MONOPOLE EQUATION AND THE $\frac{11}{8}$-CONJECTURE

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1. Introduction

Let $M$ be a 4-dimensional oriented closed spin manifold. We write $b_l(M)$ for the $l$-th Betti number of $M$ and $\text{sign}(M)$ for the signature of $M$. In [13] Y. Matsumoto conjectured the following inequality.

**Conjecture.**

$$b_2(M) \geq \frac{11}{8} |\text{sign}(M)|.$$

This conjecture is now known as *the $11/8$ conjecture*. Since the K3 surface satisfies the equality, the coefficient $11/8$ cannot be replaced by a larger number. In this paper we show a weaker inequality.

**Theorem 1.** If the intersection form of $M$ is not definite, the following inequality is satisfied.

$$b_2(M) \geq \frac{5}{4} |\text{sign}(M)| + 2.$$

Note that if the intersection form of $M$ is definite, a theorem of S.K. Donaldson implies $b_2(M) = \text{sign}(M) = 0$ ([6,7]).

V. A. Rohlin’s theorem implies that $k = -\text{sign}(M)/16$ is an integer ([15]). Let $b_+$ be the dimension of a maximal positive definite subspace of $H_2(M, \mathbb{Q})$. The $11/8$-conjecture is equivalent to the inequality $3k \leq b_+$. The inequality in Theorem 1 is equivalent to $2k + 1 \leq b_+$.

Donaldson’s theorem mentioned above says that $k \geq 1$ implies $b_+ \geq 1$. Donaldson also proved that $k \geq 1$ implies $b_+ \geq 3$ when $H_1(M, \mathbb{Z})$ is 2-torsion free ([6,7]). To obtain these estimates he used moduli spaces of instantons on $M$ with small instanton numbers. In 1994 P. B. Kronheimer explained, in his lecture in Cambridge, how to use N. Seiberg and E. Witten’s monopole equation to get similar results ([10]). In particular he showed that $k \geq 1$ always implies $b_+ \geq 3$ without any condition on $H_1(M, \mathbb{Z})$.

In this paper we consider moduli spaces of monopoles following Kronheimer’s lecture. Our key idea is to use a finite dimensional approximation of the monopole equation.

We formulate the monopole equation in Section 2 so that we can see the $\text{Pin}_2$-symmetry of the equation explicitly. The finite dimensional approximation of
the equation is constructed in Section 3 and its Pin₂-symmetry is described in Section 4. We prove Theorem 1 in Section 5 by using equivariant K-theory. In Appendix we summarize some elementary properties of the Adams operations used in Section 5.

Since the preprint version of this article appeared in 1995, there have been some related works. See [13], [4] and [8].

2. Monopole equation

To show Theorem 1, we can assume (1) \( \text{sign}(M) \) is non-positive and (2) \( b_1(M) = 0 \) without loss of generality, since (1) if we reverse the orientation of \( M \), then the sign of \( \text{sign}(M) \) changes and (2) if \( b_1(M) \) is not zero, then we can construct, by using surgery along non-trivial loops, another spin manifold with its first Betti number zero and with the same second Betti number and signature. We assume these two conditions in the rest of this paper.

In this section we formulate the monopole equation for the spin structure of \( M \). Let \( \mathbb{H} \) be the quaternion numbers and \( Sp_1 \) be the group of the quaternions with norm 1. The monopole equation has an \( S^1 \)-symmetry from definition, where \( S^1 \) is the intersection of \( Sp_1 \) with \( \mathbb{C} \) in \( \mathbb{H} \). Let \( Pin_2 \) be the normalizer of \( S^1 \) in \( Sp_1 \). An important aspect of the equation is the equation has actually a \( Pin_2 \)-symmetry ([11,16]). This extra symmetry will play a crucial role in the proof of Theorem 1.

Recall that \( Spin_4 \) is isomorphic to the product of two copies of \( Sp_1 \). We define four \( Spin_4 \times Pin_2 \)-modules \( _{-}\mathbb{H}_+ \), \( +\mathbb{H}_+ \), \( _{-}\mathbb{H} \) and \( +\mathbb{H} \) as follows. As vector spaces, they are just four copies of \( \mathbb{H} \). The actions of \( (q_-, q_+, q_0) \in Spin_4 \times Pin_2 = Sp_1 \times Sp_1 \times Pin_2 \) on \( a \in _{-}\mathbb{H}_+ \), \( \phi \in +\mathbb{H}_+ \), \( \psi \in _{-}\mathbb{H} \) and \( \omega \in +\mathbb{H} \) are defined by \( q_-aq_+^{-1} \), \( q_+\phi q_0^{-1} \), \( q_-\psi q_0^{-1} \) and \( q_+\omega q_+^{-1} \) respectively.

For a principal \( Spin_4 \)-bundle \( P \) on the 4-manifold \( M \), we have four associated vector bundles \( T \), \( S^+ \), \( S^- \) and \( \Lambda \) from the \( Spin_4 \times Pin_2 \)-modules \( _{-}\mathbb{H}_+ \), \( +\mathbb{H}_+ \), \( _{-}\mathbb{H} \) and \( +\mathbb{H} \). Then they are \( Pin_2 \)-equivariant vector bundles. Let \( \bar{R} \) be the nontrivial 1-dimensional \( Pin_2 \)-module defined by the multiplication of \( Pin_2 / S^1 = \{ \pm 1 \} \) and we write \( \bar{E} = E \otimes \bar{R} \) for \( Pin_2 \)-modules or \( Pin_2 \)-equivariant vector bundles \( E \). We shall consider \( \bar{T} \) and \( \bar{\Lambda} \) associated to \( _{-}\mathbb{H}_+ \) and \( +\mathbb{H}_+ \).

Recall that a spin structure of \( M \) is a pair of a principal \( Spin_4 \)-bundle \( P \) and an isomorphism \( TM \cong T \). When we fix this isomorphism, \( M \) has a canonical orientation and a canonical Riemannian metric.

The \( Spin_4 \times Pin_2 \)-equivariant map \( _{-}\mathbb{H}_+ \times +\mathbb{H} \to _{-}\mathbb{H} \) defined by \( (a, \phi) \mapsto a\phi \) induces the Clifford multiplication \( C: T \otimes S^+ \to S^- \). Similarly the \( Spin_4 \times Pin_2 \)-equivariant map \( _{-}\mathbb{H}_+ \times _{-}\mathbb{H}_+ \to +\mathbb{H}_+ \) defined by \( (a, b) \mapsto ab \) induces the twisted Clifford multiplication \( \bar{C}: T \otimes \bar{T} \to \bar{\Lambda} \). From the construction, \( C \) and \( \bar{C} \) are \( Pin_2 \)-equivariant.

The Riemannian connection induces the covariant derivatives \( \nabla_1 \) on \( \Gamma(S^+) \) and \( \nabla_2 \) on \( \Gamma(T) \). Let \( D_1 \) and \( D_2 \) be the (twisted) Dirac operators

\[
D_1 = C \nabla_1 : \Gamma(S^+) \to \Gamma(S^-) \quad \text{and} \quad D_2 = \bar{C} \nabla_2 : \Gamma(T) \to \Gamma(\bar{\Lambda}).
\]
We write $D$ for the direct sum of $D_1$ and $D_2$:

$$D = D_1 \oplus D_2 : \Gamma(S^+ \oplus \tilde{T}) \to \Gamma(S^- \oplus \tilde{\Lambda}).$$

The operators $D_1$, $D_2$ and $D$ are Pin$_2$-equivariant.

We also need a Pin$_2$-equivariant quadratic map

$$Q : S^+ \oplus \tilde{T} \to S^- \oplus \tilde{\Lambda}$$

induced from the Spin$_4 \times$Pin$_2$-equivariant map

$$\mathbb{H} \times \mathbb{H}_+ \to \mathbb{H} \times \mathbb{H}_+, \quad (\phi, a) \mapsto (a\phi, a\phi).$$

Then the monopole equation we shall consider is the nonlinear Pin$_2$-equivariant map

$$D + Q : V \to W,$$

where $V$ is the $L^2_3$-completion of $\Gamma(S^+ \oplus \tilde{T})$ and $W$ is the $L^2_3$-completion of $\Gamma(S^- \oplus \tilde{\Lambda})$. Note that if $u$ is in $L^2_3$, then $Du$ is in $L^2_3$ and $Qu$ is in $L^2_3$.

Remark. (1) The twisted Clifford multiplication $\tilde{C} : T \otimes \tilde{T} \to \tilde{\Lambda}$ is identified with $T^*M \otimes T^*M \to \mathbb{R} \oplus \Lambda^+$ defined by $(a, b) \mapsto \langle a, b \rangle \oplus p^+(a \wedge b)$. Here $\langle \cdot, \cdot \rangle$ is the inner product, $\Lambda^+$ is the self-dual part of $\Lambda^2 T^*M$ and $p^+ : \Lambda^2 T^*M \to \Lambda^+$ is the orthogonal projection. The real part and the imaginary part of $\tilde{\Lambda}$ are identified with $\mathbb{R}$ and $\Lambda^+$ respectively. Since $\phi i \tilde{\phi}$ is purely imaginary, the image of $Q$ is contained in $S^- \oplus \Lambda^+$.

(2) The twisted Dirac operator $D_2 : \Gamma(T) \to \Gamma(\tilde{\Lambda})$ is identified with $d^* + d^+ : \Omega^+ \to \Omega^0 \oplus \Omega^+$. Here $\Omega^k$ is the $k$-th forms on $M$, $\Omega^+$ is the self-dual 2-forms, $d^*$ is the formal adjoint of the exterior derivative $d : \Omega^0 \to \Omega^1$, and $d^+$ is the composition of $d : \Omega^1 \to \Omega^2$ with $p^+ : \Omega^2 \to \Omega^+$. In particular we have $\ker D_2 = H^1(M, \mathbb{R}) = 0$ if $b_1(M) = 0$, and also $\operatorname{coker} D_2 = H^0(M, \mathbb{R}) \oplus H^+(M, \mathbb{R})$, where $H^+(M, \mathbb{R})$ is the space of the self-dual harmonic 2-forms. Hence the index of $D_2$ is equal to $-(1 + b^+)$.

(3) Let $\mathcal{A}$ be the set of smooth connections on a complex line bundle $L$ on $M$. The monopole equation (for a spin manifold $M$) is usually defined as a map

$$\mathcal{F} : \Gamma(S^+ \otimes L) \times \mathcal{A} \to \Gamma((S^- \otimes L) \oplus \Lambda^+), \quad \mathcal{F}(\phi, A) = (D_A \phi, F^+_{\Lambda^+} + \phi i \tilde{\phi}).$$

Here $D_A$ is the Dirac operator twisted by $A$ and $F^+_{\Lambda^+}$ is the self-dual part of the curvature $F_A$ of $A$. The gauge group $\mathcal{G} = \Gamma(M, S^1)$ of $L$ naturally acts on $\Gamma(S^+ \otimes L) \times \mathcal{A}$ and $\Gamma((S^- \otimes L) \oplus \Lambda^+)$. The map $\mathcal{F}$ is $\mathcal{G}$-equivariant. Instead of dividing out by $\mathcal{G}$, we can take the slice at a base point $A_0$ in $\mathcal{A}$. Note that $\mathcal{A} = \mathcal{A}_0 + \Omega^1 i$. The slice is given by $A_0 + \ker(d^+ : \Omega^1 \to \Omega^0) i$. In particular when $L$ is the trivial bundle $M \times \mathbb{C}$ and $A_0$ is the trivial flat connection on it, $\mathcal{F}$ together with the equation of cutting slice is identified with $D + Q$. (4) When we restrict $\mathcal{F}$ on the slice, the $\mathcal{G}$-symmetry is reduced to the symmetry of...
the stabilizer of $A_0$. The stabilizer is equal to the group of the harmonic maps $\text{Harm}(M, S^1)$ from $M$ to $S^1$ whose group structure is induced from that of $S^1$. (Here we do not have to take $L$ to be trivial.) If we assume $b_1(M) = 0$, then the harmonic maps are constant, hence we have just an $S^1$-symmetry.

(5) When $L$ is trivial and $A_0$ is the trivial flat connection, we have an extra symmetry explained earlier. Actually before taking the slice, the map has the symmetry of $\langle \text{Map}(M, S^1), j \rangle$, which is a subgroup of $\text{Map}(M, \text{Sp}_1)$. Here $j$ is the constant map with value $j$. Restricted on the slice, the symmetry becomes $\langle \text{Harm}(M, S^1), j \rangle$.

3. Finite dimensional approximation

An important property of the monopole equation is compactness of the moduli space of solutions.

Lemma 3.1([11, 16]). The zero set of $D + Q$ is compact.

We introduce the norms of $V$ and $W$ explicitly. Let $\| \cdot \|_V$ and $\| \cdot \|_W$ be the $L^2$-norm on $V$ and the $L^3$-norm on $W$ defined by

$$
\|v\|_V^2 = \int_M (|D^*Dv|^2 + |v|^2), \quad \text{and} \quad \|w\|_W^2 = \int_M (|DD^*w|^2 + |w|^2)
$$

respectively. Here $D^*$ is the formal adjoint of $D$. More explicitly, $D^*$ is the sum of $D_1^*: \Gamma(S^-) \to \Gamma(S^+)$ and $D_2^*: \Gamma(\hat{\Lambda}) \to \Gamma(\hat{T})$, where $D_1^*$ and $D_2^*$ are the formal adjoint of $D_1$ and $D_2$, and identified with the (twisted) Dirac operators associated to the (twisted) Clifford multiplications $T \otimes S^- \to S^+$ and $T \otimes \hat{\Lambda} \to \hat{T}$ which are induced from the Spin$_4 \times \text{Pin}_2$-equivariant maps $\mathbb{H}_+ \times \mathbb{H} \to \mathbb{H}$, $(a, \psi) \mapsto a \psi$ and $\mathbb{H}_+ \times \mathbb{H}_+ \to \mathbb{H}_+$, $(a, \omega) \mapsto a \omega$ respectively.

These norms are preserved by the Pin$_2$-action. From Lemma 3.1, if $R$ is sufficiently large, then $(D + Q)v \neq 0$ for any $v \in V$ satisfying $\|v\|_V \geq R$. For this $R$, we have the following estimate.

Lemma 3.2. There is a positive real number $\epsilon$ such that $\|(D + Q)v\|_W \geq \epsilon$ for any $v \in V$ with $\|v\|_V = R$.

The proof of the lemma will be given later.

For a nonnegative real number $\lambda$, let $V_{\lambda}$ be the subspace of $V$ spanned by the eigenspaces of $D^*D$ with eigenvalues less than or equal to $\lambda$. Similarly we define $W_{\lambda}$ by using $DD^*$. We write $p_{\lambda}$ for the $L^2$-orthogonal projection from $W$ to $W_{\lambda}$, and $p_{\lambda}$ for $1 - p_{\lambda}$.

Lemma 3.3. For sufficiently large $\lambda$, we have the estimate $\|p_{\lambda}Qv\|_W < \epsilon$ for any $v \in V$ satisfying $\|v\|_V = R$.

The proof of the lemma will be given later.
From the lemmas 3.2 and 3.3, the map \( D + p_\lambda Q : V \to W \) has no zeros on the sphere with radius \( R \) centered in 0. The image of \( V_\lambda \) by the map \( D + p_\lambda Q \) is contained in \( W_\lambda \). We write

\[
D_\lambda + Q_\lambda : V_\lambda \to W_\lambda
\]

for this restriction, where \( D_\lambda \) is linear and \( Q_\lambda \) is quadratic. We consider this restriction as a finite dimensional approximation of \( D + Q : V \to W \). A direct consequence of this construction is:

**Lemma 3.4.** The map \( D_\lambda + Q_\lambda \) has no zeros on the sphere with radius \( R \) centered in 0.

The proofs of the lemmas 3.2 and 3.3 are immediate if we use the following facts which are easily shown from the Sobolev embedding theorem, Hölder’s inequality, the elliptic estimate for \( D \) and the spectral decomposition of \( DD^* \).

1. For any bounded sequence \( v_1, v_2, \ldots \) in \( V \), there is a subsequence \( v_1', v_2', \ldots \) weakly convergent to some \( v_\infty \) such that the sequence \( Qv_1', Qv_2', \ldots \) is strongly convergent to \( Qv_\infty \) in \( W \).
2. If the sequence \( v_1, v_2, \ldots \) is weakly convergent to \( v_\infty \) in \( V \) and the sequence \( Dv_1, Dv_2, \ldots \) is strongly convergent to \( w_\infty \) in \( W \), then \( v_1, v_2, \ldots \) is strongly convergent to \( v_\infty \) and we also have \( Dv_\infty = w_\infty \).
3. For each \( w \) in \( W \), \( \|p^\lambda w\|_W \) is decreasing and convergent to 0 as \( \lambda \to \infty \).

**Proof of Lemma 3.2.** Suppose there is a sequence \( v_1, v_2, \ldots \) in \( V \) satisfying \( \|v_d\|_V = R \) for every \( d \) and \( \|(D + Q)v_d\|_W \to 0 \) as \( d \to \infty \). From (1) above, if we replace the sequence by a subsequence, we can assume the sequence is weakly convergent to some \( v_\infty \) and the sequence \( Qv_1, Qv_2, \ldots \) is strongly convergent to \( Qv_\infty \) in \( W \). Then \( Dv_1, Dv_2, \ldots \) is strongly convergent to \( -Qv_\infty \). Hence from (2) above, the sequence \( v_1, v_2, \ldots \) is strongly convergent to \( v_\infty \). This strong convergence implies \( \|v_\infty\|_V = R \) and \( (D + Q)v_\infty = 0 \), which is a contradiction. \( \square \)

**Proof of Lemma 3.3.** Suppose there are sequences \( v_1, v_2, \ldots \) in \( V \) and \( \lambda_1, \lambda_2, \ldots \) satisfying \( \|v_d\|_V = R \) and \( \|p^{\lambda_d}Qv_d\|_W \geq \epsilon \) for every \( d \) and \( \lambda_d \to \infty \) as \( d \to \infty \). From (1) above, if we replace the sequence by a subsequence, we can assume the sequence is weakly convergent to some \( v_\infty \) and the sequence \( Qv_1, Qv_2, \ldots \) is strongly convergent to \( Qv_\infty \) in \( W \). From (3) above, we have \( \|p^{\lambda_{d_0}}Qv_\infty\|_W < \epsilon/2 \) for some large \( d_0 \). Since \( Qv_d \) is strongly convergent to \( Qv_\infty \), we have \( \|p^{\lambda_{d_0}}(Qv_{d_1} - Qv_\infty)\|_W < \epsilon/2 \) for some \( d_1 \geq d_0 \). Then, by using (3) again, we obtain

\[
\|p^{\lambda_{d_1}}Qv_{d_1}\|_W \leq \|p^{\lambda_{d_0}}Qv_{d_1}\|_W \leq \|p^{\lambda_{d_0}}(Qv_{d_1} - Qv_\infty)\|_W + \|p^{\lambda_{d_0}}Qv_\infty\|_W < \epsilon/2 + \epsilon/2 = \epsilon,
\]

which is a contradiction. \( \square \)
Remark. An alternative way to construct a finite dimensional approximation is to use an extended version of the Kuranishi map. Let $V^\lambda$ and $W^\lambda$ be the completions of the spaces spanned by the eigenspaces of $D^*D$ and $DD^*$ with eigenvalues larger than $\lambda$ respectively. Then the restriction of $D$ on $V^\lambda$ has the inverse $D^{-1}: V^\lambda \to V^\lambda$. Let $\Phi: V \to V$ be the map defined by $\Phi = Id_V + D^{-1}p^\lambda Q$. Suppose $u = \Phi v$. Then $v$ is a zero of $D + Q$ if and only if $u$ is in $V_\lambda$ and $p_\lambda(D + Q)v = 0$. It is not hard to show that $\Phi$ is close to the identity on the disk centered in 0 with a fixed radius $R$, if $\lambda$ is sufficiently large. So the map $p_\lambda(D + Q)\Phi^{-1}: V_\lambda \to W_\lambda$, which is defined only on a disk centered in 0, is a finite dimensional approximation in the sense that $\Phi$ gives a bijection between its zeros and the zeros of $D + Q: V \to W$, if we restrict these maps on the disks. When $\lambda = 0$, this construction is called the Kuranishi construction, and the approximation describes a neighborhood of 0 of the zero set of $D + Q$. If we take a larger $\lambda$, then the radius $R$ for which the approximation is valid becomes larger. Since the zero set is bounded, we eventually have an approximation to describe the whole zero set as $\lambda$ becomes large enough.

4. Pin$_2$-module structures

Since $D$ is Pin$_2$-equivariant, $V_\lambda$ and $W_\lambda$ are Pin$_2$-modules and the approximations $D_\lambda$ and $Q_\lambda$ are still Pin$_2$-equivariant. We think of $H$ as a Pin$_2$-module by using the right Pin$_2$-multiplication. Recall that we assumed $k = -\text{sign}(M)/16 \geq 0$.

Lemma 4.1. There are nonnegative integers $m$ and $n$ such that as Pin$_2$-modules

$$V_\lambda = H^{k+m} \oplus \mathbb{R}^n \quad \text{and} \quad W_\lambda = H^m \oplus \mathbb{R}^{b_*+n+1}.$$ 

Proof. Recall $D: V \to W$ is the direct sum of $D_1: L^\lambda_2(S^+) \to L^\lambda_2(S^-)$ and $D_2: L^\lambda_2(T) \to L^\lambda_2(T)$. Let $V^1_\lambda$ be the subspace of $L^\lambda_2(S^+)$ spanned by the eigenspaces of $D_1^*D_1$ with eigenvalues less than or equal to $\lambda$. We define $W^1_\lambda$, $V^2_\lambda$, and $W^2_\lambda$ similarly, then we have the decompositions $V_\lambda = V^1_\lambda \oplus V^2_\lambda$ and $W_\lambda = W^1_\lambda \oplus W^2_\lambda$ as Pin$_2$-modules. There are finitely many points $p_1, p_2, \cdots, p_l$ on $M$ such that the restriction on the fibers over these points is an injection from $V^1_\lambda$ to $\bigoplus_{i=1}^l (S^+)^i \otimes_{P_d} H^i$, which is isomorphic to $H^i$ as a Pin$_2$-module. Since the Pin$_2$-module $H$ is irreducible, $V^1_\lambda$ itself is isomorphic to a Pin$_2$-module of the form $H^{m'}$ for some $m'$. Similarly $W^1_\lambda$ is of the form $H^m$ for some $m$. The index of $D_1$ is equal to $\dim V^1_\lambda - \dim W^1_\lambda = 4m' - 4m$, which is on the other hand calculated from the Atiyah-Singer index theorem as follows.

$$\text{index } D_1 = 2\langle \hat{A}(M), [M] \rangle = -\frac{p_1(M)}{12} = -\frac{\langle L(M), [M] \rangle}{4} = -\frac{\text{sign}(M)}{4} = 4k.$$ 

Here $\hat{A}(M)$ and $L(M)$ are the $\hat{A}$-genus and the $L$-genus of $M$. Hence we have $m' = k + m$ and $V^1_\lambda = H^{k+m}$. Similarly $V^2_\lambda$ and $W^2_\lambda$ are of the form $\mathbb{R}^n$ and $\mathbb{R}^{n'}$.
for some \( n \) and \( n' \) as \( \text{Pin}_2 \)-modules. From Remark (2) in Section 2, the index of \( D_2 \) is \(-1 - b_+\), which is on the other hand equal to \( \dim V^2_\lambda - \dim W^2_\lambda = n - n' \). Hence we have \( n' = b_+ + n + 1 \) and \( W^2_\lambda = \mathbb{R}^{b_+ + n + 1} \).

**Remark.** The above argument also gives a proof of Rohlin’s theorem: the kernel and the cokernel of the Dirac operator \( D_1 \) are the sums of copies of \( \mathbb{H} \), so the index of \( D_1 \) is divisible by 4, while the index is equal to \(-\text{sign}(M)/4\) from the index theorem. Our argument to use the monopole equation would be regarded as a nonlinear version of this proof of Rohlin’s theorem.

We next show that the image of \( D + Q \) is actually contained in a subspace of \( W \) of codimension 1. Let \( s_0 \) be the parallel section of \( \hat{\Lambda} \subset \mathbb{A}^- \oplus \hat{\Lambda} \) corresponding to the \( \text{Spin}_4 \)-invariant element 1 in \( \mathbb{H}_+ \). Parallel sections are contained in the kernel of the twisted Dirac operator \( D^* \). Since the image of \( D \) is \( L^2 \)-orthogonal to the kernel of \( D^* \), the image of \( D \) is contained in the \( L^2 \)-orthogonal complement \( s_0 \) of \( s_0 \) in \( W \). The image of \( Q \) is also contained in \( s_0 \) from Remark (1) in Section 2. From the construction of the finite dimensional approximation, the image of \( D_\lambda + Q_\lambda \) is still contained in the subspace \( W_\lambda = W_\lambda \cap s_0 \) of codimension 1. Note that \( \mathbb{R}s_0 = \mathbb{R} \) as a \( \text{Pin}_2 \)-module and hence \( W_\lambda = \mathbb{H}^m \oplus \mathbb{R}^{b_+ + n} \).

**Remark.** The origin of the codimension 1 of \( W_\lambda \) is the dimension 1 of \( H^0(M, \mathbb{R}) \), which is also identified with the dimension of the symmetry group \( \text{Pin}_2 \).

We summarize the results of Section 3 and Section 4:

**Theorem 4.2.** Let \( M \) be a closed spin 4-manifold with \( b_1(M) = 0 \) and \( \text{sign}(M) \leq 0 \). Then there are finite dimensional real \( \text{Pin}_2 \)-modules \( V_\lambda \) and \( W_\lambda \) and a \( \text{Pin}_2 \)-equivariant linear map \( D_\lambda \) and a \( \text{Pin}_2 \)-equivariant quadratic map \( Q_\lambda \) from \( V_\lambda \) to \( W_\lambda \) which satisfy the following properties.

1. There are \( \text{Pin}_2 \)-module isomorphisms \( V_\lambda = \mathbb{H}^{k + m} \oplus \mathbb{R}^n \) and \( W_\lambda = \mathbb{H}^m \oplus \mathbb{R}^{b_+ + n} \) for some \( m \) and \( n \), where \( k = -\text{sign}(M)/16 \).

2. There are no zeros of \( D_\lambda + Q_\lambda \) on the sphere centered in 0 with some radius \( R \) defined by using some \( \text{Pin}_2 \)-invariant metric on \( V_\lambda \).

**Remark.** For \( \lambda_1 \) larger than \( \lambda \), we have another finite dimensional approximation \( D_{\lambda_1} + Q_{\lambda_1} : V_{\lambda_1} \to W_{\lambda_1} \), where \( V_{\lambda_1} = \mathbb{H}^{k + m_1} \oplus \mathbb{R}^{n_1} \) and \( W_{\lambda_1} = \mathbb{H}^{m_1} \oplus \mathbb{R}^{b_+ + n_1} \) for some \( m_1 \geq m \) and \( n_1 \geq n \). Restricted on the sphere of radius \( R \), \( D_{\lambda_1} + Q_{\lambda_1} \) is homotopic to the join of \( D_\lambda + Q_\lambda \) and the identity on the sphere \( S(\mathbb{H}^{m_1 - m} \oplus \mathbb{R}^{n_1 - n}) \) through \( \text{Pin}_2 \)-equivariant maps. Hence we have an inductive system of \( \text{Pin}_2 \)-equivariant maps between spheres, and can define the stable class

\[
\lim_{\lambda}[D_\lambda + Q_\lambda] \in \lim_{m,n}[S(\mathbb{H}^{k + m} \oplus \mathbb{R}^n), S(\mathbb{H}^{m} \oplus \mathbb{R}^{b_+ + n})],
\]

where \([\cdot, \cdot]\) denotes the set of \( \text{Pin}_2 \)-equivariant homotopy classes. Taking cones, we also have a stable class in the inductive limit of the set of \( \text{Pin}_2 \)-equivariant homotopy classes between certain compact pairs of disks and spheres. We can think of this stable class as a model of the *proper homotopy class* of the map \( D + Q : V \to W \), where \( W = s_0^+ \).
5. Equivariant maps

Let $V_{\lambda, C}$ and $\tilde{W}_{\lambda, C}$ be the complexifications of $V_{\lambda}$ and $\tilde{W}_{\lambda}$, which we think of as complex vector bundles over a point. In general, for a complex vector bundle $E$ over a compact space $X$ we write $BE$ for the disk bundle associated to $E$ and $SE$ for the sphere bundle which is the boundary of $BE$. If $X$ has an action of a compact Lie group $G$ and $E$ is a $G$-equivariant bundle, then we take $BE$ and $SE$ to be $G$-invariant ones.

Theorem 1 follows from:

Proposition 5.1. Suppose there is a continuous Pin$_2$-equivariant map $\tilde{f}: BV_{\lambda, C} \to B\tilde{W}_{\lambda, C}$ preserving boundaries. If $k > 0$, then we have the inequality $2k + 1 \leq b$.

Proof of Theorem 1 assuming Proposition 5.1. Since $M$ is not negative-definite, we have $b_+ \geq 1$ and the required inequality $2k + 1 \leq b_+$ is satisfied obviously when $k = 0$. Suppose $k > 0$. We use the notations in Theorem 4.2. Let $f: V_{\lambda, C} \to \tilde{W}_{\lambda, C}$ be the complexification of $D_{\lambda} + Q_{\lambda}$ defined by

$$f(u \otimes 1 + v \otimes i) = (D_{\lambda} + Q_{\lambda})u \otimes 1 + (D_{\lambda} + Q_{\lambda})v \otimes i.$$ 

Let $BV_{\lambda, C}$ be the Pin$_2$-invariant disk $\{u \otimes 1 + v \otimes i \in V_{\lambda, C} \mid \|u\|_V, \|v\|_V \leq R\}$ and $SV_{\lambda, C}$ be its boundary. The image of $BV_{\lambda, C}$ by $f$ does not contain 0. We write $SW_{\lambda, C}$ for the quotient $\tilde{W}_{\lambda, C}\{0\}/\mathbb{R}_+$ and $p: \tilde{W}_{\lambda, C}\{0\} \to SW_{\lambda, C}$ for the projection. Then the composition $\tilde{f}$ of $f|SV_{\lambda, C}: SV_{\lambda, C} \to \tilde{W}_{\lambda, C}\{0\}$ with $p: \tilde{W}_{\lambda, C}\{0\} \to SW_{\lambda, C}$ is a Pin$_2$-equivariant map. Let $\tilde{f}: BV_{\lambda, C} \to B\tilde{W}_{\lambda, C}$ be the cone of $\tilde{f}$. The restriction of $\tilde{f}$ on the boundary $SW_{\lambda, C}$ is $\tilde{f}: SV_{\lambda, C} \to SW_{\lambda, C}$. Then we can use Proposition 5.1 to obtain the required inequality.

We will use equivariant K-theory to prove Proposition 5.1. Suppose, in general, $G$ is a compact Lie group, $X$ is a compact $G$-space, $E$ and $F$ are $G$-equivariant complex vector bundles over $X$ and $\tilde{f}: BE \to BF$ is a $G$-equivariant bundle map preserving boundaries. The Thom isomorphism theorem for equivariant K-theory implies that $K_G(BE, SE)$ and $K_G(BF, SF)$ are free $K_G(X)$-modules generated by the Thom classes $\tau_E$ and $\tau_F$ respectively. Let $\tilde{f}^*: K_G(BF, SF) \to K_G(BE, SE)$ be the pullback map for $\tilde{f}$.

Definition. The degree of $\tilde{f}$ in K-theory is the unique element $\alpha_0$ of $K_G(X)$ which satisfies the relation:

$$\tilde{f}^*\tau_F = \alpha_0\tau_E.$$ 

The degree $\alpha_0$ satisfies the following equation in $K_G(X)$.

Lemma 5.2. $\Sigma(-1)^d[\Lambda^d F] = \alpha_0[\Sigma(-1)^d[\Lambda^d E]]$.

Proof. The restrictions of $\tau_E$ and $\tau_F$ on the zero sections are the Euler classes of $E$ and $F$, which are equal to $\Sigma(-1)^d[\Lambda^d E]$ and $\Sigma(-1)[\Lambda^d F]$ respectively (see
We also need other equations for $\alpha_0$. Let $l$ be an integer larger than 1 and $\psi^l$ the Adams operation. (See Appendix for the definition of $\psi^l$.) The $K$-theory characteristic class $\rho^l(E)$ is defined to be the unique element of $K_G(X)$ satisfying $\psi^l \tau_E = \rho^l(E) \tau_E$. (See (A.6) in Appendix, or [1].) The other equations for $\alpha_0$ is:

**Lemma 5.3.** $\rho^l(F)\alpha_0 = (\psi^l\alpha_0)\rho^l(E)$.

Proof. Apply $\psi^l$ on the equation $\tilde{f}^* \tau_F = \alpha_0 \tau_E$ to get $\tilde{f}^* (\rho^l(F)\tau_F) = (\psi^l\alpha_0)\rho^l(E)\tau_E$, where we used the multiplicative property of $\psi^l$ (see (A.9) in Appendix). Use the definition of $\alpha_0$ again and compare the coefficients of the generator $\tau_E$ to get the equation. □

In our case we have

- $X = \{\text{a point}\}$, $G = \text{Pin}_2$,
- $E = V_{\lambda,C} = (\mathbb{H}^{k+m} \oplus \mathbb{R}^n) \otimes \mathbb{C}$ and
- $F = \tilde{W}_{\lambda,C} = (\mathbb{H}^m \oplus \mathbb{R}^{b+n}) \otimes \mathbb{C}$.

Recall $K_G(\text{point})$ is the character ring $R(G)$ of $G$.

**Lemma 5.4.** (1) If $k > 0$, then

$$\{\alpha \in R(\text{Pin}_2) \mid \rho^l(F)\alpha = (\psi^l\alpha)\rho^l(E)\} \subset \text{Ker}(R(\text{Pin}_2) \to R(S^1)).$$

(2) There is an element $\alpha$ of $\text{Ker}(R(\text{Pin}_2) \to R(S^1))$ satisfying

$$\Sigma(-1)^d[\Lambda^d F] = \alpha \Sigma(-1)^d[\Lambda^d E]$$

if and only if $2k + 1 \leq b_+$.  

Proof of Proposition 5.1. It is an immediate consequence of the lemmas 5.2, 5.3 and 5.4. □

In the rest of this section we will show Lemma 5.4.

Proof of Lemma 5.4 (1). Let $\mathbb{C}$ be the standard 1-dimensional complex representation of $S^1$ and $t$ the class of $\mathbb{C}$ in $R(S^1)$. Let 1 be the trivial 1-dimensional complex representation of $S^1$ and we use the same notation 1 for its class in $R(S^1)$. Then $R(S^1)$ is the space of Laurent polynomials in $t$ with integer coefficients. When we regard $E$ and $F$ as $S^1$-modules, we write $E'$ and $F'$ for these representation spaces. More explicitly, $E' = 2(k + m)(\mathbb{C} \oplus \mathbb{C}^*) \oplus n$ and $F' = 2m(\mathbb{C} \oplus \mathbb{C}^*) + b_+ + n$, where we used the additive notation for direct sum. The multiplicative property of $\rho^l$ implies (see (A.9) in Appendix)

$$\rho^l(E') = \{\rho^l(\mathbb{C})\rho^l(\mathbb{C}^*)\}^{2(k+m)}\rho^l(1)^n.$$
For a line bundle \( L \), in general, we have \( \rho^l(L) = 1 + [L] + [L^2] + \cdots + [L^{l-1}] \) (see (A.8) in Appendix). From this formula, we have

\[
\rho^l(E') = \{(1 + t + \cdots + t^{l-1})(1 + t^{-1} + \cdots + t^{-(l-1)})\}^{2(k+m)} l^n.
\]

Similarly \( \rho^l(F') = \{(1 + t + \cdots + t^{l-1})(1 + t^{-1} + \cdots + t^{-(l-1)})\}^{2m b_+ + n} \). Let \( \alpha \) be an element of \( R(\text{Pin}_2) \) satisfying \( \rho^l(F') \alpha = (\psi^l \alpha) \rho^l(E) \), then the image of \( \alpha \) by the restriction map \( R(\text{Pin}_2) \rightarrow R(S^1) \) is expressed by a Laurent polynomial \( h(t) \). Suppose \( \alpha' \) is not zero. Since the action of \( j \) in \( \text{Pin}_2 \) induces the relation \( h(t) = h(t^{-1}) \), the degree \( d \) in \( t \) of the highest nonzero term in \( h(t) \) is nonnegative. Then the degree of the highest nonzero term of \( \rho^l(F') \alpha \) is \( 2(l-1)m + d \). On the other hand, from \( \psi^l h(t) = h(t^l) \) (see (A.1) and (A.2) in Appendix), the degree of the highest nonzero term of \( (\psi^l \alpha') \rho^l(E') \) is \( ld + 2(l-1)(k+m) \). If \( k > 0 \), since the degrees are not the same, \( \rho^l(F') \alpha \) cannot be equal to \( (\psi^l \alpha') \rho^l(E') \). This is a contradiction.

**Proof of Lemma 5.4 (2).** The irreducible complex representations of \( \text{Pin}_2 \) are classified as follows. First we have the trivial complex 1-dimensional representation 1 and the nontrivial complex 1-dimensional representation \( \tilde{1} = \tilde{R} \otimes 1 \). Here recall that \( \tilde{R} \) is the nontrivial real 1-dimensional representation. The other ones are complex 2-dimensional and are parameterized by positive integers \( d \), and the representation corresponding to \( d \) is characterized by the property that its class in \( R(S^1) \) is \( t^d + t^{-d} \). This classification implies:

\[
\text{Ker}(R(\text{Pin}_2) \rightarrow R(S^1)) = \{c(1 - \tilde{1}) \mid c \in \mathbb{Z}\}.
\]

Suppose \( \alpha = c(1 - \tilde{1}) \) satisfies the relation \( \Sigma(-1)^d \Lambda^d F = \alpha \Sigma(-1) \Lambda^d E \). Take the traces of \( j \) for the both hand sides, and we get

\[
2^{2m+b_+ + n} = 2c 2^{2k+2m+n},
\]

which implies \( b_+ \geq 2k + 1 \). On the contrary, when \( b_+ \geq 2k + 1 \), the relation is satisfied by \( \alpha = 2^{b_+ - 2k - 1}(1 - \tilde{1}) \).

**Remark.** When \( b_+ \geq 2k + 1 \), the above proof implies that Lemma 5.1 and Lemma 5.2 provide enough equations to determine that the degree \( \alpha_0 \) of \( \tilde{f} \) is equal to \( 2^{b_+ - 2k - 1}(1 - \tilde{1}) \).

**Appendix**

We collect some properties of the Adams operations which are used in Section 5. Let \( G \) be a compact Lie group and \( X \) a compact \( G \)-space. the *Adams operation* \( \psi^l : K_G(X) \rightarrow K_G(X) \) is defined as follows. We fix a positive integer \( l \) in Appendix. For each positive integer \( r \), let \( p_r(\sigma_1, \sigma_2, \cdots, \sigma_r) \) be the polynomial expressing \( x_1^r + x_2^r + \cdots + x_r^r \) with respect to the elementary symmetric polynomials, where \( \sigma_d \) is the \( d \)-th elementary symmetric polynomial. When \( E \)}
and $F$ are two $G$-equivariant complex vector bundles over $X$ with ranks $r$ and $s$, one can check the relation
\[
p_{r+s}([\Lambda^1(E \oplus F)], \ldots, [\Lambda^{r+s}(E \oplus F)]) = p_r([\Lambda^1 E], \ldots, [\Lambda^r E]) + p_s([\Lambda^1 F], \ldots, [\Lambda^s F])
\]
in $K_G(X)$. Then there is a unique additive homomorphism $\psi^l$ from $K_G(X)$ to itself which satisfies $\psi^l([E]) = p_r([\Lambda^1 E], \ldots, [\Lambda^r E])$. It is straightforward to check that $\psi^l$ is also a multiplicative homomorphism. When $Y$ is a $G$-invariant compact subset of $X$, the relative $K_G$-group $K_G(X,Y)$ for the compact pair $(X,Y)$ is defined to be the kernel of the natural map from $K_G(X/Y)$ to $K_G(\text{point}) = R(G)$. Since $\psi^l$ is natural for continuous maps between compact $G$-spaces, we can define $\psi^l$ on $K_G(X,Y)$ so that it is natural for continuous maps between compact pairs.

(A.1) The operation $\psi^l : K_G(X,Y) \to K_G(X,Y)$ is a ring homomorphism.

In particular we have the formula for line bundles from the definition:

(A.2) $\psi^l[L] = [L^l]$ for a line bundle $L$.

Let $E$ be an $G$-equivariant complex vector bundle over $X$, $BE$ be its $G$-invariant disk bundle and $SE$ be the boundary of $BE$. The Thom class $\tau_E$ of $E$ is an element of $K_G(BE,SE)$ which has the following properties:

(A.3) $K_G(BE,SE)$ is a free $K_G(X)$-module generated by $\tau_E$ ([2]).

(A.4) The restriction of $\tau_E$ on the zero section is equal to $e(E) = \Sigma (-1)^d [\Lambda^d E]$. The former property is the Thom isomorphism theorem in equivariant K-theory and it is the only non-elementary theorem we need in Section 5. We call the restriction $e(X)$ by the Euler class of $E$. The Thom classes have the multiplicative property:

(A.5) $\tau_E \otimes F = \tau_E \tau_F$.

The definition of the $K$-theory characteristic class $\rho^l(E) \in K_G(X)$ is given by:

(A.6) $\psi^l \tau_E = \rho^l(E) \tau_E$.

An explicit formula for $\rho^l$ is:

(A.7) $\rho^l(E) = \frac{\Sigma (-1)^d l^d \psi^l [\Lambda^d E]}{\Sigma (-1)^d l^d [\Lambda^d E]} \bigg|_{t=1}$.

From (A.7), (A.1) and (A.2) we have:

(A.8) $\rho^l(L) = 1 + [L] + [L^2] + \cdots + [L^{l-1}]$ for a line bundle $L$, and

(A.9) $\rho^l(E \oplus F) = \rho^l(E) \rho^l(F)$.

Instead of using (A.7), we can use the multiplicative properties (A.1) and (A.5) of the Thom classes and the Adams operation to show (A.9) directly from the definition of $\rho^l$. In the rest of Appendix we give a short proof of (A.7). (See also [3].)

Proof of (A.7). Since $E$ is an $G$-equivariant complex vector bundle, we have the $S^1$-action on $E$ defined by the multiplication of $S^1 \subset \mathbb{C}$, which commutes
with the $G$-action on $E$. Hence we can think of $E$ as a $G \times S^1$-equivariant bundle, where the $S^1$-action on $X$ is trivial. We write $\tilde{E}$ for this $G \times S^1$-equivariant bundle. Let $\tau_{\tilde{E}}$ and $e(\tilde{E})$ be the Thom class and the Euler class of $\tilde{E}$. Then $e(\tilde{E})$ is an element of $K_{G \times S^1}(X) = K_G(X) \otimes R(S^1)$. If we write $t$ for the class of the standard 1-dimensional representation of $S^1$, then $R(S^1)$ is the ring of the Laurent polynomials in $t$ and $e(\tilde{E})$ is not a zero-divisor. Restrict the relation $\psi^l \tau_{\tilde{E}} = \rho^l(\tilde{E}) \tau_{\tilde{E}}$ on the zero section to get $\psi^l e(\tilde{E}) = \rho^l(\tilde{E}) e(\tilde{E})$. Then, since $e(\tilde{E})$ is not a zero-divisor, we obtain $\rho^l(\tilde{E}) = (\psi^l e(\tilde{E}))/e(\tilde{E})$, from which we can deduce (A.7).

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