DYNAMICS OF RATIONAL MAPS: A CURRENT ON THE BIFURCATION LOCUS

Laura DeMarco

Abstract. Let $f_\lambda : \mathbb{P}^1 \to \mathbb{P}^1$ be a family of rational maps of degree $d > 1$, parametrized holomorphically by $\lambda$ in a complex manifold $X$. We show that there exists a canonical closed, positive (1,1)-current $T$ on $X$ supported exactly on the bifurcation locus $B(f) \subset X$. If $X$ is a Stein manifold, then the stable regime $X - B(f)$ is also Stein. In particular, each stable component in the space $\text{Poly}_d$ (or $\text{Rat}_d$) of all polynomials (or rational maps) of degree $d$ is a domain of holomorphy.

1. Introduction

It is well-known that for a rational map $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree $d > 1$, there is a natural $f$-invariant measure $\mu_f$ supported on the Julia set of $f$ [B],[Ly]. This measure can be described as the weak limit of purely atomic measures,

$$\mu_f = \lim_{n \to \infty} \frac{1}{d^n} \sum_{\{z : f^n(z) = a\}} \delta_z,$$

for any $a \in \mathbb{P}^1$ (with at most two exceptions).

There is also a potential-theoretic description of $\mu_f$, defined in terms of a homogeneous polynomial lift $F : \mathbb{C}^2 \to \mathbb{C}^2$ of $f$. The potential function on $\mathbb{C}^2$ is given by

$$h(z) = \lim_{m \to \infty} \frac{1}{d^m} \log \| F^m(z) \|,$$

and the (1,1)-current $\partial \bar{\partial} h$ satisfies

$$\pi^* \mu_f = \frac{i}{\pi} \partial \bar{\partial} h$$

where $\pi$ is the canonical projection $\mathbb{C}^2 - \{0\} \to \mathbb{P}^1$ [HP]. In particular, when $f$ is a monic polynomial, this definition reduces to

$$\mu_f = \frac{i}{\pi} \partial \bar{\partial} G = \frac{1}{2\pi} \Delta G \, dx \wedge dy,$$

where $G : \mathbb{C} \to [0, \infty)$ is the Green’s function for the complement of the filled Julia set $K(f) = \{ z : f^n(z) \not\to \infty \text{ as } n \to \infty \}$.

Received March 2, 2000.
In this paper, we construct a (1,1)-current on the parameter space of a holomorphic family of rational maps, supported exactly on the bifurcation locus (just as $\mu_f$ is supported exactly on the Julia set).

Let $X$ be a complex manifold. A holomorphic family of rational maps $f$ over $X$ is a holomorphic map $f : X \times \mathbb{P}^1 \to \mathbb{P}^1$. For each parameter $\lambda \in X$, we obtain a rational map $f_\lambda : \mathbb{P}^1 \to \mathbb{P}^1$ with Julia set $J(f_\lambda)$. The bifurcation locus $B(f)$ of the family $f$ over $X$ is the set of all $\lambda_0 \in X$ for which $\lambda \mapsto J(f_\lambda)$ is a discontinuous function (in the Hausdorff topology) in any neighborhood of $\lambda_0$ ($\S$2).

**Theorem 1.1.** Let $f : X \times \mathbb{P}^1 \to \mathbb{P}^1$ be a holomorphic family of rational maps on $\mathbb{P}^1$ of degree $d > 1$. Then there exists a canonical closed, positive (1,1)-current $T(f)$ on $X$ such that the support of $T(f)$ is $B(f)$, the bifurcation locus of $f$.

By general properties of positive currents (Lemma 3.3), we have

**Corollary 1.2.** If $X$ is a Stein manifold, then $X - B(f)$ is also Stein.

Let $\text{Rat}_d$ and $\text{Poly}_d$ denote the “universal families” of all rational maps and of all monic polynomials of degree exactly $d > 1$. We have $\text{Poly}_d \simeq \mathbb{C}^d$ and $\text{Rat}_d \simeq \mathbb{P}^{2d+1} - V$, where $V$ is a resultant hypersurface. In particular, $\text{Rat}_d$ and $\text{Poly}_d$ are Stein manifolds.

**Corollary 1.3.** Every stable component in $\text{Rat}_d$ and $\text{Poly}_d$ is a domain of holomorphy (i.e. a Stein open subset).

Corollary 1.3 answers a question posed by McMullen in [M2], motivated by analogies between rational maps and Teichmüller space. Bers and Ehrenpreis showed that finite-dimensional Teichmüller spaces are domains of holomorphy [BE].

**Sketch proof of Theorem 1.1.** Consider a holomorphic family of homogeneous polynomial maps $\{F_\lambda\}$ on $\mathbb{C}^2$, locally lifting the holomorphic family $f$ over $X$. Let $\{h_\lambda\}$ be the corresponding potential functions on $\mathbb{C}^2$ defined by equation (1). The function $h_\lambda(z)$ is plurisubharmonic in both $\lambda \in X$ and $z \in \mathbb{C}^2$, and it is pluriharmonic in $z$ away from $\pi^{-1}(J(f_\lambda))$. Suppose for simplicity that we have holomorphic functions $c_j : X \to \mathbb{P}^1$, $j = 1, \ldots, 2d - 2$, parametrizing the critical points of $f_\lambda$ in $\mathbb{P}^1$. We choose lifts $\tilde{c}_j$ from a neighborhood in $X$ to $\mathbb{C}^2$ so that $c_j = \pi \circ \tilde{c}_j$ and define the plurisubharmonic function

$$H(\lambda) = \sum_j h_\lambda(\tilde{c}_j(\lambda)).$$

The desired (1,1)-current on $X$ is defined by

$$T(f) = i \frac{\partial \bar{\partial} H}{2\pi},$$

independent of the choices of $\{F_\lambda\}$ and $\tilde{c}_j$. It is supported on $B(f)$ since $H$ fails to be pluriharmonic exactly when a critical point $c_j(\lambda)$ passes through the Julia set $J(f_\lambda)$.
I would like to thank C. McMullen, J.E. Fornaess, and X. Buff for helpful comments and ideas.

2. Stability

Let \( f : X \times \mathbb{P}^1 \to \mathbb{P}^1 \) be a holomorphic family of rational maps of degree \( d > 1 \). The Julia sets of such a family are said to move holomorphically at a point \( \lambda_0 \in X \) if there is a family of injections \( \phi_\lambda : J_{\lambda_0} \to \mathbb{P}^1 \), holomorphic in \( \lambda \) near \( \lambda_0 \) with \( \phi_{\lambda_0} = \text{id} \), such that \( \phi_\lambda(J_{\lambda_0}) = J_\lambda \) and \( \phi_\lambda \circ f_{\lambda_0}(z) = f_\lambda \circ \phi_\lambda(z) \). In other words, \( \phi_\lambda \) provides a conjugacy between \( f_{\lambda_0} \) and \( f_\lambda \) on their Julia sets.

The family of rational maps \( f \) over \( X \) is stable at \( \lambda_0 \in X \) if any of the following equivalent conditions are satisfied [M1, Theorem 4.2]:

1. The number of attracting cycles of \( f_\lambda \) is locally constant at \( \lambda_0 \).
2. The maximum period of an attracting cycle of \( f_\lambda \) is locally bounded at \( \lambda_0 \).
3. The Julia set moves holomorphically at \( \lambda_0 \).
4. For all \( \lambda \) sufficiently close to \( \lambda_0 \), every periodic point of \( f_\lambda \) is attracting, repelling, or persistently indifferent.
5. The Julia set \( J_\lambda \) depends continuously on \( \lambda \) (in the Hausdorff topology) in a neighborhood of \( \lambda_0 \).

Suppose also that each of the \( 2d - 2 \) critical points of \( f_\lambda \) are parametrized by holomorphic functions \( c_j : X \to \mathbb{P}^1 \). Then the following conditions are equivalent to those above:

6. For each \( j \), the family of functions \( \{ \lambda \mapsto f_\lambda^n(c_j(\lambda)) \}_{n \geq 0} \) is normal in some neighborhood of \( \lambda_0 \).
7. For all nearby \( \lambda \), \( c_j(\lambda) \in J_\lambda \) if and only if \( c_j(\lambda_0) \in J_{\lambda_0} \).

We let \( S(f) \subset X \) denote the set of stable parameters and define the bifurcation locus \( B(f) \) to be the complement \( X - S(f) \). Mañé, Sad, and Sullivan showed that \( S(f) \) is open and dense in \( X \) [MSS, Theorem A].

Example. In the family \( f_c(z) = z^2 + c \), the bifurcation locus is \( B(f) = \partial M \), where \( M = \{ c \in \mathbb{C} : f_c^n(0) \not\to \infty \text{ as } n \to \infty \} \) is the Mandelbrot set [M1, Theorem 4.6].

Lemma 2.1. If \( B(f) \) is contained in a complex hypersurface \( D \subset X \), then \( B(f) \) is empty.

Proof. Suppose there exists \( \lambda_0 \in B(f) \). By characterization (4) of stability, any neighborhood \( U \) of \( \lambda_0 \) must contain a point \( \lambda_1 \) at which the multiplier \( m(\lambda) \) of a periodic cycle for \( f_\lambda \) is passing through the unit circle. In other words, the holomorphic function \( m(\lambda) \) defined in a neighborhood \( N \) of \( \lambda_1 \) is non-constant with \( |m(\lambda_1)| = 1 \). The set \( \{ \lambda \in N : |m(\lambda)| = 1 \} \) lies in the bifurcation locus and cannot be completely contained in a hypersurface. \( \square \)
3. Stein manifolds and positive currents

Let $X$ be a paracompact complex manifold and $\mathcal{O}(X)$ its ring of holomorphic functions. Then $X$ is a **Stein manifold** if the following three conditions are satisfied:

- for any $x \in X$ there exists a neighborhood $U$ of $x$ and $f_1, \ldots, f_n \in \mathcal{O}(X)$ defining local coordinates on $U$;
- for any $x \neq y \in X$, there exists an $f \in \mathcal{O}(X)$ such that $f(x) \neq f(y)$; and
- for any compact set $K$ in $X$, the holomorphic hull

$$\hat{K} = \{ x \in X : |f(x)| \leq \sup_K |f| \text{ for all } f \in \mathcal{O}(X) \}$$

is also compact in $X$.

An open domain $\Omega$ in $X$ is **locally Stein** if every boundary point $p \in \partial \Omega$ has a neighborhood $U$ such that $U \cap \Omega$ is Stein.

**Properties of Stein manifolds.** The Stein manifolds are exactly those which can be embedded as closed complex submanifolds of $\mathbb{C}^N$. If $\Omega$ is an open domain in $\mathbb{C}^n$ then $\Omega$ is Stein if and only if $\Omega$ is pseudoconvex if and only if $\Omega$ is a domain of holomorphy. An open domain in a Stein manifold is Stein if and only if it is locally Stein. Also, an open domain in complex projective space $\mathbb{P}^n$ is Stein if and only if it is locally Stein and not all of $\mathbb{P}^n$. See, for example, [H] and the survey article by Siu [S].

**Examples.** (1) $\mathbb{C}^N$ is Stein. (2) The space of all monic polynomials of degree $d$, $\text{Poly}_d \simeq \mathbb{C}^d$, is Stein. (3) $\mathbb{P}^n - V$ for a hypersurface $V$ is Stein. If $V$ is the zero locus of degree $d$ homogeneous polynomial $F$ and $\{g_j\}$ a basis for the vector space of homogeneous polynomials of degree $d$, then the map $(g_1/F, \ldots, g_N/F)$ embeds $\mathbb{P}^n - V$ as a closed complex submanifold of $\mathbb{C}^N$. (4) The space $\text{Rat}_d$ of all rational maps $f(z) = P(z)/Q(z)$ on $\mathbb{P}^1$ of degree exactly $d$ is Stein. Indeed, parameterizing $f$ by the coefficients of $P$ and $Q$ defines an isomorphism $\text{Rat}_d \simeq \mathbb{P}^{2d+1} - V$, where $V$ is the resultant hypersurface given by the condition $\gcd(P, Q) \neq 1$.

A $(p, q)$-**current** $T$ on a complex manifold of dimension $n$ is an element of the dual space to smooth $(n - p, n - q)$-forms with compact support. See [HP], [Le], and [GH] for details. The wedge product of a $(p, q)$-current $T$ with any smooth $(n - p, n - q)$-form $\alpha$ defines a distribution by $(T \wedge \alpha)(f) = T(f\alpha)$ for $f \in C^\infty_c(X)$. Recall that a distribution $\delta$ is positive if $\delta(f) \geq 0$ for functions $f \geq 0$. A $(p, p)$-current is **positive** if for any system of $n - p$ smooth $(1, 0)$-forms with compact support, $\{\alpha_1, \ldots, \alpha_{n-p}\}$, the product

$$T \wedge (i\alpha_1 \wedge \bar{\alpha}_1) \wedge \cdots \wedge (i\alpha_{n-p} \wedge \bar{\alpha}_{n-p})$$

is a positive distribution.

An upper-semicontinuous function $h$ on a complex manifold $X$ is **plurisubharmonic** if $h|D$ is subharmonic for any complex analytic disk $D^1$ in $X$. The current $T = i\partial\bar{\partial}h$ is positive for any plurisubharmonic $h$, and $T \equiv 0$ if and only
if \( h \) is pluriharmonic. The “\( \partial \bar{\partial} \)-Poincaré Lemma” says that any closed, positive (1,1)-current \( T \) on a complex manifold is locally of the form \( i\partial \bar{\partial} h \) for some plurisubharmonic function \( h \) [GH].

The next three Lemmas show that the “region of pluriharmonicity” of a plurisubharmonic function is locally Stein. See [C, Theorem 6.2], [U, Lemma 2.4], [FS, Lemma 5.3], and [R, Theorem II.2.3] for similar statements.

**Lemma 3.1.** Suppose \( h \) is plurisubharmonic on the open unit polydisk \( D^2 \) in \( \mathbb{C}^2 \) and \( h \) is pluriharmonic on the “Hartogs domain”

\[
\Omega_\delta = \{ (z, w) : |z| < 1, |w| < \delta \} \cup \{ (z, w) : 1 - \delta < |z| < 1, |w| < 1 \}.
\]

Then \( h \) is pluriharmonic on \( D^2 \).

**Proof.** Let \( H \) be a holomorphic function on \( \Omega_\delta \) such that \( h = \text{Re} \, H \). Any holomorphic function on \( \Omega_\delta \) extends to \( D^2 \), and extending \( H \) we have \( h \leq \text{Re} \, H \) on \( D^2 \) since \( h \) is plurisubharmonic. The set

\[
A = \{ z \in D^2 : h = \text{Re} \, H \}
\]

is closed by upper-semi-continuity of \( h \). If \( A \) has a boundary point \( w \in D^2 \), then for any ball \( B(w) \) about \( w \), we have

\[
h(w) = \text{Re} \, H(w) = \frac{1}{|B(w)|} \int_{B(w)} \text{Re} \, H \\
> \frac{1}{|B(w)|} \int_{B(w)} h
\]

since \( \text{Re} \, H > h \) on a set of positive measure in \( B(w) \). This inequality, however, contradicts the sub-mean-value property of the subharmonic function \( h \).

Therefore \( A = D^2 \) and \( h \) is pluriharmonic on the polydisk. \( \square \)

**Lemma 3.2.** Let \( X \) be a complex manifold. If an open subset \( \Omega \subset X \) is not locally Stein, there is a \( \delta > 0 \) and an embedding

\[
e : D^2 \to X
\]

so that \( e(\Omega_\delta) \subset \Omega \) but \( e(D^2) \not\subset \Omega \).

**Proof.** Suppose \( \Omega \) is not locally Stein at \( x \in \partial \Omega \). By choosing local coordinates in a Stein neighborhood \( U \) of \( x \) in \( X \), we may assume that \( U \) is a pseudoconvex domain in \( \mathbb{C}^n \). Then \( \Omega_0 = U \cap \Omega \) is not pseudoconvex and the function \( \phi(z) = -\log d_0(z) \) is not plurisubharmonic near \( x \in \partial \Omega_0 \). Here, \( d_0 \) is the Euclidean distance function to the boundary of \( \Omega_0 \).

If \( \phi \) is not plurisubharmonic at the point \( z_0 \in U \cap \Omega \), then there is a one-dimensional disk \( \alpha : D^1 \to \Omega \) centered at \( z_0 \) such that \( \int_{\partial D^1} \phi < \phi(z_0) \) (identifying the disk with its image \( \alpha(D^1) \)). Let \( \psi \) be a harmonic function on \( D^1 \) so that \( \psi = \phi \) on \( \partial D^1 \). Then \( \psi(z_0) < \phi(z_0) \). Let \( \Psi \) be a holomorphic function on \( D^1 \) with \( \psi = \text{Re} \, \Psi \).

Now, let \( p \in \partial \Omega \) be such that \( d_0(z_0) = |z_0 - p| \). Let \( e : D^2 \to U \) be given by

\[
e(z_1, z_2) = \alpha(z_1) + z_2(1 - \varepsilon)e^{-\Psi(z_1)}(p - z_0).
\]
That is, the two-dimensional polydisk is embedded so that at each point \( z_1 \in D^1 \) there is a disk of radius \( |(1 - \varepsilon) \exp(-\Psi(z_1))| \) in the direction of \( p - z_0 \). If \( \varepsilon \) is small enough we have a Hartogs-type subset of the polydisk contained in \( \Omega \) but the polydisk is not contained in \( \Omega \) since \( d_0(z_0, \partial \Omega) = \exp(-\phi(z_0)) < \exp(-\psi(z_0)) \).

Lemma 3.3. Let \( T \) be a closed, positive \((1,1)\)-current on a complex manifold \( X \). Then \( \Omega = X - \text{supp}(T) \) is locally Stein.

Proof. Let \( p \) be a boundary point of \( \Omega \). Choose a Stein neighborhood \( U \) of \( p \) in \( X \) so that \( T = i\partial \bar{\partial}h \) for some plurisubharmonic function \( h \) on \( U \). By definition of \( \Omega \), \( h \) is pluriharmonic on \( U \cap \Omega \).

If \( \Omega \) is not locally Stein at \( p \), then by Lemma 3.2, we can embed a two-dimensional polydisk into \( U \) so that a Hartogs-type domain \( \Omega_\delta \) lies in \( \Omega \), but the polydisk is not contained in \( \Omega \). By Lemma 3.1, \( h \) must be pluriharmonic on the whole polydisk, contradicting the definition of \( \Omega \).

Corollary 3.4. If \( X \) is Stein, then so is \( X - \text{supp}T \).

Example. If \( X \) is a Stein manifold and \( V \) a hypersurface, then \( V = \text{supp}T \) for a positive \((1,1)\)-current \( T \) given locally by \( T = \frac{1}{\pi} \partial \bar{\partial} \log |f| \), where \( V \) is the zero set of \( f \). Lemma 3.3 shows that \( X - V \) is locally Stein, and thus Stein. Similarly, \( P^n - V \) is Stein for any hypersurface \( V \).

4. The potential function of a rational map

Let \( f : P^n \to P^n \) be a holomorphic map. Let \( F : C^{n+1} \to C^{n+1} \) be a lift of \( f \) to a homogeneous polynomial, unique up to scalar multiple, so that \( \pi \circ F = f \circ \pi \) where \( \pi \) is the projection \( C^{n+1} \setminus \{0\} \to P^n \). Let \( d \) be the degree of the components of \( F \); then \( f \) has topological degree \( d^n \).

Assume that \( d > 1 \). Following [HP], we define the potential function of \( F \) by

\[
h_F(z) = \lim_{m \to \infty} \frac{1}{d^m} \log \|F^m(z)\|.
\]

The limit converges uniformly on compact subsets of \( C^{n+1} \setminus 0 \), and \( h_F(z) \) is plurisubharmonic on \( C^{n+1} \) since \( \log \| \cdot \| \) is plurisubharmonic. Let \( \Omega_F \subset C^{n+1} \) be the basin of attraction of the origin for \( F \); that is,

\[
\Omega_F = \{ x \in C^{n+1} : F^m(x) \to 0 \text{ as } m \to \infty \}.
\]

Note that \( \Omega_F \) is open and bounded.

From the definition, we obtain the following properties of the potential function \( h_F \) [HP]:

1. \( h_F(\alpha z) = h_F(z) + \log |\alpha| \) for \( \alpha \in C^* \);
2. \( \Omega_F = \{ z : h_F(z) < 0 \} \); and
3. \( h_F \) is independent of the choice of norm \( \| \cdot \| \) on \( C^{n+1} \).
Theorem 4.1. (Hubbard-Papadopol, Ueda, Fornaess-Sibony) The support of the positive (1,1)-current
\[ \omega_f = \frac{i}{\pi} \partial \bar{\partial} h_f \]
on \mathbb{C}^{n+1} - 0 is equal to the preimage of the Julia set \( \pi^{-1}(J(f)) \). If \( n = 1 \), then the Brolin-Lyubich measure \( \mu_f \) satisfies \( \pi^* \mu_f = \omega_f \).

Proof. See [HP, Theorem 4.1] for \( n = 1 \) and [U, Theorem 2.2], [FS, Theorem 2.12] for \( n > 1 \).

From Corollary 3.4, we obtain the following ([U, Theorem 2.3], [FS, Theorem 5.2]):

Corollary 4.2. (Ueda, Fornaess-Sibony) The Fatou components of \( f : \mathbb{P}^n \to \mathbb{P}^n \) are Stein.

5. The bifurcation current

In this section we complete the proof of Theorem 1.1. Let \( f : X \times \mathbb{P}^1 \to \mathbb{P}^1 \) be a holomorphic family of rational maps on \( \mathbb{P}^1 \) of degree \( d > 1 \). Let \( \{F_\lambda\} \) be a holomorphic family of homogeneous polynomials on \( \mathbb{C}^2 \), locally lifting the family \( f \), and let \( h_\lambda \) denote the potential function of \( F_\lambda \) (§4). The potential function \( h_\lambda(z) \) is plurisubharmonic as a function of the pair \( (\lambda, z) \).

Fix \( \lambda_0 \in X \). In a neighborhood \( U \) of \( \lambda_0 \), we can choose coordinates on \( \mathbb{P}^1 \) so that \( \infty \) is not a critical point of \( f_\lambda, \lambda \in U \). For \( z \in \mathbb{P}^1 - \{\infty\} \), let \( \tilde{z} = (z, 1) \in \mathbb{C}^2 \). Define a function \( H \) on \( U \) by
\[ H(\lambda) = \sum_{\{c : f_\lambda^j(c) = 0\}} h_\lambda(\tilde{c}) , \]
where the critical points are counted with multiplicity. Now, let \( N(\lambda) \) be the number of critical points of the rational map \( f_\lambda \) (counted without multiplicity). Let
\[ D(f) = \{\lambda_0 \in X : N(\lambda) \text{ does not have a local maximum at } \lambda = \lambda_0\} . \]

Then \( D(f) \) is a complex hypersurface in \( X \), since it is defined by the vanishing of a discriminant. If \( \lambda_0 \not\in D(f) \), there exists a neighborhood \( U \) of \( \lambda_0 \) and holomorphic functions \( c_j : U \to \mathbb{P}^1, j = 1, \ldots, 2d - 2 \), parametrizing the critical points of \( f_\lambda \), such that \( \infty \not\in c_j(U) \) for all \( j \). In this case, we can express \( H \) as the sum
\[ H(\lambda) = \sum_j H_j(\lambda) \]
of the plurisubharmonic functions
\[ H_j(\lambda) = \frac{1}{d^m} \log \| F_\lambda^m(\tilde{c}_j(\lambda)) \| . \]

For any \( \lambda_0 \in X \), then, \( H \) is defined and continuous in a neighborhood \( U \) of \( \lambda_0 \) and plurisubharmonic on \( U - D(f) \); therefore \( H \) is plurisubharmonic on \( U \).
The bifurcation current $T$ is the positive $(1,1)$-current on parameter space $X$ given locally by

$$T = \frac{i}{\pi} \partial \bar{\partial} H.$$ 

The next Lemma shows that $T$ is globally well-defined on $X$.

**Lemma 5.1.** The current $T = \frac{i}{\pi} \partial \bar{\partial} H$ is independent of (a) the choice of lifts $\tilde{c}_j$ of $c_j$ and (b) the choice of lifts $F_\lambda$ of $f_\lambda$.

**Proof.** Suppose we define a new lift $\tilde{c}_j(\lambda) = t(\lambda) \cdot \tilde{c}(\lambda)$ for some holomorphic function $t$ taking values in $C^*$. Property (1) of the potential function $h_\lambda$ (§4) implies that $h_\lambda(\tilde{c}(\lambda)) = h_\lambda(\tilde{c}(\lambda)) + \log |t(\lambda)|$ and $i \partial \bar{\partial} H$ is unchanged since $\log |t(\lambda)|$ is pluriharmonic, proving (a). If the lifted family $\{F_\lambda\}$ is similarly replaced by $\{t(\lambda) \cdot F_\lambda\}$, a computation shows that $h_\lambda$ is changed only by the addition of the pluriharmonic term $\frac{1}{d-1} \log |t(\lambda)|$ where $d$ is the degree of the $f_\lambda$. This proves (b). \qed

**Lemma 5.2.** A parameter $\lambda_0$ lies in the stable regime $S(f) \subset X$ if and only if the function $H$ is pluriharmonic in a neighborhood of $\lambda_0$.

**Proof.** Let us first suppose that $\lambda_0 \in S(f)$ is not in $D(f)$ (in the notation above). By characterization (6) of stability (§2), for each $j$, the family of functions $\{\lambda \mapsto f_\lambda^m(c_j(\lambda))\}$ is normal in a neighborhood $V$ of $\lambda_0$; hence, there exists a subsequence converging uniformly on compact subsets to a holomorphic function $g_j(\lambda)$. As in [HP, Prop 5.4], we can shrink our neighborhood $V$ if necessary to find a norm $\|\cdot\|$ on $C^2$ so that $\log \|\cdot\|$ is pluriharmonic on $\pi^{-1}(g_j(V))$; e.g., if $g_j(V)$ is disjoint from $\{|x| = |y|\}$, we can choose norm $\|(x,y)\| = \max\{|x|,|y|\}$. Then, on any compact set in $V$, the functions

$$\lambda \mapsto \frac{1}{dm_k} \log \|F_\lambda^{m_k}(\tilde{c}_j(\lambda))\|$$

are pluriharmonic if $k$ is large enough. By property (3) of the potential function $h_\lambda$ (§4), this subsequence converges uniformly to $H_j$. Therefore, $H$ is pluriharmonic on $V$.

If $\lambda_0$ lies in $D(f) \cap S(f)$, then $H$ is defined and continuous on a neighborhood $V$ of $\lambda_0$ and pluriharmonic on $V - D(f)$. As $D(f)$ has codimension 1, $H$ must be pluriharmonic on all of $V$.

For the converse, let us suppose again that $\lambda_0 \notin D(f)$ and that $H$ is pluriharmonic in a neighborhood of $\lambda_0$. Each $H_j$ is pluriharmonic and so we may write $H_j = \text{Re} \ G_j$ in a neighborhood $V$ of $\lambda_0$. In analogy with [U, Prop. 2.1], we define new lifts $\tilde{c}_j(\lambda) = e^{-G_j(\lambda)} \cdot \tilde{c}(\lambda)$ of $c_j$ and compute

$$h_\lambda(\tilde{c}_j(\lambda)) = h_\lambda(\tilde{c}(\lambda)) + \log |e^{-G_j(\lambda)}|$$

$$= h_\lambda(\tilde{c}(\lambda)) - \text{Re} \ G_j$$

$$= H_j - H_j$$

$$= 0.$$
By property (2) of $h_\lambda$, this implies that $\hat{c}_j(\lambda)$ lies in $\partial\Omega_\lambda$ for all $\lambda \in V$. If $V$ is small enough, the set $\bigcup_{\lambda \in V} (\{\lambda\} \times \partial\Omega_\lambda)$ has compact closure in $X \times \mathbb{C}^2$. As the functions $F_\lambda$ preserve $\partial\Omega_\lambda$, the family $\{\lambda \mapsto F_\lambda^n(\hat{c}_j(\lambda))\}$ is uniformly bounded and thus normal. Of course, $f_\lambda^n \circ c_j = \pi \circ F_\lambda^n \circ \hat{c}_j$ demonstrating that $\lambda_0$ is a stable parameter by (6) of Section 2.

Finally suppose that $H$ is pluriharmonic in a neighborhood $U$ of parameter $\lambda_0 \in D(f)$. Then $U - D(f)$ lies in the stable regime and Lemma 2.1 shows that all of $U$ must belong to $S(f)$.

**Proof of Theorem 1.1.** Let $T$ be the bifurcation current defined above for the family of rational maps $f$ over $X$. By Lemma 5.2, the support of $T$ is the bifurcation locus $B(f)$.

Corollaries 1.2 and 1.3 now follow immediately from Corollary 3.4.

### 6. Examples

**Example 6.1.** In the family $\{f_\lambda(z) = z^d + \lambda\}, \lambda \in \mathbb{C}$, the bifurcation current $T$ takes the form

$$ T = \frac{d-1}{d} \left( \frac{i}{\pi} \partial\bar{\partial} G \right) $$

where $G$ is the Green’s function for the complement of the “degree $d$ Mandelbrot set” $M_d = \{\lambda : f_\lambda^n(0) \not\to \infty \text{ as } n \to \infty\}$. That is, $T$ is a multiple of harmonic measure supported on $\partial M_d$. The $T$-mass of $\partial M_d$ is $(d - 1)/d$.

**Proof.** If $G_\lambda$ denotes the Green’s function for the complement of the filled Julia set $K(f_\lambda) = \{z : f_\lambda^n(z) \not\to \infty \text{ as } n \to \infty\}$, then $G(\lambda) = G_\lambda(\lambda)$ (see e.g. [CG, VIII.4]). By [HP, Prop 8.1], we have

$$ h_\lambda(x, y) = G_\lambda(x/y) + \log |y| $$

where $(x, y), y \neq 0$, is a point of $\mathbb{C}^2$. Note that $d - 1$ of the critical points of $f_\lambda$ are at $z = 0$ and the other $d - 1$ are at $z = \infty$. Computing, we find

$$ T = \frac{i}{\pi} \sum_j \partial\bar{\partial} h_\lambda(\hat{c}_j(\lambda)) $$

$$ = (d-1) \frac{i}{\pi} \partial\bar{\partial} h_\lambda(0, 1) $$

$$ = (d-1) \frac{i}{\pi} \partial\bar{\partial} G_\lambda(0) $$

$$ = \frac{d-1}{d} \frac{i}{\pi} \partial\bar{\partial} G_\lambda(\lambda). $$

**Example 6.2.** Let $f$ be a polynomial of degree $d$ and $G_f$ the Green’s function for the complement of the filled Julia set. The Lyapunov exponent of $f$ (for the Brolin-Lyubich measure) satisfies ([Prz],[Mn])

$$ L(f) = \log d + \sum_{c \in \mathbb{C}, f'(c) = 0} G_f(c). $$
If \( \{ f_z \} \) is any holomorphic family of polynomials, the Lyapunov exponent as a function of the parameter is a potential function for the bifurcation current; that is,

\[
T = \frac{i}{\pi} \partial \bar{\partial} L.
\]

In the sequel, we examine further the connection between the bifurcation current and the Lyapunov exponent.

References


