THE BLOCH-KATO CONJECTURE FOR ADJOINT MOTIVES OF MODULAR FORMS

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Abstract. The Tamagawa number conjecture of Bloch and Kato describes the behavior at integers of the $L$-function associated to a motive over $\mathbb{Q}$. Let $f$ be a newform of weight $k \geq 2$, level $N$ with coefficients in a number field $K$. Let $M$ be the motive associated to $f$ and let $A$ be the adjoint motive of $M$. Let $\lambda$ be a finite prime of $K$. We verify the $\lambda$-part of the Bloch-Kato conjecture for $L(A, 0)$ and $L(A, 1)$ when $\lambda \nmid Nk!$ and the mod $\lambda$ representation associated to $f$ is absolutely irreducible when restricted to the Galois group over $\mathbb{Q} \left( \sqrt{(-1)^{(\ell-1)/2}} \right)$ where $\lambda | \ell$.

1. Introduction

This is a summary of results on the Tamagawa number conjecture of Bloch and Kato [B-K] for adjoint motives of modular forms of weight $k \geq 2$. The conjecture relates the value at 0 of the associated $L$-function to arithmetic invariants of the motive. We prove in [D-F-G] that it holds up to powers of certain “bad primes.” The strategy for achieving this is essentially due to Wiles [Wi], as completed with Taylor in [T-W]. The Taylor-Wiles construction yields a formula relating the size of a certain module measuring congruences between modular forms to that of a certain Galois cohomology group. This was carried out in [Wi] and [T-W] in the context of modular forms of weight 2, where it was used to prove results in the direction of the Fontaine-Mazur conjecture [F-M]. While it was no surprise that the method could be generalized to higher weight modular forms and that the resulting formula would be related to the Bloch-Kato conjecture, there remained many technical details to verify in order to accomplish this. In particular, the very formulation of the conjecture relies on a comparison isomorphism between the $\ell$-adic and de Rham realizations of the motive provided by theorems of Faltings [Fa] or Tsujii [Ts], and verification of the conjecture requires the careful application of such a theorem. We also need to generalize results on congruences between modular forms to higher weight, and to compute certain local Tamagawa numbers.

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2. The adjoint motive

Suppose \( f \) is a newform of weight \( k \geq 2 \), level \( N \) and character \( \psi \) with coefficients in the ring of integers \( \mathcal{O} \) of a number field \( K \). For \( * = B, \text{dr} \) or \( \lambda \) (a prime of \( K \)), we let \( M_* \) denote the corresponding realization of the motive attached to \( f \) (see [De2]). This is a two-dimensional \( K_* \)-subspace (where \( K_B = K_\text{dr} = K \)) of the corresponding cohomology group of the modular curve \( X_1(N) \) with coefficients in a sheaf depending on \( k \). We have certain additional structure on each \( M_* \) (an involution, filtration or \( G_\mathbb{Q} \)-action) and comparison isomorphisms relating certain pairs of realizations. Let \( S \) be a set of primes containing those dividing \( Nk! \), and let \( \mathcal{O}_B = \mathcal{O}_\text{dr} = \mathcal{O}_S \) denote the \( S \)-integers in \( K \). For \( * = B, \text{dr} \) or \( \lambda \) for \( \lambda \notin S \), we also have natural \( \mathcal{O}_* \)-lattices \( M_* \) in \( M_* \) and integral versions of the comparison isomorphisms involving the \( M_\lambda \).

Let \( D_* \) denote the \(*\)-realization of the \((1 - k)\)-Tate twist of the motive associated to \( \psi \). Poincaré duality for \( X_1(N) \) gives rise to a perfect alternating pairing on \( M_* \) with values in \( D_* \), i.e., an isomorphism
\[
\mu_* : \wedge^2 M_* \rightarrow D_* ,
\]
respecting the various additional structures and comparison isomorphisms. We also have natural \( \mathcal{O}_* \)-lattices \( D_* \) in \( D_* \), but the map \( \wedge^2 M_* \rightarrow D_* \) need not be an isomorphism; the image is \( \eta D_* \) for a certain ideal \( \eta \) of \( \mathcal{O}_S \). More generally, if \( \Sigma \) is a finite set of primes contained in \( S \), we analogously define \( M_*^{\Sigma} \), \( \mu_*^{\Sigma} \) and \( \eta^{\Sigma} \) using level structure which is, roughly speaking, minimal outside \( \Sigma \).

We let \( A_* \) denote the set of trace-zero elements in \( \text{End}_{K_*} M_* \). So \( A_* \) is three-dimensional over \( K_* \) and we have comparison isomorphisms, including
\[
I_\infty : A_\text{dr} \otimes \mathbb{C} \rightarrow A_B \otimes \mathbb{C} ,
\]
induced by those for the \( M_* \). Replacing \( f \) by a twist does not change this data, so we always assume \( f \) has minimal conductor among its twists. The \( L \)-function associated to \( A_\lambda \) is independent of \( \lambda \); we denote it \( L(A, s) \) and regard it as taking values in \( K \otimes \mathbb{C} \). We know also that \( L(A, s) \) extends to an entire function on the complex plane and satisfies the functional equation (see [G-J], [Sc]),
\[
\Lambda(A, s) = \epsilon(A, s) \Lambda(A, 1 - s) ,
\]
where
\[
\Lambda(A, s) = L(A, s) \Gamma_R(s + 1) \Gamma_C(s + k - 1) \]
and \( \epsilon(A, s) \) is as defined by Deligne [De1]. Moreover \( L(A, 0) \) and \( L(A, 1) \) are invertible elements of \( \mathbb{R} \otimes K \), and \( \epsilon(A, s) = \pm C^{1 - 2s} \) for some positive integer \( C \) dividing \( N \).

3. The Bloch-Kato conjecture

We show that the conjecture of Bloch and Kato [B-K] correctly predicts the values of \( L(A, 0) \) and \( L(A, 1) \) up to an element of \( \mathcal{O}_S^\times \), where \( S \) is a certain finite set of primes containing those dividing \( Nk! \). We use the formulation of the
conjecture given by Fontaine and Perrin-Riou [F-P], which is easily generalized to the setting of $K$-coefficients.

The fundamental line for $A$ is the $K$-line defined by

$$
\Delta = \text{Hom}_K(A^\vee_K, A_{\text{tr}}/\text{Fil}^0 A_{\text{tr}}) \\
\cong \text{Hom}_K(\text{Fil}^{k-1} M_{\text{tr}}, M_{\text{tr}}/\text{Fil}^{k-1} M_{\text{tr}}).
$$

The Deligne period for $A$ is the natural basis $e^+$ for $\Delta \otimes R$ gotten from $I_\infty$. One finds that $c^+ L(A,0)^{-1}$ is an element $\delta \in \Delta$ characterized by

$$(3.1) \quad \mu_{\text{dr}}(f \wedge \delta(f)) = \frac{i^k \eta((k-2)!)^2 \epsilon(M) \prod_{\ell \in P}(1 + p^{-1})}{2 \epsilon(D) \epsilon(A)} \cdot G_{\text{dr}},$$

where $\eta \in \{0, 1\}$ has the same parity as $k$, $P$ is a certain exceptional set of primes dividing $N$ and $G_{\text{dr}}$ (for Gauss sum) is the usual basis for $D_{\text{dr}}$. This is proved by relating the Deligne period to the Petersson product, applying a method of Rankin and Shimura (for Gauss sum) and using the functional equation.

To state the $\lambda$-part of the Bloch-Kato conjecture, one chooses a Galois-stable $O_\lambda$-lattice $A_\lambda$ in $A_L \cong A_B \otimes_K K_\lambda$ and an $O_\lambda$-lattice $\omega$ in $(A_{\text{tr}}/\text{Fil}^0 A_{\text{tr}}) \otimes_K K_\lambda$. The conjecture, which turns out to be independent of these choices, states that

- $H^1_f(G_Q, A_\lambda/A_\lambda)$ is finite;
- $\delta$ generates the $O_\lambda$-submodule

$$
\frac{\text{Fitt}_{O_\lambda} H^0(G_Q, A_\lambda/A_\lambda) \cdot \text{Fitt}_{O_\lambda} H^0(G_Q, B_\lambda/B_\lambda)}{\text{Tam}_{\text{dr}}(A_\lambda) \cdot \text{Fitt}_{O_\lambda} H^1_f(G_Q, A_\lambda/A_\lambda)} \cdot \text{Hom}_{O_\lambda}(A_\lambda^+, \omega)
$$

of $\Delta \otimes_K K_\lambda$, where $B_\lambda = \text{Hom}_{Z}_f(A_\lambda, Z_f(1))$, $B_\lambda = B_\lambda \otimes Q$ and $\text{Tam}_{\text{dr}}(A_\lambda)$ is the Tamagawa ideal of $A_\lambda$ relative to $\omega$.

4. Methods

The two main problems are to compute the local contribution at $\ell$ to the Tamagawa ideal, $\text{Tam}_{\ell, \omega}(A_\lambda)$, and the length of $H^1_f(G_Q, A_\lambda/A_\lambda)$. We restrict our attention to primes $\lambda$ not in $S$, and let $A_\ast$ denote the set of elements of $\text{End}_{O}(M_\ast)$ of trace zero. One can then use existing machinery to show that $\text{Tam}_{\ell, \omega}(A_\lambda) = \partial_{\lambda}$, where $\omega = (A_{\text{tr}}/\text{Fil}^0 A_{\text{tr}}) \otimes_{O_{\lambda}} O_{\lambda}$. The proof requires only that $\lambda$ not divide $Nk!$ and uses the integral version of Faltings’ comparison theorem [Fa]. For primes between $k$ and $2k$, the argument is slightly delicate since the filtration length of $A_{\text{tr}}$ is greater than $\ell - 2$.

The computation of the $H^1_f$ relies on the methods of Wiles [Wi] and Taylor-Wiles [T-W], as modified in [Di1] and [Di2]. Now we have to impose another hypothesis on $\lambda$:

- $\bar{\rho} : G_Q \to \text{Aut}_{O_{\lambda}/\lambda}(\mathbb{M}_{\lambda}/\lambda \mathbb{M}_{\lambda})$ has absolutely irreducible restriction to $G_L$,

where $L = \mathbb{Q}((\sqrt{-1})^{(\ell - 1)/2})$. 


We give an axiomatic formulation of the method of the following nature: Suppose

\[ R = \{ \rho : G_\mathbb{Q} \to \text{Aut}_{K_\rho} V_\rho \} \]

is a set of liftings of \( \bar{\rho} \), where for each \( \rho \) in \( R \), \( K_\rho \) is a finite extension of \( K_\lambda \) in \( \bar{K}_\lambda \) and the restriction of \( \rho \) to \( G_\mathbb{Q} \bar{\rho} \) is crystalline of Hodge-Tate type \((0, k - 1)\) (see [Fo]). For any finite set of primes \( \Sigma \) not containing \( \ell \), let \( R^\Sigma \) denote the set of \( \rho \in R \) such that \( \rho \) is minimally ramified outside \( \Sigma \). Suppose that for each \( \rho \in R^\Sigma \) we are given an isomorphism

\[ \mu^\Sigma_\rho : \wedge^2_{K_\rho} V_\rho \to D_\lambda \otimes_{K_\lambda} K_\rho. \]

We assume that for fixed \( \rho \) and varying \( \Sigma \), the ratios of the \( \mu^\Sigma_\rho \) are determined by Euler factors. We assume also that \( \bigoplus_{\rho \in R}(V_\rho \otimes_{K_\rho} K_\lambda) \) contains a lattice \( V \) having a certain self-duality property with respect to the pairings. We show that if the numbers \( \#R^\Sigma \) satisfy a certain numerical criterion, then

- every lifting of \( \bar{\rho} \) which is crystalline of Hodge-Tate type \((0, k - 1)\) is isomorphic to an element of \( R \);
- if \( \rho \in R^\Sigma \) and \( K_\rho = K_\lambda \), then

\[ \text{Fitt}_{\mathcal{O}_\lambda} H^1_\Sigma(G_\mathbb{Q}, W_\rho \otimes_{\mathcal{O}_\lambda} (K_\lambda/\mathcal{O}_\lambda)) = \text{Fitt}_{\mathcal{O}_\lambda} D_\lambda/\mu^\Sigma_\rho(\wedge^2_{\mathcal{O}_\lambda} V_\rho), \]

where \( V_\rho = V \cap V_\rho \) and \( W_\rho \) is the set of elements of \( \text{End}_{\mathcal{O}_\lambda} V_\rho \) of trace zero.

### 5. Main results

Generalizing results and methods of Ribet, Wiles and others ([Ri], [D-T], [Wi], [T-W], [Di1]), we verify that these hypotheses are satisfied by the set \( R \) of liftings of \( \bar{\rho} \) associated to newforms of weight \( k \), level prime to \( \ell \) and character \( \psi \). We thus obtain the following result in the direction of the Fontaine-Mazur conjecture [F-M]:

**Theorem 1.** Suppose \( \rho : G_\mathbb{Q} \to \text{Aut}_{K_\lambda} V_\lambda \) is continuous, two-dimensional, unramified outside a finite set of primes and has crystalline restriction to \( G_{\mathbb{Q}_\ell} \) of Hodge-Tate type \((0, d)\) with \( 0 < d < \ell - 1 \). If \( \bar{\rho} \) is modular and has absolutely irreducible restriction to \( G_L \), then \( \rho \) is modular.

We also find that \( \text{Fitt}_{\mathcal{O}_\lambda} H^1_\Sigma(G_\mathbb{Q}, A_\lambda/A_\lambda) = \eta^\Sigma \mathcal{O}_\lambda \). Using the modified Poitou-Tate sequence (see [Fl2]), we can then compute the length of \( H^1_\Sigma(G_\mathbb{Q}, A_\lambda/A_\lambda) \), which is dual to \( H^1(G_\mathbb{Q}, B_\lambda/B_\lambda) \) [Fl]. Note that (2) implies that \( H^0(G_\mathbb{Q}, A_\lambda/A_\lambda) \) and \( H^0(G_\mathbb{Q}, B_\lambda/B_\lambda) \) both vanish. Applying equation (1) and its analogue for \( L(A, 1) \), we conclude:

**Theorem 2.** Let \( f \) be a newform of weight \( k \geq 2 \) and level \( N \) with coefficients in a number field \( K \). Let \( S \) denote the set of primes \( \lambda \) such that \( \lambda \) divides \( NK! \) or (2) fails. For \( \lambda \) not in \( S \), the \( \lambda \)-part of the Bloch-Kato conjecture holds for \( L(A, 0) \) and \( L(A, 1) \).
Finally we remark that the set $S$ is finite. In fact, if $\lambda$ does not divide $Nk!$ and (2) fails, then $\ell = 2k - 1$, $\ell = 2k - 3$ or $M_{\lambda}/\lambda M_{\lambda}$ defines a reducible representation of $G_{\mathbb{Q}}$, and reducibility for infinitely many $\lambda$ would violate the Ramanujan conjecture.

References


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