaire sur les pinceaux de courbes de genre au moins deux, Asterisique **86** (1981), 1–34.

[Fu] W. Fulton, *Intersection theory*, Springer-Verlag, New York, 1984.

[Ha] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977.

[Le] F. Lecomte, *Rigidité des groupes de Chow*, Duke Math. J. **53** (1986), 405–426.

[Me-Su] A. Mercurev and A. Suslin, *K-cohomology of Severi-Brauer varieties and the norm residue homomorphism*, Izv. Akad. Nauk. USSR Ser. Mat. **46** (1982), 1011–1046.

[Mi] J. Milne, *Etale cohomology*, Princeton University Press, Princeton, NJ., 1980.

[Ro] A.A. Roitman, *The torsion group of zero cycles modulo rational equivalence*, Ann. of Math. **111** (1980), 553–569.

[Rot] J. Rotman, *An introduction to the theory of groups, 3^{rd} edition*, Allyn and Bacon, Inc., Boston, 1984.

[Sch] C. Schoen, *Complex varieties for which the Chow Group mod n is not finite*, preprint, 1996 (revised 1999).

[Sch2] ______, *Zero cycles modulo rational equivalence for some varieties over fields of transcendence degree one*, Proc. Symp. Pure Math. **46** (1987), 463–473.

[Sch3] ______, *The Chow group modulo l for the triple product of a general elliptic curve*, preprint, 1999.

[Scholl] A.J. Scholl, *Classical motives*, Proc. Symp. Pure Math. **55** (1994), 163–187.

[Sil] J. Silverman, *The arithmetic of elliptic curves*, Springer-Verlag, New York, 1986.

[Sil2] ______, *Advanced topics in the arithmetic of elliptic curves*, Springer-Verlag, New York, 1994.

[Su] A. Suslin, *On the K-theory of algebraically closed fields*, Inv. Math. **73** (1983), 241–245.

[To] B. Totaro, *Torsion algebraic cycles and complex cobordism*, J. Amer. Math. Soc. **10** (1997), 467–493.

[To2] ______, letter to the author, 1997.

Department of Mathematics, Duke University, Box 90320, Durham, NC 27708-0320

E-mail address: `schoen@ math.duke.edu`