PROPER HOLOMORPHIC DISCS IN $\mathbb{C}^2$

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1. Results

Let $U$ denote the open unit disc in $\mathbb{C}$ and $T = bU$ the unit circle. Let $X$ be a Stein manifold of dimension at least two. It was proved in [Glo] that for any point $p \in X$ there exists a proper holomorphic map $f: U \to X$ satisfying $f(0) = p$. We shall call such maps proper holomorphic discs in $X$. For smoothly bounded pseudoconvex domains in $\mathbb{C}^n$ this was proved earlier in [FG], and the essential addition in [Glo] was a method for crossing critical points of a strongly plurisubharmonic exhaustion function on $X$. The methods developed in [FG] and [Glo] actually show the following.

Theorem 1.1. Let $X$ be a Stein manifold with $\text{dim } X \geq 2$, let $\rho: X \to \mathbb{R}$ be a smooth exhaustion function which is strongly plurisubharmonic on $\{\rho > M\}$ for some $M \in \mathbb{R}$, and let $d$ be a metric on $X$. Given a continuous map $h: \overline{U} \to X$ which is holomorphic on $U$ and satisfies $\rho(h(e^{i\theta})) > M$ for $e^{i\theta} \in T$, there exists for any pair of numbers $0 < r < 1$, $\epsilon > 0$, and for any finite set $A \subset U$ a proper holomorphic map $f: U \to X$ satisfying

(i) $\lim_{|\zeta| \to 1} \rho(f(\zeta)) = +\infty$,
(ii) $\rho(f(\zeta)) > \rho(h(\zeta)) - \epsilon$ for $\zeta \in U$,
(iii) $d(f(\zeta), h(\zeta)) < \epsilon$ for $|\zeta| \leq r$, and
(iv) $f(\zeta) = h(\zeta)$ for $\zeta \in A$.

We are interested to what extent does theorem 1.1 hold if $\rho$ is a (strongly) plurisubharmonic function whose sub-level sets are not necessarily relatively compact. Besides its intrinsic interest, we are motivated by the question whether it is possible to avoid any closed complex hypersurface $L$ in a Stein manifold by proper holomorphic discs. Such $L$ is the zero set of a smooth plurisubharmonic function $\rho: X \to \mathbb{R}_+$ which is strongly plurisubharmonic on $\{\rho > 0\} = X\setminus L$; therefore a positive answer to the first question gives proper holomorphic discs in $X$ avoiding $L$. In this paper we obtain positive results in certain model situations in $\mathbb{C}^2$. We begin with the following result.

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Theorem 1.2. For each $c < 1$ and $M \in \mathbb{R}$ the conclusion of theorem 1.1 holds with $X = \mathbb{C}^2$ and the function $\rho_c : \mathbb{C}^2 \to \mathbb{R}$ given by

\begin{equation}
\rho_c(z_1, z_2) = \rho_c(x_1 + iy_1, x_2 + iy_2) = x_1^2 + x_2^2 - c(y_1^2 + y_2^2).
\end{equation}

If on the other hand $c \geq 1$ then for any proper holomorphic map $f : U \to \mathbb{C}^2$ the function $\rho_c \circ f$ is unbounded from below on $U$; hence there exist no proper holomorphic discs satisfying theorem 1.1 (i) for $\rho = \rho_c$ with $c \geq 1$.

Note that $\rho_c$ is strongly plurisubharmonic if $c < 1$, strongly plurisuperharmonic if $c > 1$, and $\rho_1(z_1, z_2) = \Re(z_1^2 + z_2^2)$ is pluriharmonic.

The second statement in theorem 1.2 (for $c \geq 1$) can be seen by applying theorem 1.5 (d) below to the function $g = f_1^2 + f_2^2$: since its range at any boundary point $e^{i\theta} \in T$ omits at most a polar set in $\mathbb{C}$, its real part $\Re g = \rho_1(f_1, f_2)$ is unbounded from below. Since $\rho_c \leq \rho_1$ for $c \geq 1$, the same is true for $\rho_c \circ f$. The first part of theorem 1.2 (for $c < 1$) is proved in section 3.

When $c > 0$, $\rho_c$ is not an exhaustion function on $\mathbb{C}^2$. For $0 < c < 1$ theorem 1.2 gives proper holomorphic maps $f : U \to \mathbb{C}^2$ with images $f(U)$ contained in the real cone $\Gamma_c = \{\rho_c > 0\}$ with axis $\mathbb{R}^2 = \{y = 0\}$. Moreover, when $c > 1$ we can apply theorem 1.2 with $-\rho_c(z)/c = y_1^2 + y_2^2 - \frac{1}{c}(x_1^2 + x_2^2)$ to obtain a proper holomorphic map $f : U \to \mathbb{C}^2$ whose image avoids $\Gamma_c$. This gives proper holomorphic discs in $\mathbb{C}^2$ avoiding relatively large real cones. On the other hand, no proper holomorphic disc (in fact, no transcendental complex curve) in $\mathbb{C}^2$ can avoid a nonempty open complex cone; see theorem 2 in [SW] and theorem 1.5 below.

Our next result concerns discs avoiding pairs of complex lines in $\mathbb{C}^2$.

Theorem 1.3. There exists a proper holomorphic map $f = (f_1, f_2) : U \to \mathbb{C}^2$ whose image $f(U)$ is contained in $(\mathbb{C}^*)^2 = \mathbb{C}^2 \setminus \{zw = 0\}$.

Writing $f : U \to (\mathbb{C}^*)^2$ as $f = (e^{g_1}, e^{g_2}) = (e^{u_1 + iv_1}, e^{u_2 + iv_2})$, we have $|f|^2 = |f_1|^2 + |f_2|^2 = e^{2u_1} + e^{2u_2}$, and $f$ is proper as a map into $\mathbb{C}^2$ if and only if $\max\{u_1, u_2\}$ tends to $+\infty$ at the boundary of $U$. Thus theorem 1.3 is equivalent to

Theorem 1.4. There exists a pair of harmonic functions $u_1, u_2$ on the disc $U$ such that

$$\lim_{|z| \to 1} \max\{u_1(\zeta), u_2(\zeta)\} = +\infty.$$
in 1975 that for parallel lines in \( \mathbb{C}^2 \) this is the only obstruction: If \( E \subset \mathbb{C} \) is a closed polar set containing at least two points, there exists a proper holomorphic map \( f = (f_1, f_2): U \to \mathbb{C}^2 \) such that \( f_1: U \to \mathbb{C} \setminus E \) is a universal covering map of the disc onto \( \mathbb{C} \setminus E \). We don’t know whether an analogue of Alexander’s result holds for complex lines through the origin.

In the remainder of this section we discuss the boundary behavior of proper holomorphic maps \( f = (f_1, f_2): U \to \mathbb{C}^2 \) at the circle \( T = \{ |\zeta| = 1 \} \). We must recall some basic notions from the theory of cluster sets of meromorphic functions on the disc; we refer to Chapter 8 in the monograph [CL] (see section 5 below for more details).

Let \( g \) be a meromorphic function on \( U \). A point \( e^{i\theta} \in T \) at which the (unrestricted) cluster set of \( g \) equals \( \overline{\mathbb{C}} = \mathbb{C} \cup \{ \infty \} \) is called a Weierstrass point of \( g \). If the restricted cluster set of \( g \) at \( e^{i\theta} \) within each conical region in \( U \) with vertex \( e^{i\theta} \) equals \( \overline{\mathbb{C}} \) then \( e^{i\theta} \) is called a Plessner point of \( g \). A point \( e^{i\theta} \) at which \( g \) has a non-tangential limit (a limit as \( \zeta \to e^{i\theta} \) within any cone in \( U \) with vertex \( e^{i\theta} \)) is called a Fatou point of \( g \), and the set of all Fatou point is the Fatou set of \( g \). The range of \( g \) at \( e^{i\theta} \), denoted \( R(g, e^{i\theta}) \), consists of all \( \alpha \in \overline{\mathbb{C}} \) such that \( g(\zeta_j) = \alpha \) for points in a sequence \( \zeta_j \in U \) with \( \lim_{j \to \infty} \zeta_j = e^{i\theta} \).

**Theorem 1.5.** Let \( f = (f_1, f_2): U \to \mathbb{C}^2 \) be a proper holomorphic map of the disc to \( \mathbb{C}^2 \). Let \( P, Q \) be nonconstant holomorphic polynomials on \( \mathbb{C}^2 \) whose leading order homogeneous parts have no common divisor. Denote by \( g \) any of the following (meromorphic) functions: (i) \( f_1 \) or \( f_2 \), (ii) \( f_1/f_2 \), (iii) \( P(f_1, f_2) \), (iv) \( P(f_1, f_2)/Q(f_1, f_2) \). Then

(a) the Fatou set of \( g \) has Lebesgue measure zero in \( T \),
(b) every point of \( T \) is a Weierstrass point of \( g \),
(c) almost every point of \( T \) is a Plessner point of \( g \), and
(d) for every \( e^{i\theta} \in T \) the set \( \mathbb{C} \setminus R(g, e^{i\theta}) \) is polar.

Theorem 1.5 is proved in section 5. Part (d) can be interpreted as a result on polynomial hulls as follows. We define the polynomial hull \( \hat{K} \) of an arbitrary subset \( K \subset \mathbb{C}^n \) as the intersection of all closed set in \( \mathbb{C}^n \) of the form \( \{ \Re P \leq 0 \} \) containing \( K \), where \( P \) is a holomorphic polynomial. For compact sets this coincides with the usual definition of the polynomial hull. Clearly \( \hat{K} \) is contained in the closed convex hull of \( K \). Theorem 1.5 (d) implies

**Corollary 1.6.** If \( f: U \to \mathbb{C}^2 \) is a proper holomorphic map then for each open set \( D \subset \mathbb{C} \) intersecting \( T \) the polynomial hull of \( f(U \cap D) \) equals \( \mathbb{C}^2 \) (and hence its closed convex hull also equals \( \mathbb{C}^2 \) ).
In the Appendix we comment on the proof of theorem 1.1 in [Glo]. Let \( \rho: X \to \mathbb{R} \) be a strongly plurisubharmonic Morse exhaustion function on a Stein manifold \( X \) of dimension \( \geq 2 \). We show that one can push the boundary of an analytic disc in \( X \) over a critical level of \( \rho \) by using the gradient flow of \( \rho \). This creates a non-holomorphic contribution which can be cancelled off during a later stage of the lifting procedure (this was the crucial observation in [Glo]).

2. Lifting holomorphic discs

In this section we describe a general method for lifting the boundary of an analytic disc in \( \mathbb{C}^n \) to a higher level set of a strongly plurisubharmonic function \( \rho: \mathbb{C}^n \to \mathbb{R} \). This method was developed in [FG], but for our present needs we need more precise estimates for the amount of possible lifting at each step of the process.

**Proposition 2.1.** Let \( \lambda: T \times \mathbb{U} \to \mathbb{C}^n \) be a continuous map such that for each \( \zeta \in T \) the map \( \lambda_\zeta = \lambda(\zeta, \cdot) : \mathbb{U} \to \mathbb{C}^n \) is holomorphic in \( \mathbb{U} \) and \( \lambda_\zeta(0) = 0 \). Given numbers \( \epsilon > 0 \) and \( 0 < r < 1 \), there exists a holomorphic polynomial map \( h: \mathbb{C} \to \mathbb{C}^n \) satisfying

(i) \( \text{dist}(h(\zeta), \lambda_\zeta(\mathbb{U})) < \epsilon \) \( (\zeta \in T) \),

(ii) \( \text{dist}(h(t\zeta), \lambda_\zeta(\mathbb{U})) < \epsilon \) \( (\zeta \in T, r \leq t \leq 1) \), and

(iii) \( |h(\zeta)| < \epsilon \) \( (|\zeta| \leq r) \).

**Proof.** It suffices to show that \( \lambda \) can be approximated uniformly on \( T \times \overline{\mathbb{U}} \) by maps of the form

\[
(2.1) \quad \tilde{\lambda}(\zeta, w) = \frac{w}{\zeta^M} \sum_{j=1}^{N} A_j(\zeta)w^{j-1},
\]

where the \( A_j \)’s are holomorphic polynomials and \( M, N \) are positive integers. The polynomial map

\[
h(\zeta) = \tilde{\lambda}(\zeta, \zeta^K) = \zeta^{K-M} \sum_{j=1}^{N} A_j(\zeta)\zeta^{(j-1)K}
\]

then satisfies proposition 2.1 provided that the approximation of \( \lambda \) by \( \tilde{\lambda} \) is sufficiently close and the integer \( K \geq M \) is chosen sufficiently large.

We begin by replacing \( \lambda \) by \( (\zeta, w) \mapsto \lambda(\zeta, sw) \) for a suitable \( s < 1 \) sufficiently close to 1. Denoting the new map again by \( \lambda \) we may thus assume that \( \lambda_\zeta \) is holomorphic in a larger disc \( |w| < 1/s \) for each \( \zeta \in T \). We expand \( \lambda \) in Taylor series with respect to \( w \) and approximate it uniformly on \( bU \times T \) by a Taylor polynomial \( \lambda_N(\zeta, w) = \sum_{j=1}^{N} a_j(\zeta)w^j \) with continuous coefficients \( a_j : T \to \mathbb{C}^n \). (The coefficient \( a_0 \) is zero since \( \lambda(\zeta, 0) = 0 \).) Finally we approximate each \( a_j \) uniformly on \( T \) by a map \( A_j(\zeta)/\zeta^M \) for some holomorphic polynomial \( A_j \) and some integer \( N \) which can be chosen to be independent of \( j \). This gives the desired approximation of \( \lambda \) by a map of the form (2.1). \( \square \)
Corollary 2.2. Let \( g_0: \overline{U} \to \mathbb{C}^n \) be a continuous map that is holomorphic in \( U \) and let \( \lambda \) be as in proposition 2.1. Suppose that \( \rho: \mathbb{C}^n \to \mathbb{R} \) is a real continuous function such that for some constants \( C_0 < C_1 \) and \( 0 < r < 1 \) we have

\[
\begin{align*}
(a) \quad & \rho(g_0(\zeta) + \lambda(\zeta, w)) = C_1 \quad (\zeta \in T, \ w \in T), \\
(b) \quad & \rho(g_0(\zeta) + \lambda(\zeta, w)) > C_0 \quad (\zeta \in T, \ w \in \overline{U}), \text{ and} \\
(c) \quad & \rho(g_0(\zeta)) > C_0 \quad (r \leq |\zeta| \leq 1).
\end{align*}
\]

Then for each \( \epsilon > 0 \) there exists a holomorphic polynomial map \( g: \mathbb{C} \to \mathbb{C}^n \) satisfying

\[
\begin{align*}
(i) \quad & |\rho(g(\zeta)) - C_1| < \epsilon \quad (\zeta \in T), \\
(ii) \quad & \rho(g(\zeta)) > C_0 \quad (r \leq |\zeta| \leq 1), \text{ and} \\
(iii) \quad & |g(\zeta) - g_0(\zeta)| < \epsilon \quad (|\zeta| < r).
\end{align*}
\]

Proof. Take \( g(\zeta) = \tilde{g}_0(\zeta) + h(\zeta) \), where \( \tilde{g}_0 \) is a polynomial approximation of \( g_0 \) and \( h \) is a suitably chosen map provided by proposition 2.1. \( \square \)

Assume now that \( \rho: \mathbb{C}^n \to \mathbb{R} \) is a function of class \( C^2 \). For each fixed \( z \) we write

\[
(2.2) \quad \rho_z(w) = \rho(z + w) - \rho(z) = \Re Q_z(w) + L_z(w) + o(|w|^2),
\]

where

\[
Q_z(w) = 2 \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(z)w_j + \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial z_k}(z)w_j w_k,
\]

\[
L_z(w) = \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(z)w_j \overline{w}_k,
\]

\((Q_z \text{ is the Levi polynomial and } L_z \text{ is the Levi form of } \rho \text{ at } z)\). The set

\[
(2.3) \quad \Lambda_z = \{ w \in \mathbb{C}^n : Q_z(w) = 0 \}
\]

is a quadratic complex hypersurface in \( \mathbb{C}^n \) and we have \( \rho_z(w) = L_\rho(z; w) + o(|w|^2) \) for \( w \in \Lambda_z \). For \( c > 0 \) we denote by \( B(z; c) \) the connected component of the sublevel set \( \{ w \in \Lambda_z : \rho_z(w) < c \} \) which contains the point \( 0 \in \Lambda_z \). If \( \rho \) is strongly plurisubharmonic near \( z \) (i.e., its Levi form \( L_z \) at \( z \) is positive definite) and if \( \partial \rho(z) \neq 0 \) (so that the hypersurface \( \Lambda_z \) is smooth near 0), then for all sufficiently small \( c > 0 \) the set \( B(z; c) \) is diffeomorphic to the real \( (2n - 2) \)-dimensional ball. Moreover, if \( C > 0 \) is such that the function \( \rho_z|\Lambda_z \) has no critical points on \( B(z; C) \) other than the point 0, Morse theory shows that for \( 0 < c \leq C \) the sets \( B(z; c) \) are complex manifolds diffeomorphic to the \((2n - 2)\)-ball. (We include the singularities of \( \Lambda_z \) among the critical points of \( \rho_z|\Lambda_z \).) In particular, when \( n = 2 \), these sets are complex one-dimensional and hence conformally equivalent to the disc. We state the next proposition only for \( n = 2 \) since we shall only need this case.
Proposition 2.3. Let $g_0: \overline{U} \to \mathbb{C}^2$ be a continuous map that is holomorphic in $U$ and let $\rho: \mathbb{C}^2 \to \mathbb{R}$ be a $\mathbb{C}^2$ function which is strongly plurisubharmonic in a neighborhood of $g_0(T)$ and has no critical points on $g_0(T)$. Suppose that $C: T \to (0, \infty)$ is a continuous function such that the function $\rho_{g_0(\zeta)}|A_{g_0(\zeta)}$ has no critical points on $B(g_0(\zeta); C(\zeta)) \setminus \{0\}$ for each $\zeta \in T$. Then for each $\epsilon > 0$ and $0 < r < 1$ there is polynomial map $g: \mathbb{C} \to \mathbb{C}^2$ satisfying

(i) $|\rho(g(\zeta)) - \rho(g_0(\zeta)) - C(\zeta)| < \epsilon$ $(\zeta \in T)$,
(ii) $\rho(g(\zeta)) > \rho(g_0(\zeta)) - \epsilon$ $(\zeta \in U)$, and
(iii) $|g(\zeta) - g_0(\zeta)| < \epsilon$ $(|\zeta| \leq r)$.

Proof. We have seen above that for each $\zeta \in T$ the set $B(g_0(\zeta); C(\zeta)) \subset A_{g_0(\zeta)}$ is conformally equivalent to the disc $U$. Decreasing $C(\zeta)$ slightly (so that $B(g_0(\zeta); C(\zeta))$ is still biholomorphic to $U$ for some $r > C(\zeta)$) we can obtain a parametrization $\lambda_\zeta: \overline{U} \to B(g_0(\zeta); C(\zeta))$ $(\zeta \in T)$, depending continuously on $(\zeta, w) \in T \times \overline{U}$, such that $\lambda_\zeta$ is holomorphic in $U$ and $\lambda_\zeta(0) = g_0(\zeta)$ for each $\zeta \in T$. The result now follows from proposition 2.1 applied to the family of discs $\lambda_\zeta$.

If $K_0 \subset K_1 \subset \mathbb{C}^2$ is a pair of compact sets such that $\rho$ is strongly plurisubharmonic and has no critical points on $K_1$, there is a constant $C > 0$ such that $\rho|A_z$ has no critical points on $B(z; C) \setminus \{0\}$ for each $z \in K_0$. Hence proposition 2.3 provides a uniform lifting of the boundary of an analytic disc (with respect to $\rho$) as long as the boundary remains in $K_0$. If the set $A(c_0, c_1) = \{x \in X: c_0 \leq \rho(x) \leq c_1\}$ is compact for some $c_0 < c_1$ and if $\rho$ is strongly plurisubharmonic and without critical points on this set, proposition 2.3 allows us to lift the boundary of an analytic disc in $X$ from the level $\rho = c_0$ to the level $\rho = c_1$. Unfortunately this breaks down in general if the level sets of $\rho$ are not compact. In this case we need a more precise analysis which we shall do for the function (1.1).

Proposition 2.4. Let $\rho_c$ be the function (1.1). If $c < 1$ there exists a number $a = a(c) > 0$ with the following property: For each continuous map $h: \overline{U} \to \mathbb{C}^2$, holomorphic in $U$, such that $m(h) = \inf\{\rho_c(h(\zeta)); |\zeta| = 1\} > 0$, and for each pair of numbers $\epsilon > 0$ and $0 < r < 1$ there exists a holomorphic polynomial map $g: \mathbb{C} \to \mathbb{C}^2$ satisfying

(i) $m(g) \geq (1 + a)m(h)$,
(ii) $\rho_c(g(\zeta)) > \rho_c(h(\zeta)) - \epsilon$ $(|\zeta| \leq 1)$, and
(iii) $|g(\zeta) - h(\zeta)| < \epsilon$ $(|\zeta| \leq r)$.

Proof. Note that $\rho_c$ is strongly plurisubharmonic when $c < 1$. Fix such a $c$ and write $\rho = \rho_c$. The only critical point of $\rho$ is $z_1 = z_2 = 0$. Proposition 2.4 follows immediately from proposition 2.3 and the following

Lemma 2.5. Let $p = \rho_c$ for some $c < 1$ be given by (1.1). There is a constant $a = a(c) > 0$ such that for each $z \in \mathbb{C}^2$ with $\rho(z) > 0$ the function $\rho_p|A_z$ has no critical points on $B(z; a \rho(z)) \setminus \{0\}$. 
Proof. A calculation shows that \( \rho_z(w) = \Re Q_z(w) + \mathcal{L}_z(w) \), where
\[
Q_z(w) = 2(x_1 + icy_1)w_1 + 2(x_2 + icy_2)w_2 + \frac{1}{2}(1 + c)(w_1^2 + w_2^2)
\]
\[
\mathcal{L}_z(w) = \frac{1}{2}(1 - c)(|w_1|^2 + |w_2|^2) = \frac{1}{2}(1 - c)|w|^2.
\]

It suffices to consider the case \( 0 < c < 1 \). If \( w \in \Lambda_z \) then
\[
(2.4) \quad \rho_z(w) = \rho(z + w) - \rho(z) = \frac{1}{2}(1 - c)|w|^2
\]

The critical points of \( \rho_z|\Lambda_z \) are precisely those points \( w \in \Lambda_z \) at which the complex gradients \( \partial Q_z \) and \( \partial \rho_z \) (with respect to the variable \( w = (w_1, w_2) \in \mathbb{C}^2 \)) are \( \mathbb{C} \)-linearly dependent. This set will include any singular points of \( \Lambda_z \). By (2.4) we may replace \( \partial \rho_z \) by \( \partial |w|^2 \). Set \( h(x + iy) = x + icy \), so \( |h(x + iy)|^2 = x^2 + c^2 y^2 \). We have
\[
\partial Q_z(w) = (2h(z_1) + (1 + c)w_1, 2h(z_2) + (1 + c)w_2), \quad \partial |w|^2 = (\overline{w}_1, \overline{w}_2).
\]

This gives the following system of two equations for \( w \), in which the first is the colinearity equation between \( \partial Q_z \) and \( \partial |w|^2 \) (after conjugation) and the second is \( Q_z(w) = 0 \):
\[
(2.5) \quad \begin{align*}
2\overline{h(z_2)}w_1 - 2\overline{h(z_1)}w_2 &= -(1 + c)(w_1 \overline{w}_2 - \overline{w}_1 w_2) \\
4h(z_1)w_1 + 4h(z_2)w_2 &= -(1 + c)(w_1^2 + w_2^2).
\end{align*}
\]

It suffices to obtain a good lower estimate for the norm \( |w| \) of any nonzero solution of (2.5) in terms of \( |z| \). We apply Cramer’s formula to express \( w_1 \) and \( w_2 \) from the linear part in terms of the right hand side terms in (2.5). The determinant of the matrix of coefficients is \( W(z) = 8(|h(z_1)|^2 + |h(z_2)|^2) \geq c'|z|^2 \) where \( c' > 0 \) depends only on \( c \). If we replace one of the columns of the coefficient matrix by the right hand side then each term in the corresponding determinant is of the form constant times \( h(z_j)w_k w_l \) for some \( j, k, l \in \{1, 2\} \). Hence we can estimate the determinant from above by the Cauchy-Schwarz inequality and thus obtain the following estimate for the solutions of (2.5):
\[
|w_j| \leq \frac{c_2(|h(z_1)|^2 + |h(z_2)|^2)^{1/2}|w|^2}{W(z)} \leq \frac{c_3|w|^2}{|z|} \quad (j = 1, 2).
\]

This gives \( |w| \leq c_4|w|^2/|z| \) and therefore \( |w| \geq c_5|z| \) for any nonzero solution \( w \) of (2.5), where \( c_5 > 0 \) depends only on \( c \). Since \( w \in \Lambda_z \), (2.4) gives
\[
\rho(z + w) \geq \rho(z) + c_6|z|^2 \geq \rho(z) + c_7 \rho(z)
\]
for some \( c_7 > 0 \). Thus any constant \( a < c_7 \) satisfies lemma 2.5. \qed
3. Proper discs in cones in $\mathbb{C}^2$ with real axis

In this section we prove theorem 1.2. If the constant $M$ in the theorem is negative, we first apply the procedure described in [Glo] to cross the critical point of $\rho_c$ at $(0,0)$ and thus push the boundary of the given initial analytic disc $h$ to the set $\rho_c > 0$ while changing $h$ as little as desired on $\{|\zeta| \leq r\}$. Hence it suffices to prove theorem 1.2 for $M \geq 0$. In this case the result follows immediately from the following.

**Theorem 3.1.** Let $c < 1$, $M > 0$, and let $\rho = \rho_c$ be the function (1.1). Given a continuous map $h: \overline{U} \to \mathbb{C}^2$, holomorphic in $U$, such that $\rho(h(\zeta)) > M$ for $|\zeta| = 1$, there exists for each $\epsilon > 0$ and $0 < r_1 < 1$ a proper holomorphic map $f: U \to \mathbb{C}^2$ satisfying

- (i) $\lim_{|\zeta| \to 1} \rho_c(f(\zeta)) = +\infty$,
- (ii) $\rho(f(\zeta)) > \rho(h(\zeta)) - \epsilon$ $(|\zeta| < 1)$, and
- (iii) $|f(\zeta) - h(\zeta)| < \epsilon$ $(|\zeta| \leq r_1)$.

**Proof.** It suffices to consider the case $0 < c < 1$. Fix numbers $M > 0$, $0 < r < 1$, $\epsilon > 0$ and a map $h$ as in the statement of theorem 3.1 and write $M_1 = M$, $\epsilon_1 = \epsilon$, $f_1 = h$. Let $a > 0$ be the number given by proposition 2.4 for the pair $c$ and $M_1$. Set

$$M_k = (1 + a)^{k-1} M_1, \quad \epsilon_k = \epsilon / 2^{k-1}, \quad k = 2, 3, 4, \ldots$$

We inductively construct a sequence of polynomial maps $f_k: \overline{U} \to \mathbb{C}^2$ and a sequence of numbers $0 < r_1 < r_2 < r_3 < \ldots < 1$ with $\lim_{k \to \infty} r_k = 1$ such that the following hold for each $k \geq 2$:

- (a_k) $\rho(f_k(\zeta)) > M_k \left( r_k \leq |\zeta| \leq 1 \right)$,
- (b_k) $\rho(f_k(\zeta)) > \rho(f_{k-1}(\zeta)) - \epsilon_{k-1} \left( |\zeta| \leq 1 \right)$, and
- (c_k) $|f_k(\zeta) - f_{k-1}(\zeta)| < \epsilon_{k-1} \left( |\zeta| \leq r_{k-1} \right)$.

The construction proceeds as follows. By assumptions the condition (a_1) holds for $|\zeta| = 1$. By continuity we can increase $r_1$ such that (a_1) holds for $r_1 \leq |\zeta| \leq 1$. Proposition 2.4 gives a map $f_2$ such that $\rho(f_2(\zeta)) > M_2$ for $|\zeta| = 1$ and such that (b_2) and (c_2) hold. By continuity we can choose a number $r_2 < 1$ sufficiently close to 1 such that (a_2) holds for $r_2 \leq |\zeta| \leq 1$.

This process can be continued inductively. If we already have $f_{k-1}$, proposition 2.4 gives the next map $f_k$ which satisfies (a_k) initially only for $|\zeta| = 1$, and it satisfies (b_k) and (c_k). By continuity we can choose $r_k < 1$ sufficiently close to 1 so that (a_k) holds. We can thus insure that $\lim_{k \to \infty} r_k = 1$.

Condition (c) insures that $f = \lim_{k \to \infty} f_k: U \to \mathbb{C}^2$ exists uniformly on compacts in $U$. For $|\zeta| \leq r_1$ we have

$$|f(\zeta) - f_1(\zeta)| \leq \sum_{k=1}^{\infty} |f_{k+1}(\zeta) - f_k(\zeta)| < \sum_{k=1}^{\infty} \epsilon_{k+1} = \epsilon.$$
This proves (iii) since \( h = f_1 \). For a fixed \( \zeta \in U \) and \( k \geq 1 \) we have

\[
\rho(f(\zeta)) = \lim_{j \to \infty} \rho(f_j(\zeta)) = \rho(f_k(\zeta)) + \sum_{j=k}^{\infty} (\rho(f_{j+1}(\zeta)) - \rho(f_j(\zeta)))
\]

\[
> \rho(f_k(\zeta)) - \sum_{j=k}^{\infty} \epsilon_{j+1} = \rho(f_k(\zeta)) - \epsilon_k.
\]

For \( k = 1 \) we get (ii) in the theorem. For points \( \zeta \) in the annulus \( r_k \leq |\zeta| < 1 \) we get \( \rho(f(\zeta)) > \rho(f_k(\zeta)) - \epsilon_k > M_k - \epsilon \). Since \( \lim_{k \to \infty} M_k = \infty \), this implies (i) and completes the proof of theorem 3.1. \( \square \)

4. Proper discs in \( \mathbb{C}^2 \) which omit a pair of lines

Theorem 1.3 follows from the following more precise result.

**Theorem 4.1.** Let \( n \geq 2 \). Given a continuous map \( h = (h_1, h_2, \ldots, h_n): U \to \mathbb{C}^n \) which is holomorphic in \( U \) and given a number \( 0 < r < 1 \) such that the components \( h_j \) have no zeros in \( \{ \zeta: r \leq |\zeta| \leq 1 \} \), there exists for each \( \epsilon > 0 \) a proper holomorphic map \( f = (f_1, f_2, \ldots, f_n): U \to \mathbb{C}^n \) such that the \( f_j \)’s have no zeros in \( \{ \zeta: r \leq |\zeta| < 1 \} \) and \( |f(\zeta) - h(\zeta)| < \epsilon \) for \( |\zeta| \leq r \).

We shall give details only for \( n = 2 \). By factoring out the (finitely many) zeros of the \( h_j \)’s we can reduce to the case when the \( h_j \)’s have no zeros on \( U \). We seek a solution in the form \( f = (e^{g_1}, e^{g_2}) = (e^{u_1 + iv_1}, e^{u_2 + iv_2}) \) for some holomorphic map \( g = (g_1, g_2): U \to \mathbb{C}^2 \). Set

\[
(4.1) \quad \rho(x_1 + iy_1, x_2 + iy_2) = \max\{x_1, x_2\}.
\]

Since \( |f|^2 = |f_1|^2 + |f_2|^2 = e^{2u_1} + e^{2u_2} \), \( f \) is proper into \( \mathbb{C}^2 \) if and only if \( \rho(g(\zeta)) = \max\{u_1(\zeta), u_2(\zeta)\} \) tends to \( +\infty \) as \( |\zeta| \to 1 \). Such map \( g \) will be obtained as the limit \( g = \lim_{k \to \infty} g_k \) of an inductively constructed sequence \( g_k \), where the inductive step from \( g_{k-1} \) to \( g_k \) will be furnished by corollary 2.2. To this end we need a suitable family of analytic discs which we now construct.

**Proposition 4.2.** Let \( \rho \) be the function (4.1). Given a compact set \( K \subset \subset \mathbb{C}^2 \) and constants \( C_0, C_1 \in \mathbb{R} \) such that \( C_0 < \rho(z) < C_1 \) \( (z \in K) \), there is a continuous map \( \lambda: K \times U \to \mathbb{C}^2 \) such that for each \( z \in K \) the map \( \lambda(z, \cdot): U \to \mathbb{C}^2 \) is holomorphic and

(i) \( \rho(\lambda(z, w)) = C_1 \) \( (z \in K, |w| = 1) \),

(ii) \( \rho(\lambda(z, w)) > C_0 \) \( (z \in K, |w| \leq 1) \).

**Proof.** We follow the proof of Bochner’s tube theorem (see [Hör], p. 41). We first describe the model situation. Write the coordinates on \( \mathbb{C}^2 \) in the form \( z = x + iy \),
with \( x, y \in \mathbb{R}^2 \), and identify \( \mathbb{R}^2 \) with \( \{ y = 0 \} \subset \mathbb{C}^2 \). Set
\[
\begin{align*}
  k &= \{(x_1, 0) : 0 \leq x_1 \leq 1\} \cup \{(0, x_2) : 0 \leq x_2 \leq 1\} \\
  K_\epsilon &= \{ x + iy \in \mathbb{C}^2 : x \in k, \ |y|^2 \leq 1/\epsilon \} \\
  \co(k) &= \{(x_1, x_2) : x_1 \geq 0, \ x_2 \geq 0, \ x_1 + x_2 \leq 1\} \\
  \gamma_\epsilon &= \{(x_1, x_2) \in \co(k) : x_1 + x_2 - \epsilon(x_1^2 + x_2^2) = 1 - \epsilon\} \\
  \Gamma_\epsilon &= \{(z_1, z_2) \in \mathbb{C}^2 : (x_1, x_2) \in \co(k), \ z_1 + z_2 - \epsilon(z_1^2 + z_2^2) = 1 - \epsilon\}
\end{align*}
\]

**Lemma 4.3.** (Notation as above) There is an \( \epsilon_0 > 0 \) such that for each \( \epsilon \) with \( 0 < \epsilon < \epsilon_0 \) the set \( \Gamma_\epsilon \) is a holomorphic disc with boundary contained in \( K_\epsilon \), \( \Gamma_\epsilon \cap \mathbb{R}^2 = \gamma_\epsilon \), and \( \gamma_\epsilon \) is a smooth real-analytic curve contained in the convex hull \( \co(k) \) of \( k \). The union \( \bigcup_{0 < \epsilon < \epsilon_0} \gamma_\epsilon \) contains every point in the interior of \( \co(k) \) and sufficiently close to the open segment \( \gamma_0 = \{(x_1, 1 - x_1) : 0 < x_1 < 1\} \).

**Proof.** Observe that \( \gamma_\epsilon = \{ F_\epsilon = 0 \} \cap \co(k) \) where \( F_\epsilon(x_1, x_2) = x_1 + x_2 - \epsilon(x_1^2 + x_2^2) - 1 + \epsilon \).

Simple calculations show that for \( 0 < \epsilon < 1/2 \) we have \( F_\epsilon(x_1, 0) < 0 \) for \( 0 \leq x_1 < 1 \), \( F_\epsilon(1, 0) = F_\epsilon(0, 1) = 0 \), \( F(x_1, 1 - x_1) > 0 \) when \( 0 < x_1 < 1 \), and \( \partial F_\epsilon(x_1, x_2) / \partial x_2 = 1 - 2\epsilon x_2 > 0 \) for \( 0 \leq x_2 \leq 1 \). These properties imply that \( \gamma_\epsilon \) is a graph \( y_1 = h_\epsilon(x_1) \) of a real-analytic function \( h_\epsilon \) over the segment \( 0 \leq x_1 \leq 1 \), with the endpoints \((1, 0)\) and \((0, 1)\). Since \( \partial F_\epsilon / \partial \epsilon = 1 - (x_1^2 + x_2^2) \geq 0 \) on \( \co(k) \) we conclude that, as \( \epsilon \) decreases to 0, the functions \( h_\epsilon \) increase to \( h_0(x_1) = 1 - x_1 \). This gives the last claim in lemma 4.3.

We will show that for sufficiently small \( \epsilon > 0 \) there exists a bounded, simply connected region \( D_\epsilon \subset \{ z_2 = 0 \} \) with piecewise smooth boundary such that \( \Gamma_\epsilon \) is the graph of a holomorphic function over \( D_\epsilon \). The equation for \( \Gamma_\epsilon \) is equivalent to
\[
(4.2) \quad x_1 + x_2 - \epsilon(x_1^2 + x_2^2) + \epsilon(y_1^2 + y_2^2) = 1 - \epsilon \\
(1 - 2\epsilon x_1)y_1 + (1 - 2\epsilon x_2)y_2 = 0.
\]

When \( y_1 = y_2 = 0 \) we get the equation for \( \gamma_\epsilon \), and hence \( \Gamma_\epsilon \cap \mathbb{R}^2 = \gamma_\epsilon \). On \( \co(k) \) we have \( x_1 + x_2 \geq 0 \) and \( x_1^2 + x_2^2 \leq 1 \), with equality only at the points \((1, 0)\) and \((0, 1)\). Rewriting the first equation in (4.2) in the form
\[
(x_1 + x_2) + \epsilon(y_1^2 + y_2^2) = 1 - \epsilon(1 - (x_1^2 + x_2^2)) \leq 1
\]
we see that (4.2) has no solutions for \( |y|^2 = y_1^2 + y_2^2 > 1/\epsilon \), and it has no solutions on \( \gamma_0 \cap \mathbb{R}^2 \) (\( \gamma_0 \) was defined in lemma 4.3). Hence the boundary of \( \Gamma_\epsilon \) is contained in \( K_\epsilon \) and therefore \( \Gamma_\epsilon \subset \co(K_\epsilon) \). From the second equation in (4.2) we get
\[
(4.3) \quad y_2 = -y_1 \frac{1 - 2\epsilon x_1}{1 - 2\epsilon x_2}
\]
Let in the model case. Lemma 4.3, applied to a slightly larger triangle \( \tilde{T} \) such that the set points in \( \lambda \) we have

\[
G(e, x_1, y_1, x_2) := x_1 + x_2 - \epsilon(x_1^2 + x_2^2) + \epsilon y_1^2 \left( 1 + \frac{(1 - 2\epsilon x_1)^2}{(1 - 2\epsilon x_2)^2} \right) - 1 + \epsilon = 0.
\]

Consider first its restriction to \( x_2 = 0 \):

\[
G(e, x_1, y_1, 0) = x_1 - \epsilon x_1^2 + \epsilon y_1^2 \left( 1 + (1 - 2\epsilon x_1)^2 \right) - 1 + \epsilon = 0.
\]

Let \( a_\epsilon > 0 \) be the solution of the equation \( G(0, a_\epsilon, 0) = 2\epsilon a_\epsilon^2 - 1 + \epsilon = 0 \). Calculations show that \( G(e, 0, y_1, 0) < 0 \) for \( |y_1| < a_\epsilon \), \( G(e, 1, y_1, 0) \geq 0 \) (with equality only at \( y_1 = 0 \)), and \( \frac{\partial G}{\partial x_1}(x_1, y_1, 0) > 0 \) for \( 0 \leq x_1 \leq 1 \). This shows that the set

\[
\sigma_\epsilon = \{ x_1 + iy_1 : 0 \leq x_1 \leq 1, \ G(e, x_1, y_1, 0) = 0 \}
\]

is a smooth real-analytic curve which can be written as a graph \( x_1 = g_\epsilon(y_1) \) over the interval \( |y_1| \leq a_\epsilon \), and the set

\[
D_\epsilon = \{ x_1 + iy_1 \in \mathbb{C} : 0 < x_1 < 1, \ G(e, x_1, y_1, 0) < 0 \}
\]

\[
= \{ x_1 + iy_1 : 0 < x_1 < g_\epsilon(y_1), \ |y_1| < a_\epsilon \}
\]

(with piecewise smooth boundary) is conformally equivalent to the disc. A calculation shows that for \( \epsilon > 0 \) sufficiently small we have \( \frac{\partial G}{\partial x_2}(x_1, y_1, x_2) > 0 \) on \( 0 \leq x_1 \leq 1 \) and \( y_1^2 \leq 1/\epsilon \), and \( G(e, x_1, y_1, 1) > 0 \) for \( x_1 + iy_1 \in D_\epsilon \). Since \( G(e, x_1, y_1, 0) < 0 \) for \( x_1 + iy_1 \in D_\epsilon \), it follows that (4.4) has a unique solution \( x_2 = \xi_\epsilon(x_1, y_1) \in [0, 1] \) for each \( z_1 = x_1 + iy_1 \in \partial D_\epsilon \), and it has no solutions for points in \( \{ 0 \leq x_1 \leq 1 \} \setminus \partial D_\epsilon \). From (4.3) we also calculate \( y_2 \) and thus obtain a unique analytic solution \( z_2 = f_\epsilon(z_1) \) \( (z_1 \in \partial D_\epsilon) \) of the system (4.2). This proves that \( \Gamma_\epsilon \) is an analytic disc with boundary in \( K_\epsilon \).

We continue with the proof of proposition 4.2. For each \( y \in \mathbb{R}^2 \) and \( C \in \mathbb{R} \) we have

\[
\{ x \in \mathbb{R}^2 : \rho(x + iy) = C \} = \{ (x_1, C) : x_1 \leq C \} \cup \{ (C, x_2) : x_2 \leq C \}.
\]

For each point \( z = x + iy \in \mathbb{C}^2 \) with \( C_0 < \rho(z) < C_1 \) we can choose a line segment \( l_z \subset R^2 + iy \) passing through \( z \) such that \( \rho > C_0 \) on \( l_z \) and the endpoints of \( l_z \) belong to \( \{ \rho = C_1 \} \). We can choose such \( l_z \) depending smoothly on \( z \) in the region \( C_0 < \rho(z) < C_1 \). The segment \( l_z \) together with the two bounded segments in the level set \( \rho = C_1 \) (in \( \mathbb{R}^2 + iy \)) determines a closed triangle \( T_z \subset \mathbb{R}^2 + iy \) which corresponds (after a rotation and dilation of coordinates) to the set \( co(k) \) in the model case. Lemma 4.3, applied to a slightly larger triangle \( \tilde{T}_z \supset T_z \) obtained by a small parallel translation of the segment \( l_z \) so as to include the point \( z \) in the interior of \( T_z \), gives an analytic disc \( \Gamma_z \subset \mathbb{C}^2 \) passing through \( z \) such that \( \rho > C_0 \) on \( \Gamma_z \) and \( \rho = C_1 \) on \( b\Gamma_z \). We can parametrize \( \Gamma_z \) by a map \( \lambda(z, \cdot) : \overline{U} \to \Gamma_z \), holomorphic in \( U \) and depending continuously on \( z \in K \).

Combining proposition 4.2 an corollary 2.2 we obtain
Corollary 4.4. Let \( \rho \) be the function (4.1). Given a continuous map \( g_0: \overline{U} \to \mathbb{C}^2 \), holomorphic in \( U \), and constants \( 0 < r < 1 \), \( C_0, C_1 \in \mathbb{R} \) that \( C_0 < \rho(g_0(\zeta)) < C_1 \) for \( r \leq |\zeta| \leq 1 \), there is for each \( \epsilon > 0 \) a holomorphic polynomial map \( g: \overline{U} \to \mathbb{C}^2 \) satisfying

i) \( |\rho(g(\zeta)) - C_1| < \epsilon \) \( (|\zeta| = 1) \),
ii) \( \rho(g(\zeta)) > C_0 \) \( (r \leq |\zeta| \leq 1) \), and
iii) \( |g(\zeta) - g_0(z)| < \epsilon \) \( (|\zeta| \leq r) \).

Proof of theorem 4.1. Choose a sequence \( \epsilon_k > 0 \), \( \sum_{k=1}^{\infty} \epsilon_k < 1 \). We begin by an arbitrary continuous map \( g_1: \overline{U} \to \mathbb{C}^2 \) that is holomorphic in \( U \) and a number \( 0 < r_1 < 1 \). Choose numbers \( M_0, M_1 \in \mathbb{R} \) such that \( M_0 < \rho(g_0(\zeta)) < M_1 \) for \( r_1 \leq |\zeta| \leq 1 \). Choose a number \( M_2 \geq M_1 + 1 \) and apply corollary 4.4 to get a polynomial map \( g_2: \mathbb{C} \to \mathbb{C}^2 \) and a number \( r_2, r_1 < r_2 < 1 \), such that the following hold for \( k = 2 \):

(a) \( M_{k-1} < \rho(g_k(\zeta)) < M_k \) \( (r_k \leq |\zeta| \leq 1) \),
(b) \( \rho(g_k(\zeta)) > M_{k-2} \) \( (r_{k-1} \leq |\zeta| \leq 1) \), and
(c) \( |g_k(\zeta) - g_{k-1}(\zeta)| < \epsilon_{k-1} \) \( (|\zeta| \leq r_{k-1}) \).

This process can be continued inductively as follows. Suppose that we have already constructed \( g_k \) for some \( k \geq 2 \). Choose \( M_k \geq M_{k-1} + 1 \) and apply corollary 4.4 to get a map \( g_k \) which satisfies (a) for \( |\zeta| = 1 \) and it satisfies (b) and (c). By continuity we can choose \( r_k < 1 \) such that \( 1 - r_k < (1 - r_{k-1})/2 \) and such that (a) holds for \( r_k \leq |\zeta| \leq 1 \).

By construction we have \( \lim_{k \to \infty} r_k = 1 \), \( \lim_{k \to \infty} M_k = +\infty \), and \( g = \lim_{k \to \infty} g_k \) exists uniformly on compacts in \( U \) by (c). It remains to show that \( \rho(g(\zeta)) \to +\infty \) as \( |\zeta| \to 1 \). Fix \( k \geq 2 \) and consider points in \( A_k = \{ \zeta: r_{k-1} \leq |\zeta| \leq r_k \} \). For \( l \geq k \) we have \( |g_{l+1}(\zeta) - g_l(\zeta)| < \epsilon_l \), so \( |g(\zeta) - g_k(\zeta)| \leq \sum_{i=k}^{l} |g_{i+1}(\zeta) - g_i(\zeta)| < \sum_{i=k}^{\infty} \epsilon_i < 1 \). From this and (b) we get \( \rho(g(\zeta)) > \rho(g_k(\zeta)) - 1 > M_{k-2} - 1 \) for \( \zeta \in A_k \). Since \( M_{k-2} \to +\infty \) as \( k \to \infty \), the result follows.

Remark. One can give an alternative proof of theorem 1.4 as follows. One can construct a family of holomorphic maps \( F_p: \mathbb{C} \to \mathbb{C}^2 \), depending continuously on \( p \in (\mathbb{C}^*)^2 \), such that (i) \( F_p(0) = p \), (ii) \( |F_p'(\zeta)| \geq |p| - \epsilon_p \) for all \( \zeta \in \mathbb{C} \) (where \( \epsilon_p > 0 \) can be made independent of \( p \) in any compact set \( K \subset \subset (\mathbb{C}^*)^2 \)), (iii) \( F_p(\mathbb{C}) \) misses \( zw = 0 \), and (iv) \( \lim_{|\zeta| \to \infty} |F_p'(\zeta)| = +\infty \). The discs \( \zeta \to F(\zeta) \), \( |\zeta| \leq R \) with \( R \) large enough, can be taken as building blocks to construct proper holomorphic discs \( U \to \mathbb{C}^2 \) whose image avoids both coordinate axes (compare with proposition 4.2). Similar method can used to construct proper holomorphic discs in \( \mathbb{C}^2 \) avoiding the curve \( zw = 1 \).

5. Boundary behavior of proper holomorphic discs

In this section we prove theorem 1.5. We begin by recalling some classical results on boundary behavior of meromorphic functions on \( U = \{ |\zeta| < 1 \} \) (see e.g. [CL] and [Pri]). Let \( \overline{\mathbb{C}} = \mathbb{C} \cup \{ 0 \} \) denote the Riemann sphere. For \( a \in \mathbb{C} \) and
$r > 0$ set $D(a; r) = \{ \zeta \in \mathbb{C} : |\zeta - a| < r \}$. In what follows let $f$ be a meromorphic function on $U$. We denote by $C(f, e^{i\theta})$ its unrestricted cluster set at $e^{i\theta} \in T$:

$$C(f, e^{i\theta}) = \bigcap_{r > 0} f(U \cap D(e^{i\theta}; r)).$$

Equivalently, $a \in \mathbb{C}$ belongs to $C(f, e^{i\theta})$ if and only if there exists a sequence $\zeta_j \in U$ such that $\lim_{j \to \infty} \zeta_j = e^{i\theta}$ and $\lim_{j \to \infty} f(\zeta_j) = a$. If $D \subset U$ is a subset with $e^{i\theta} \in \overline{D}$, we denote by $C_D(f, e^{i\theta})$ the restricted cluster set of $f$ at $e^{i\theta}$, defined as the set of limits of $f$ along sequences $\zeta_j \in D$ with $\lim_{j \to \infty} \zeta_j = e^{i\theta}$.

A point $e^{i\theta}$ for which $C(f, e^{i\theta}) = \mathbb{C}$ is called a Weierstrass point of $f$, and the set of all such points is the Weierstrass set $W(f)$ [CL, p. 149].

For each $e^{i\theta} \in T$ and $0 < \alpha < 1$ we set

$$\Gamma_{\alpha}(e^{i\theta}) = \{ \zeta \in U : |\Im(1 - \zeta e^{-i\theta})| < \alpha |\zeta - e^{i\theta}| \}.$$

This is an angle in $U$ with vertex at $e^{i\theta}$ and opening $2 \arcsin \alpha$, bisected by the radius that terminates at $e^{i\theta}$. If the limit

$$(5.1) \quad f^*(e^{i\theta}) = \lim_{\Gamma_{\alpha}(e^{i\theta}) \ni \zeta \to e^{i\theta}} f(\zeta) \in \overline{\mathbb{C}}$$

exists and is independent of $\alpha$, it is called the nontangential limit of $f$ at $e^{i\theta}$ and $e^{i\theta}$ is called a Fatou point of $f$. The set of all Fatou points the Fatou set $F(f)$ [CL, p. 21].

A point $e^{i\theta} \in T$ is called a Plessner point of $f$ if for every angle $\Gamma$ with vertex $e^{i\theta}$ the partial cluster set $C_{\Gamma}(f, e^{i\theta})$ equals $\overline{\mathbb{C}}$ (i.e., it is total). The set of all Plessner points is the Plessner set $I(f)$ [CL, p. 147]. Clearly $I(f) \subset W(f)$.

The Nevanlinna characteristic of a holomorphic function $f$ on $U$ is defined by

$$T(r, f) = \int_0^{2\pi} \log^+ |f(re^{i\theta})| \frac{dt}{2\pi} \quad (0 \leq r < 1);$$

for meromorphic functions see [CL, p. 39] or [Nev].

The range of $f$, denoted $R(f)$, is the set of all $\alpha \in \mathbb{C}$ for which there exists a sequence $\zeta_j \in U$ with $\lim_{j \to \infty} \zeta_j = 1$ such that $f(\zeta_j) = \alpha$ for all $j \in \mathbb{N}$. By restricting the attention only to sequences $z_j \in U$ with $\lim_{j \to \infty} \zeta_j = e^{i\theta}$ we get the range of $f$ at $e^{i\theta}$, denoted $R(f, e^{i\theta})$.

The notion of logarithmic capacity of a Borel set $E \subset \mathbb{C}$ can be found in [CL, p. 9]. Such a set is of capacity zero if and only if it is polar, i.e., it is contained in the $-\infty$ level set of a non-constant subharmonic function on $\mathbb{C}$ [Lan, Tsu]. The following summarizes some of the known results which we shall need in the proof of theorem 1.5.
Theorem 5.1. Let $f$ be a meromorphic function on the disc $U$.

(a) If $e^{i\theta} \in T$ is not a Weierstrass point of $f$ then there is an open arc $\gamma \subset T$ containing $e^{i\theta}$ such that almost every point in $\gamma$ is a Fatou point of $f$.

(b) If $f$ has bounded Nevanlinna characteristic on $U$ then almost every point of $T$ is a Fatou point of $f$.

(c) Almost every point in $T$ belongs to $F(f) \cup I(f)$.

(d) If $f(U) \subset \overline{C} \setminus E$ for some set $E$ of positive capacity then $f$ has bounded Nevanlinna characteristic.

(e) If $R(f, e^{i\theta})$ omits a set $E \subset \overline{C}$ of positive capacity then there is an open arc $\gamma \subset T$ containing $e^{i\theta}$ such that almost every point of $\gamma$ is a Fatou point of $f$.

Proof. (a) If $e^{i\theta}$ is not a Weierstrass point of $f$, there is a disc $D(e^{i\theta}; r)$ such that $f(D(e^{i\theta}; r) \cap U)$ omits a disc $D(a; \delta) \subset \mathbb{C}$. The function $g(\zeta) = 1/(f(\zeta) - a)$ is then bounded holomorphic in $D(e^{i\theta}; r) \cap U$ and hence by Fatou’s theorem it has nontangential limit $g^*(e^{it})$ at almost every point $e^{it} \in \gamma = T \cap D(e^{i\theta}; r)$ [CL, p. 21]. The same is then true for $f$ and hence almost every point of $\gamma$ belongs to the Fatou set $F(f)$. (See also [CL, Theorem 8.4].) Part (b) follows by combining Fatou’s theorem with a theorem of R. Nevanlinna to the effect that a meromorphic function with bounded Nevanlinna characteristic on $U$ is the quotient of two bounded holomorphic functions [CL, p. 41]. Part (c) is a classical theorem due to Plessner ([Ple], [Pri, p. 217] or [CL, p. 147]).

Part (d) is due to Frostman [Fro]; the following simple proof was shown to us by D. Marshall. After a fractional linear transformation we may assume that $\infty \in E \subset \{|z| > 1\}$. Let $g_0(z)$ be the Green’s function for $\mathbb{C} \setminus E$ with a logarithmic pole at 0 (so $g_0(z) \to 0$ as $z \to E$). Then $\log^+ \frac{1}{|z|} \leq g_0(z)$. The function $u(z) := g_0(z) + \log |z|$ is harmonic on $\mathbb{C} \setminus E$ since both summands are harmonic on $\mathbb{C} \setminus (E \cup \{0\})$ and the pole at 0 cancels off. If $f: U \rightarrow \mathbb{C} \setminus E$ is a holomorphic function then $u \circ f$ is harmonic on $U$ and we have

$$\int_0^{2\pi} \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi} = \int_0^{2\pi} \left( \log^+ \frac{1}{|f(re^{i\theta})|} + \log |f(re^{i\theta})| \right) \frac{d\theta}{2\pi} \leq \int_0^{2\pi} (g_0(f(re^{i\theta})) + \log |f(re^{i\theta})|) \frac{d\theta}{2\pi} = \int_0^{2\pi} u(f(re^{i\theta})) \frac{d\theta}{2\pi} = u(f(0)).$$

Thus $\int_0^{2\pi} \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi} \leq u(f(0))$ for $r \in (0, 1)$ which proves (d).

For (e) observe that $R(f, e^{i\theta}) = \cap_{n \in \mathbb{N}} f(D_n)$, where $D_n = D(e^{i\theta}; 1/n) \cap U$. The sets $f(D_n)$ are decreasing with $n$. If $R(f, e^{i\theta})$ omits a set $E$ of positive capacity then $f(D_n)$ omits a set $E'$ of positive capacity for some sufficiently large $n \in \mathbb{N}$. We may assume that $\infty \in E'$. Observe that $D_n$ is conformally equivalent to the disc. From (d) and (b) applied to the holomorphic function
$f : D_n \to \mathbb{C} \setminus E'$ it follows that almost every point of the arc $\gamma = D_n \cap T$ is a Fatou point of $f$. \hfill \Box$

We shall frequently use the following uniqueness theorem due to Plessner ([Ple], [CL, p. 146]) and to Lusin and Priwalow [Pri, p. 212].

**Theorem 5.2.** If a meromorphic function $f$ on $U$ has an angular limit equal to zero at each point in a set $E \subset T$ of positive Lebesgue measure then $f$ is the zero function.

**Remark.** In theorem 5.2 we cannot replace angular limits with radial limits, see examples due to Lusin and Priwalow in [Pri], sec. IV.5. Here we use the term ‘angular limit’ rather than ‘nontangential limit’ since the latter usually means the existence of the limit within every angle with the given vertex.

**Proof of theorem 1.5.** Let $(f_1, f_2) : U \to \mathbb{C}^2$ be a proper holomorphic map and let $g$ be any of the functions as in theorem 1.5. It suffices to show that the Fatou set $F(g)$ has measure zero. From theorem 5.1 (a) it will then follow that $W(g) = T$, theorem 5.1 (c) will imply that the Plessner set $I(g)$ has full measure in $T$, and theorem 5.1 (e) will imply that the complement of the range $R(g, e^{i\theta})$ in $\mathbb{C}$ has capacity zero for each $e^{i\theta} \in T$. Since sets of capacity zero in $\mathbb{C}$ coincide with polar sets ([Tsu], [Lan]), theorem 1.5 (d) follows.

To prove that $F(g)$ has measure zero we consider separately each case.

**Case (i).** Suppose that $f_1$ has an angular limit $f_1^*(e^{i\theta}) \in \mathbb{C}$ (5.1) at all points $e^{i\theta}$ in a set $A \subset T$. Then $A$ is Lebesgue measurable and can be written as $A = A_1 \cup A_2$, where $A_1$ is the set of all $e^{i\theta} \in A$ such that $f_1^*(e^{i\theta}) \in \mathbb{C}$ and $A_2$ is the set of all $e^{i\theta} \in A$ with $f_1^*(e^{i\theta}) = \infty$. Then $1/f_1$ has angular limits zero at each point of $A_2$. If $A_2$ is of positive measure, theorem 5.2 implies that $1/f_1$ is identically zero in $U$, a contradiction. Thus $A_2$ has measure zero. Consider now $A_1$. Since $(f_1, f_2) : U \to \mathbb{C}^2$ is proper, max{$|f_1(\zeta)|, |f_2(\zeta)|$} tends to $+\infty$ as $|\zeta| \to 1$. Since $f_1$ has a finite angular limit at each $e^{i\theta} \in A_1$, $|f_2|$ has an angular limit $\infty$ at each point of $A_1$. If $A_1$ is of positive measure, Plessner’s theorem, applied to $1/f_2$, gives a contradiction as before. This shows that $A_1$ is of measure zero as well, and therefore the Fatou set of $f_1$ is of measure zero. The same applies to $f_2$.

**Case (ii).** Suppose that $g = f_1/f_2$ has an angular limit $g^*(e^{i\theta}) \in \overline{\mathbb{C}}$ (5.1) within an angle $\Gamma_\theta$ at each point $e^{i\theta}$ in a set $A \subset T$. As in part (i) we write $A = A_1 \cup A_2$, where $g^*$ is finite on $A_1$ and equals $\infty$ on $A_2$. Theorem 5.2 shows as above that $A_2$ must be of measure zero for otherwise $g$ would be constant. If $A_1$ is of positive measure, there is a set $A_0 \subset A_1$ of positive measure and a number $0 < M < \infty$ such that $|g^*(e^{i\theta})| < M$ for each $e^{i\theta} \in A_0$. Hence there is a disc $U_\theta$ centered at $e^{i\theta}$ such that $|f_1(\zeta)/f_2(\zeta)| \leq M$ for $\zeta \in \Gamma_\theta \cap U_\theta$. Hence $|f_1(\zeta)| \leq M|f_2(\zeta)|$ and therefore

$$\max\{|f_1(\zeta)|, |f_2(\zeta)|\} \leq \max\{M|f_2(\zeta)|, |f_2(\zeta)|\} \quad (\zeta \in \Gamma_\theta \cap U_\theta).$$
Since this maximum tends to $+\infty$ as $\zeta \to e^{i\theta}$, it follows that $|f_2(\zeta)| \to \infty$ as $\zeta \to e^{i\theta}$ within $\Gamma_\vartheta$. Thus $1/f_2$ has angular limits zero at each point of $A_0$, a contradiction to theorem 5.2. This proves that $A_1$ must be of measure zero as well.

**Case (iii).** This follows from case (i) by observing that for each nonconstant holomorphic polynomial $P$ on $C^2$ there exists another holomorphic polynomial $Q$ such that $(P,Q):C^2 \to C^2$ is a proper map, and hence $P(f_1,f_2):U \to C^2$ is the first component of a proper map $U \to C^2$. In fact, we have

**Lemma 5.3.** Let $P$ and $Q$ be nonconstant holomorphic polynomials on $C^2$ whose leading order homogeneous parts $P'$ resp. $Q'$ have no common zero on $C^2\{0\}$. Then $(P,Q):C^2 \to C^2$ is a proper map.

We leave out the simple proof. Observe that the zero set of $P'$ is a finite union of complex lines, so it suffices to choose $Q$ to be a linear function which does not vanish on $P' = 0$ except at the origin; the pair $(P,Q)$ then provides a proper self-map of $C^2$.

**Case (iv).** Apply (i) and lemma 5.3 to the map $(P(f_1,f_2),Q(f_1,f_2)):U \to C^2$.

**Appendix:** Crossing a critical level by analytic discs

Let $X$ be a Stein manifold of dimension at least two and $\rho:X \to \mathbb{R}$ a strongly plurisubharmonic Morse exhaustion function. Let $p \in X$ be a critical point of $\rho$. Choose constants $c_0,c_1$ such that $c_0 < \rho(p) < c_1$ and $p$ is the only critical point of $\rho$ in $A(c_0,c_1) = \{x \in X : c_0 \leq \rho(x) \leq c_1\}$. Suppose that $f_0:U \to X$ is a holomorphic map such that $c_0 < \rho(f_0(e^{i\theta})) < \rho(p)$ for each $e^{i\theta} \in T$. In [Glo] the second author showed how to construct a smooth map $f_1:U \to X$ which is close to being holomorphic on $U$ such that $\rho(p) < \rho(f_1(e^{i\theta})) < c_1$ for $e^{i\theta} \in T$ and such that $f_1$ approximates $f_0$ on a smaller disc $|\zeta| \leq r < 1$. The map $f_1$ is obtained by adding to $f_0$ a small non-holomorphic contribution which can be controlled by the data. Once the boundary curve $f_1(T)$ passes the critical level of $\rho$ at $p$ we can use the procedure described in sect. 2 above (or in [FG]) to continue pushing it higher towards the next critical level of $\rho$. It was shown in [Glo] that the non-holomorphic contribution made at the initial step may be cancelled off during a later stage of the construction, once the boundary of the disc is sufficiently far above the critical level at $p$. The reason is that the modification process is a linear one, and we obtain the final solution as the sum of a convergent series. (Here it is convenient to embed $X$ into a Euclidean space $\mathbb{C}^N$.)

Here we wish to point out that the transition from $f_0$ to $f_1$ as above can also be accomplished by applying to $f_0$ the gradient flow $\theta_t$ of $\rho$ (in the direction of increasing $\rho$). Unless a point $x \in A(c_0,c_1)$ belongs to the stable manifold $W^s(p)$ of $p$ (see e.g. Shub [Sh]), we have $\rho(\theta_t(x)) > \rho(p)$ for sufficiently large $t > 0$. Thus, if $f_0(T) \cap W^s(p) = \emptyset$, we can choose a smooth positive function $a$ on $U$ such that the map $f_1(\zeta) = \theta_{a(\zeta)}(f_0(\zeta))$ ($|\zeta| \leq 1$) satisfies $\rho(f_1(e^{i\theta})) > \rho(p)$ for $e^{i\theta} \in T$. 
If the number $c_0$ is sufficiently close to $\rho(p)$ as we may assume to be the case, the set $W^*(p) \cap A(c_0, c_1)$ is a closed real submanifold of $A(c_0, c_1)$ whose dimension equals the index $i(p)$ (the number of negative eigenvalues of the Hessian) of $\rho$ at $p$. Since $\rho$ is strongly plurisubharmonic, we have $i(p) \leq \dim \mathbb{C} X$ (see [AF]) and therefore $\dim f_0(T) + \dim W^*(p) \leq 1 + \dim \mathbb{C} X < \dim \mathbb{R} X$. By transversality a generic small holomorphic perturbation of $f_0$ satisfy the required condition $f_0(T) \cap W^*(p) = \emptyset$ which makes it possible to obtain $f_1$ as above. The rest of the procedure remains as in [Glo].

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