A COUNTEREXAMPLE IN UNIQUE CONTINUATION

CARLOS E. KENIG AND NIKOLAI NADIRASHVILI

1. Introduction

In 1939, T. Carleman [Car39] showed that if $\Delta u - Vu = 0$ in $\mathbb{R}^2$, $V \in L^\infty_{\text{loc}}(\mathbb{R}^2)$, and $u$ vanishes of infinite order at $x_0 \in \mathbb{R}^2$, then $u = 0$. This was extended to $n \geq 3$ by C. Müller [Müll54]. In the late 70’s and early 80’s, there was considerable interest, in view of applications to the absence of embedded eigenvalues, in extending the above result to $V \in L^p_{\text{loc}}$, $p < \infty$ (see the surveys [Ken87] and [Ken89] and [Wol95]). In this direction, we want to recall the result in [JK85], where it is shown that, if $n > 2$ and $V \in L^{\frac{n}{2}}_{\text{loc}}$, an analogous conclusion can be obtained, and if $n = 2$, $V \in L^p_{\text{loc}}$, $p > 1$, the same is true. Moreover, in [Ste85], it is shown that it $n > 2$, the same conclusion can be reached if $V \in L^{\frac{n}{2}, \infty}$, the ‘weak-type’ Lorentz space, provided that the $L^{\frac{n}{2}, \infty}$ norm is small enough.

From several points of view, these results are optimal. Easy examples can be obtained (see [JK85]) for which, for $n > 2$, $V \in L^p_{\text{loc}}$, for all $p < \frac{n}{2}$, $u$ vanishes of infinite order at $x_0$, but $u$ is not identically zero. More subtle examples are due to T. Wolff [Wol92b], who shows that the smallness condition on the $L^{\frac{n}{2}, \infty}$-norm, $n > 2$ cannot be removed, and that when $n = 2$, there are $V \in L^1$, and $u$ vanishing of infinite order at $x_0$, for which $u$ is not identically zero. Nevertheless, for the applications mentioned above, it would suffice to know that, if $\Delta u - Vu = 0$, and $u$ has compact support, then $u \equiv 0$. Up to now, as was mentioned in [Ken87], [Ken89] and [Wol92a], it was not known if there are examples of $V \in L^1$, with non-zero $u$ of compact support, verifying this equation. In this note we close this gap in our knowledge, producing such an example, in all dimensions $n \geq 2$. The $L^1$-norm of the potential $V$ can be taken as small as one likes.

Remark. After this paper was written, T. Wolff informed us of related work by Niculae Mandache [Man], for equations of the form $\Delta u = \vec{V} \cdot \nabla u$.

2. Main theorem

Theorem 1. There are measurable functions $u, V$ defined on $\mathbb{R}^2$, both supported in $B_1$, where $B_1$ is the open unit disc, which are smooth in $B_1$, such that $u, V, Vu \in L^1(\mathbb{R}^2)$, and such that

$$\Delta u - Vu = 0 \text{ in } \mathcal{D}'.$$
In order to prove the theorem, we will need an inductive construction. Let

\[ r_{k+1}^0 = 1 - \frac{1}{5^k}, \quad r_{k}^1 = 1 - \frac{1}{5^k}, \]
\[ r_{k+1}^2 = 1 - \frac{1}{5^k+2}, \quad r_{k}^2 = 1 - \frac{1}{5^k+3}, \]
\[ r_{k+1}^3 = 1 - \frac{1}{5^k+4}, \quad r_{k}^3 = 1 - \frac{1}{5^k+5}, \]

so that

\[ r_{k+1}^0 < r_k^1 < r_{k+1}^2 < r_k^2 < r_{k+1}^3 < r_k^3 < r_{k+1}^4, \]

for \( k = 1, 2, \ldots \).

Let

\[ B_{4k} = \{ x : |x| < r_{k}^4 \}, \]
\[ B_{3k} = \{ x : |x| < r_{k}^3 \}, \]
\[ B_{2k} = \{ x : |x| < r_{k}^2 \}, \]
\[ B_{1k} = \{ x : |x| < r_{k}^1 \}, \]
\[ A_k = \{ x : r_{k+1}^0 < |x| < r_{k}^2 \}, \]
\[ D_k = \{ x : r_{k+1}^3 < |x| < r_{k}^4 \}. \]

Finally, let \( \phi_k \in C^\infty_0 (B_1), 0 \leq \phi_k \leq 1 \), with \( \phi_k = 1 \) in \( B_{3k} \), supp\( \phi_k \subset B_{4k} \). Note that supp\( \nabla \phi_k \subset D_k \), supp\( \Delta \phi_k \subset D_k \). We make a few remarks about these sets:

\[ \text{dist}(A_k, \partial B_1) \simeq \frac{1}{k}, \quad \text{dist}(A_k, D_k) \simeq \frac{1}{k}, \]
\[ \text{dist}(A_k, \partial B_{4k}) \simeq \frac{1}{k}, \quad \text{dist}(D_k, \partial B_1) \simeq \frac{1}{k}, \]
\[ \text{dist}(D_k, A_{k+1}) \simeq \frac{1}{k}, \quad \text{dist}(D_k, D_{k+1}) \simeq \frac{1}{k}. \]

3. The construction

We define \( u_1 \equiv 1 \) and now, for \( k = 1, 2, \ldots \), we define \( u_k \) inductively. Thus, assume that \( u_k \) has been defined, and we proceed to construct \( u_{k+1} \).

Let \( v_k = \phi_k u_k, f_k = \Delta (\phi_k u_k) \), so that \( v_k \) solves

\[
\begin{cases}
\Delta v_k = f_k \text{ in } B_1 \\
v_k|_{\partial B_1} \equiv 0.
\end{cases}
\]

Let now \( \alpha_n, n = 1, 2, \ldots \) be a sequence of distributions of the form

\[ \alpha_n = \sum_{i=1}^{i_n} a_i \delta_{x_i^n}, \]

where \( \delta_{x_i^n} \) is the delta mass at \( x_i^n \in D_k \), and chosen so that

\[ \alpha_n \rightharpoonup f_k \text{ weakly in } \overline{D_k} \text{ as } n \to \infty. \]

For fixed \( n \), set

\[ \alpha_n^\epsilon = \sum_{i=1}^{i_n} a_i \delta_{x_i^n}^\epsilon, \]
where \( \delta_{\epsilon} \) is a smoothing of \( \delta_{\epsilon} \), by a non-negative smooth function, supported in an \( \epsilon \) neighborhood of \( x^n_i \). We will always choose \( \epsilon \) small so that
\[
\text{supp} \alpha^n_{\epsilon} \subset D_k.
\]
Let now \( v^n_{\epsilon} \) solve
\[
\begin{cases}
\Delta v^n_{\epsilon} = f_k \text{ on } B_1 \setminus D_k \\
v^n_{\epsilon}|_{\partial B_1} = 0
\end{cases}
\]
Note that as \( n \to \infty \), and then \( \epsilon \to 0 \), \( v^n_{\epsilon} \to v_k \). Now, choose first \( n_0 \) so large, and then \( \epsilon_0 \) so small that
\[
|v^n_{\epsilon_0} - v_k| \leq \frac{1}{8k} \text{ on } B_k^2 \cup A_{k+1}
\]
and so that
\[
\|\Delta (\phi_{k+1} v^n_{\epsilon_0})\| \leq \frac{1}{2k+3}.
\]
The first condition is a direct consequence of the weak convergence of \( \alpha^n \). For the second one, note that on \( D_{k+1} \), \( f_k \equiv 0 \), and \( v_k \equiv 0 \), \( \nabla v_k \equiv 0 \).
\[
\Delta (\phi_{k+1} v^n_{\epsilon_0}) = \phi_{k+1} \Delta v^n_{\epsilon_0} + 2 \nabla \phi_{k+1} \nabla v^n_{\epsilon_0} + (\Delta \phi_{k+1}) v^n_{\epsilon_0} = 2 \nabla \phi_{k+1} \nabla v^n_{\epsilon_0} + \Delta \phi_{k+1} v^n_{\epsilon_0},
\]
and so the second condition also follows from the weak convergence.
We may also assume, without loss of generality, that
\[
\|\alpha^n_{\epsilon_0}\| \leq \|f_k\| \text{ on } D_k.
\]
and since \( |v^n_{\epsilon_0}| \to \infty \) on \( \text{supp} \alpha^n_{\epsilon_0} \), as \( \epsilon_0 \to 0 \), we may assume that
\[
|v^n_{\epsilon_0}| \geq 1 \text{ on } \text{supp} \alpha^n_{\epsilon_0}.
\]
We will now define \( u_{k+1} = v^n_{\epsilon_0} \).
We will next deduce a few properties of \( u_k \).

4. Properties of \( u_k \)

(P1) \( u_{k+1} \in C^\infty(B_1) \). Moreover, \( \text{supp} \Delta u_{k+1} \subset \cup_{j=1}^k D_k \).

Proof. We will prove the two statements inductively. For \( k = 1 \), recall that
\[
\Delta u_2 = \begin{cases}
 f_1 \text{ in } B_1 \setminus \overline{D}_1 \\
 \alpha^n_{\epsilon_0} \text{ in } \overline{D}_1.
\end{cases}
\]
But, \( f_1 = \Delta (u_1 \phi_1) = \Delta (\phi_1) \), and since \( \text{supp} \Delta (\phi_1) \subset D_1 \), \( f_1 \) is 0 in \( B_1 \setminus \overline{D}_1 \).
Moreover, \( \text{supp} \alpha^n_{\epsilon_0} \subset D_1 \), and so, clearly, \( \Delta u_2 \) is supported in \( D_1 \), and is smooth.
But then \( u_2 \) is also smooth in \( B_1 \). Assume that both statements hold up to \( k \).
\[
\Delta u_{k+1} = \begin{cases}
 f_k \text{ on } B_1 \setminus \overline{D}_k \\
 \alpha^n_{\epsilon_0} \text{ on } \overline{D}_k.
\end{cases}
\]
In $B_1 \setminus \overline{B}_{k+1}^1$, $\phi_k \equiv 0$, and so $f_k \equiv 0$. In $B_k^2$, $\phi_k \equiv 1$, and so $f_k = \Delta u_k$. Hence, both statements hold up to $k + 1$.

\[
|u_{k+1}| \leq \frac{1}{8k} \quad \text{on } A_{k+1}.
\]

**Proof.** On $A_{k+1}$, $\phi_k \equiv 0$ and so,

\[
|u_{k+1}| = |u_1 - \phi_k u_k| \leq \frac{1}{8k}.
\]

\[
\int_{B_1} |\Delta(\phi_{k+1} u_{k+1})| \leq C, \quad \text{for all } k.
\]

**Proof.** We know that

\[
\|\Delta(\phi_{k+1} u_{k+1})\|_{L^1(D_{k+1})} \leq \frac{1}{2k+3}.
\]

Moreover, in $B_1 \setminus \overline{B}_{k+1}^4$, $\phi_{k+1} \equiv 0$, so $\Delta(\phi_{k+1} u_{k+1}) = 0$. By construction, inside $B_k^3$, $\phi_k \equiv 1$ and so $\Delta(\phi_{k+1} u_{k+1}) = \Delta(u_{k+1})$. But in $B_{k+1}^3 \setminus B_k^4$, $\phi_k \equiv 0$, and so $\Delta(u_{k+1}) = \Delta(\phi_k u_k) = 0$ there. In $D_k$, $\Delta u_{k+1} = \alpha_{\epsilon_0}$, and so,

\[
\int_{D_k} |\Delta u_{k+1}| \leq 2 \int_{D_k} |\Delta(\phi_k u_k)| \leq \frac{2}{2(k-1)+3} \leq \frac{1}{2k}.
\]

Gathering the information, we obtain

\[
\|\Delta(\phi_{k+1} u_{k+1})\|_{L^1(B_1)} \leq \|\Delta(\phi_k u_k)\|_{L^1(B_1)} + \frac{1}{2k+3} + \frac{1}{2k},
\]

and (P3) follows.

\[
\int_{B_1} |\phi_{k+1} u_{k+1}| \leq C \quad \text{for all } k.
\]

This is immediate from (P3).

**Proof of the theorem.** We first claim that $\{u_k\}$ converges uniformly on compact subsets of $B_1$, to a function $u$, which is smooth in $B_1$ and for which $\text{supp}\Delta u \subset \bigcup_{k=1}^{\infty} D_k$, and such that $|u| > \frac{1}{2}$ on $\text{supp}\Delta u$.

**Proof of claim.** Fix $r < 1$, and choose $k_0$ so that $\overline{B_r} \subset B_{k_0}^2$, and hence, $\overline{B_r} \subset B_k^2$ for all $k \geq k_0$. For $n, m \geq k_0$, $n > m$, we have that $\phi_j \equiv 1$ on $\overline{B_r}$, $j = m, \ldots, n - 1$, and so

\[
|u_m - u_n| \leq \sum_{k=m}^{\infty} \frac{1}{8k},
\]

and thus we have the uniform convergence. Note also that (P1) implies that all the $u_k$’s are harmonic outside of $\bigcup_{j=1}^{\infty} D_j$, and hence, so is $u$. Next, note that $\Delta u_k = \Delta u_{k_0}$ in $\overline{B_r}$, for $k \geq k_0$. This is because, for $k > k_0$, $D_k \subset B_1 \setminus \overline{B_r}$, and
\( \phi_{k-1} \equiv 1 \) on \( \overline{B_r} \). From this it follows that \( \Delta u = \Delta u_{k_0} \) in \( \overline{B_r} \), and hence, by (P1), \( \Delta u \) is smooth in \( B_r \), and hence so is \( u \).

We finally need to check that \( |u| > \frac{1}{2} \) on \( \text{supp} \Delta u \). It is enough to do it on \( \text{supp} \Delta u \cap D_k \), for each \( k \). Fix such a \( k \), and note that, as before, we have for \( j > k \), \( \Delta u_j = \Delta u_{k+1} \) on \( D_k \): since \( D_k \subset B_j^3 \), and so \( \Delta u_j = \Delta (\phi_{j-1}u_{j-1}) = \Delta u_{j-1} \), where the last equality holds as long as \( D_k \subset B_{j-1}^3 \), or \( k < j - 1 \). The last valid case is when \( j - 1 = k + 1 \), as claimed. On \( \partial D_k \), \( \Delta u_{k+1} = \alpha_n^{\phi} + 1 \), and so, on \( D_k \cap \text{supp} \Delta u = D_k \cap \text{supp} \Delta u_{k+1} \), we have that \( |u^0_n| > 1 \), i.e., \( |u_{k+1}| > 1 \). If \( j > k + 1 \), \( D_k \subset B_j^3 \), \( D_k \subset B_{j+1}^3 \), and so \( |u_j - u_{j-1}| < \frac{1}{5^j} \). Thus, if \( j > k + 1 \), \( |u_j - u_{k+1}| \leq \sum_{j=k+2}^{\infty} \frac{1}{r^j} \leq \frac{1}{2} \), and the last claim follows. Next, we claim that

\[
\int_{B_1} |\Delta u| \leq C,
\]

\[
\int_{B_1} |u| \leq C.
\]

These are immediate consequences of (P3) and (P4).

Finally, we define \( u = 0 \) outside \( B_1 \). We let \( V = \Delta u / u \) in \( \text{supp} \Delta u \cap B_1 \), and 0 elsewhere. Note that, since \( |u| > \frac{1}{2} \) on \( \text{supp} \Delta u \cap B_1 \), \( V \) is well defined, and \( \Delta u = Vu \) pointwise in \( B_1 \). Note also that since \( \Delta u \in L^1(B_1) \), \( |V| \leq 2|\Delta u| \), we have that \( V \in L^1(B_1) \), \( Vu \in L^1(B_1) \). Finally, we will check that \( \Delta u - Vu = 0 \) in \( D'(\mathbb{R}^2) \). In order to check this, we first note that \( |u| < \frac{1}{\sqrt{x}} \) on \( A_{k+1} \). Indeed, by (P2), \( |u_{k+1}| \leq \frac{1}{\sqrt{x}} \) on \( A_{k+1} \), and if \( j > k + 1 \), \( A_{k+1} \subset B_j^3 \), and hence \( |u_j - \phi_{j-1} u_{j-1}| < \frac{1}{5^j} \), and also \( A_{k+1} \subset B_{j-1}^3 \), and so \( \phi_{j-1} \equiv 1 \) there.

Note also that \( u \) is harmonic in \( A_{k+1} \), and hence, by interior estimates we have \( |\nabla u| \leq \frac{C}{r^2} \) in \( \partial B_{k+1}^3 \). Let \( \psi \in C^\infty_0(\mathbb{R}^2) \). We need to check that

\[
\int_{\mathbb{R}^2} [u \Delta \psi - Vu \psi] = 0.
\]

The above integral equals

\[
\int_{B_1} [u \Delta \psi - Vu \psi] = \lim_{k \to \infty} \int_{B_{k+1}^3} [u \Delta \psi - Vu \psi],
\]

since \( u \in L^1(B_1) \), \( Vu \in L^1(B_1) \), \( \psi \in C^\infty_0(\mathbb{R}^2) \). Now,

\[
\int_{B_{k+1}^3} [u \Delta \psi - Vu \psi] = \int_{B_{k+1}^3} [u \Delta \psi - \Delta Vu]
\]

\[
= \int_{\partial B_{k+1}^3} \left[ \frac{\partial \psi}{\partial n} \frac{\partial u}{\partial n} \psi \right],
\]

and so

\[
\left| \int_{B_{k+1}^3} [u \Delta \psi - Vu \psi] \right| \leq \frac{C}{4^k} + \frac{C}{2^k},
\]

and the desired result follows.
Remark. Since we can make $v_{n_0}^\epsilon$ as large as we please on $\text{supp} \alpha_{n_0}^\epsilon$, we can take the $L^1$ norm of $V$ as small as we like.

References


