EQUIDIMENSIONALITY OF LAGRANGIAN FIBRATIONS ON HOLOMORPHIC SYMPLECTIC MANIFOLDS

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ABSTRACT. We prove that every irreducible component of every fibre of Lagrangian fibrations on holomorphic symplectic manifolds is a Lagrangian subvariety. Especially, Lagrangian fibrations are equidimensional.

1. Introduction

We begin with the definition of Lagrangian subvarieties.

Definition 1. Let $X$ be a complex manifold with a holomorphic symplectic form $\omega$. A subvariety $Y$ is said to be a Lagrangian subvariety if $\dim Y = \left(\frac{1}{2}\right) \dim X$ and there exists a resolution $\nu : \tilde{Y} \to Y$ such that $\nu^*\omega$ is identically zero on $\tilde{Y}$.

Note that this notion does not depend on the choice of $\nu$. We prove the following theorem.

Theorem 1. Let $X$ be a Kähler manifold and $f : X \to B$ a proper surjective morphism over a normal variety $B$. Assume that there exists a $d$-closed holomorphic symplectic form $\omega$ on $X$ and a general fiber of $f$ is a Lagrangian subvariety with respect to $\omega$. Then every irreducible component of every fibre of $f$ is a Lagrangian subvariety. Especially $f$ is equidimensional.

Since every holomorphic form on a compact Kähler manifold is $d$-closed, we obtain the following result from combining Theorem 1 with [2, Theorem 2] and [3, Theorem 1].

Corollary 1. Let $f : X \to B$ be a surjective morphism from an irreducible symplectic manifold $X$ to a normal projective variety $B$. Assume that $0 < \dim B < \dim X$. Then every irreducible component of every fibre of $f$ is a Lagrangian subvariety.

Remark. If we drop the condition of properness, then $f$ is not necessarily equidimensional. Let $f$ be a morphism from $\mathbb{C}^4$ to $\mathbb{C}^2$ defined by

$$f(x, y, z, w) := (x, xy),$$

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and $\omega := dx \wedge dz + dy \wedge dw$. Then $\omega$ is a $d$-closed holomorphic symplectic form and a general fibre of $f$ is a Lagrangian subvariety with respect to $\omega$. Since $\dim f^{-1}(0) = 3$, $f$ is not equidimensional.

From Theorem 1, we obtain some information of the singularities of $B$.

**Corollary 2.** For every point $p$ of $B$, there exists a Stein neighborhood $U$ of $p$ and a finite morphism $\pi : \tilde{U} \to U$ from a smooth Stein manifold $\tilde{U}$.

**Proof.** For a point $p$ of $B$, we choose a point $q \in f^{-1}(p)$ and a smooth Stein neighborhoods $W$ of $q$. Since $f$ is equidimensional, we obtain a finite morphism $\pi : \tilde{U} \to U$ from a smooth Stein manifold $\tilde{U}$ by cutting $W$ with hypersurfaces.

**Remark.** The author does not know whether there exists an example such that $B$ is not smooth.

2. **Proof of Theorem 1**

We refer the following theorem due to Kollár [1, Theorem 2.2] and Mo. Saito [4, Theorem 2.3, Remark 2.9].

**Theorem 2.** Let $f : X \to B$ be a proper surjective morphism from a smooth Kähler manifold $X$ to a normal variety $B$. Then $R^i f_* \omega_X$ is torsion free, where $\omega_X$ is the dualizing sheaf of $X$.

**Proof.** Let $\bar{\omega}$ be the complex conjugate of $\omega$. Since $\omega$ is $d$-closed, $\bar{\omega}$ can be considered as an element of $H^2(X, \mathcal{O}_X)$. By Leray spectral sequence, there exists a morphism $$H^2(X, \mathcal{O}_X) \to H^0(B, R^2 f_* \mathcal{O}_X).$$ Then $\bar{\omega}$ is a torsion element in $H^0(B, R^2 f_* \mathcal{O}_X)$ since a general fibre of $f$ is a Lagrangian subvariety. In addition, $\omega_X \cong \mathcal{O}_X$. Hence $\bar{\omega}$ is zero in $H^0(B, R^2 f_* \mathcal{O}_X)$ by Theorem 2. We derive a contradiction assuming that there exists an irreducible component of a fibre of $f$ which is not a Lagrangian subvariety. The letter $V$ denotes an non Lagrangian component. We take an embedding resolution $\pi : \tilde{X} \to X$ of $V$. Let $\tilde{V}$ be the proper transform of $V$. We will show that $\pi^* \omega$ is not zero in $H^0(\tilde{V}, \Omega^2_{\tilde{V}})$. If $\dim V > (1/2) \dim X$, it is obvious by the definition. If $\dim V > (1/2) \dim X$, we take a smooth point $q \in V$ such that $\pi$ is isomorphic in a neighborhood of $q$. Since $\dim V > (1/2) \dim X$ and $\omega$ is nondegenerate, the restriction of $\omega$ on the tangent space of $V$ at $q$ is nonzero. Because $\pi$ is isomorphic in a neighborhood of $q$, $\pi^* \omega$ is not zero in $H^0(\tilde{V}, \Omega^2_{\tilde{V}})$. Take the complex conjugate, $\pi^* \bar{\omega}$ is not zero in $H^2(\tilde{V}, \mathcal{O}_{\tilde{V}})$. Therefore $\bar{\omega}$ is not zero in $H^2(V, \mathcal{O}_V)$. Let $p := f(V)$ and $X_p := f^{-1}(p)$. We consider the following morphism: $$R^2 f_* \mathcal{O}_X \otimes k(p) \to H^2(X_p, \mathcal{O}_{X_p}) \to H^2(V, \mathcal{O}_V).$$ Then $\bar{\omega}$ is zero in $R^2 f_* \mathcal{O}_X \otimes k(p)$ and nonzero in $H^2(V, \mathcal{O}_V)$. That is a contradiction.

$\square$
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References


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