0. Introduction

Let $X$ be a locally finite tree and let $G = \text{Aut}(X)$. Then $G$ is naturally a locally compact group ([BL], Ch. 3). A discrete subgroup $\Gamma \leq G$ is called an $X$-lattice if

\begin{equation}
\text{Vol}(\Gamma \backslash X) := \sum_{x \in V(\Gamma \backslash X)} \frac{1}{|\Gamma_x|}
\end{equation}

is finite, and a uniform $X$-lattice if $\Gamma \backslash X$ is a finite graph, non-uniform otherwise ([BL], Ch. 3). Bass and Kulkarni have shown ([BK], (4.10)) that $G = \text{Aut}(X)$ contains a uniform $X$-lattice if and only if $X$ is the universal covering of a finite connected graph, or equivalently, that $G$ is unimodular and $G \backslash X$ is finite. In this case, we call $X$ a uniform tree.

Following ([BL], (3.5)) we call $X$ rigid if $G$ itself is discrete, and we call $X$ minimal if $G$ acts minimally on $X$, that is, there is no proper $G$-invariant subtree. If $X$ is uniform then there is always a unique minimal $G$-invariant subtree $X_0 \subseteq X$ ([BL] (5.7), (5.11), (9.7)). We call $X$ virtually rigid if $X_0$ is rigid (cf. ([BL], (3.6))).

Let $X$ be a locally finite tree, and let $\Gamma \leq \Gamma'$ be an inclusion of $X$-lattices. Then by ([BL], (1.7)) we have:

\begin{equation}
\text{Vol}(\Gamma' \backslash \backslash X) = \frac{\text{Vol}(\Gamma \backslash X)}{[\Gamma' : \Gamma]}.
\end{equation}

We call an infinite ascending chain

\begin{equation}
\Gamma_1 < \Gamma_2 < \Gamma_3 < \ldots
\end{equation}

of $X$-lattices an infinite tower of $X$-lattices. By (0.2), the lattice inclusions of (0.3) are of finite index, and $\text{Vol}(\Gamma_i \backslash \backslash X) \to 0$ as $i \to \infty$.

The Kazhdan-Margulis property for lattices in Lie groups ([KM]) states that the covolume of a lattice is bounded away from zero. Hence the existence of
infinite towers of X-lattices in \( G = \text{Aut}(X) \) shows that the Kazhdan-Margulis property is violated for X-lattices.

Bass and Kulkarni have given ([BK], (Sec. 7)) several examples of uniform trees such that \( G = \text{Aut}(X) \) contains infinite towers of uniform X-lattices. The second author has extended the results and techniques of Bass-Kulkarni to all uniform trees that are not rigid ([R]).

Here our main result is that, with one exception (see §5), if \( G = \text{Aut}(X) \) contains a non-uniform X-lattice, then \( G \) contains an infinite tower of non-uniform X-lattices.

The authors would like to thank H. Bass for many helpful discussions and suggestions.

1. The setting

An edge-indexed graph \((A, i)\) consists of an underlying graph \( A \), and an assignment of a positive integer \( i(e) > 0 \) to each oriented edge \( e \in EA \). Our underlying graph \( A \) will always be understood to be locally finite. In [BK] and [BL] one allows \( i(e) \) to be any positive cardinal, but our interest here is only in finite \( i(e) \). If \( i(e) > 1 \), we call \( e \) ramified and unramified otherwise.

Let \( \mathcal{A} = (A, \mathcal{A}) \) be a graph of groups, with underlying graph \( A \), vertex groups \((\mathcal{A}_a)_{a \in VA} \), edge groups \((\mathcal{A}_e = \mathcal{A}_e)_{e \in EA} \) and monomorphisms \( \alpha_e : \mathcal{A}_e \hookrightarrow \mathcal{A}_{\partial_0 e} \). A graph of groups \( \mathcal{A} \) naturally gives rise to an edge-indexed graph \( \mathcal{I}(\mathcal{A}) = (A, i) \) whose indices are the indices of the edge groups as subgroups of the adjacent vertex groups: that is, \( i(e) = [\mathcal{A}_{\partial_0 e} : \alpha_e \mathcal{A}_e] \), which we assume to be finite, for all \( e \in EA \).

Given an edge-indexed graph \((A, i)\), a graph of groups \( \mathcal{A} \) such that \( \mathcal{I}(\mathcal{A}) = (A, i) \) is called a grouping of \((A, i)\). We call \( \mathcal{A} \) a finite grouping if the vertex groups \( \mathcal{A}_a \) are finite and a faithful grouping if \( \mathcal{A} \) is a faithful graph of groups, that is, if \( \pi_1(\mathcal{A}, a) \) acts faithfully on \( X = (\mathcal{A}, a) \).

Let \( \mathcal{A}' \) and \( \mathcal{A} \) be groupings of \((A, i)\). Then \( \mathcal{A}' = (A, \mathcal{A}') \) is called a full graph of subgroups of \( \mathcal{A} = (A, \mathcal{A}) \) (as in ([B], (1.14))) if \( \mathcal{A}'_a \leq \mathcal{A}_a \) for \( a \in A' \), and for \( e \in EA' \), \( \mathcal{A}'_e \leq \mathcal{A}_e \), and \( \alpha'_e = \alpha_e \mathcal{A}' e \). We further assume that for \( e \in EA' \), with \( \partial_0 e = a \), \( \mathcal{A}_e' \cap \alpha_e \mathcal{A}_e = \alpha_e \mathcal{A}_e' \), that is \( \mathcal{A}_e' / \alpha_e \mathcal{A}_e' \longrightarrow \mathcal{A}_e / \alpha_e \mathcal{A}_e \) is injective, and hence bijective. This assumption implies that \( \mathcal{I}(\mathcal{A}') = (A, i) \), and that \( \pi_1(\mathcal{A}', a') \leq \pi_1(\mathcal{A}, a) \) ([B], (1.14)).

Let \((A, i)\) be an edge-indexed graph. A tower of groupings on \((A, i)\) is a semi-infinite sequence \((\mathcal{A}_i)_{i \in \mathbb{Z}_{> 0}}\) of groupings of \((A, i)\) such that each \( \mathcal{A}_i \) is a full graph of proper subgroups of \( \mathcal{A}_{i+1} \). A tower of faithful groupings induces an infinite ascending chain of fundamental groups:

\[
(1) \quad \pi_1(\mathcal{A}_1, a_0) \leq \pi_1(\mathcal{A}_2, a_0) \leq \pi_1(\mathcal{A}_3, a_0) \leq \ldots
\]

For an edge \( e \in EA \), define:

\[
(2) \quad \Delta(e) := \frac{i(\overline{e})}{i(e)}.
\]
If $\gamma = (e_1, \ldots, e_n)$ is a path, set:

$$\Delta(\gamma) := \Delta(e_1) \cdots \Delta(e_n).$$

**Definition.** An edge-indexed graph $(A, i)$ is called unimodular if $\Delta(\gamma) = 1$ for all closed paths $\gamma$ in $A$.

Now assume that $(A, i)$ is unimodular. Pick a base point $a_0 \in VA$, and define, for $a \in VA$,

$$N_{a_0}(a) := \frac{\Delta a}{\Delta a_0} (= \Delta(\gamma) \text{ for any path } \gamma \text{ from } a_0 \text{ to } a) \in \mathbb{Q}_{>0}. \quad (3)$$

For $e \in EA$, put

$$N_{a_0}(e) := \frac{N_{a_0}(\partial_0(e))}{i(e)}.$$

Following ([BL], (2.6)), we say that $(A, i)$ has bounded denominators if

$$\{N_{a_0}(e) \mid e \in EA\}$$

has bounded denominators, that is, if for some integer $D > 0$, $D \cdot N_{a_0}$ takes only integer values on edges. Since

$$N_{a_1} = \frac{\Delta a_0}{\Delta a_1} N_{a_0},$$

this condition is independent of $a_0 \in VA$.

**Theorem ([BK], (2.4)).** An indexed graph $(A, i)$ admits a finite grouping if and only if $(A, i)$ is unimodular and has bounded denominators. The grouping can further be taken to be faithful.

As in ([BL], Ch.2) we define the volume of an indexed graph $(A, i)$ at a basepoint $a_0 \in VA$:

$$\text{Vol}_{a_0}(A, i) := \sum_{a \in VA} \left( \frac{1}{\Delta a} \right) = \sum_{a \in VA} \left( \frac{\Delta a_0}{\Delta a} \right). \quad (4)$$

Then

$$\text{Vol}_{a_1}(A, i) = \frac{\Delta a_0}{\Delta a_1} \text{Vol}_{a_0}(A, i),$$

([BL], Ch.2) We write $\text{Vol}(A, i) < \infty$ if $\text{Vol}_a(A, i) < \infty$ for some, and hence every $a \in VA$.

If $\mathcal{A}$ is a finite grouping of $(A, i)$, then we have ([BL], (2.6.15)):

$$\text{Vol}(\mathcal{A}) = \frac{1}{|\mathcal{A}|} \text{Vol}_a(A, i), \quad (5)$$
which is automatically finite if Vol \((A, i) < \infty\).

We now describe a method for constructing \(X\)-lattices which follows naturally from the fundamental theory of Bass-Serre ([B], [S]), and was first suggested in ([BK]). We begin with an edge-indexed graph \((A, i)\). Then \((A, i)\) determines \(X = (A, i, a_0)\) up to isomorphism ([BL], Ch. 2).

We say that \((A, i)\) admits a lattice if \((A, i)\) admits a grouping \(A\) such that \(\pi_1(A, a_0)\) is an \(X\)-lattice. This happens if and only if \((A, i)\) satisfies:

(U) \((A, i)\) is unimodular, and
(BD) \((A, i)\) has bounded denominators, and
(FV) \((A, i)\) has finite volume.

Assume that \((A, i)\) is unimodular and has bounded denominators (which is automatic if \(A\) if finite). By ([BK], (2.4)) we can find a finite faithful grouping \(A\) of \((A, i)\) and a group \(\Gamma = \pi_1(A, a_0)\) acting faithfully on \(X\). Then

(a) \(\Gamma\) is discrete, since \(A\) is a graph of finite groups.
(b) \(\Gamma\) is a uniform \(X\)-lattice if and only if \(A\) is finite.
(c) \(\Gamma\) is a non-uniform \(X\)-lattice if and only if \(A\) is infinite, and

\[\text{Vol}(\Gamma \setminus X) = \text{Vol}(A)(:= \sum_{a \in VA} \frac{1}{|A_a|} = \frac{1}{|A_a|} \text{Vol}_a(A, i)) < \infty.\]

Our task is the following: given an edge-indexed graph \((A, i)\) of finite volume, construct an infinite tower of finite faithful groupings of \((A, i)\). This induces an infinite tower

\[\Gamma_1 < \Gamma_2 < \Gamma_3 < \ldots\]

of \(X\)-lattices in \(\text{Aut}(X)\), for \(X = (\sim A, i)\), with \(\Gamma_i \setminus X = A, i = 1, 2, \ldots\).

An edge \(e \in EA\) is called separating if \(A - \{e, \overline{e}\}\) has two connected components \(A_0(e)\) and \(A_1(e)\), where \(A_0(e)\) and \(A_1(e)\) contain \(\partial_0(e)\) and \(\partial_1(e)\) respectively.

Let \((A, i)\) be any connected edge-indexed graph. A subset \(\beta \subset EA\) of \(n \geq 2\) (oriented) edges is called an arithmetic bridge for \((A, i)\) (as in ([C1], Sec. 4)) if:

1. \(\beta \cap \overline{\beta} = \emptyset\), \(A - (\beta \cup \overline{\beta})\) has two connected components, \(A_0\) and \(A_1\),
2. For every \(e \in \beta, \partial_0 e \in A_0\) and \(\partial_1 e \in A_1\),
3. There exists an integer \(d > 1\) such that \(d | i(e)\) for every \(e \in \beta\).

Following [BT] we say that \((A, i)\) is discretely ramified if for \(e \in EA\)

\[i(e) > 1 \implies i(e) = 2, e\) is separating, and \((A_0(e), i)\) is an unramified tree.

We call \((A, i)\) a dominant-rooted edge-indexed tree if \(A\) is a tree and if there exists an \(a \in VA\) such that \(i(e) = 1\) for all edges \(e \in EA\) directed towards \(a\). Let \((A, i)\) be an edge-indexed graph. We say that \((A, i)\) is restricted if \((A, i)\) satisfies any one of the following conditions:

(DR) \((A, i)\) is discretely ramified, or
(F) \((A, i)\) is a dominant-rooted edge-indexed tree, or
(GS) \(A\) is a tree, and \((A, i)\) contains a prime-prime interval (see [R]) and no other ramified edges.
We say that \((A, i)\) is permissible if \((A, i)\) admits a lattice and if \((A, i)\) is not restricted. We note that an infinite edge-indexed graph \((A, i)\) with finite volume is automatically non-discretely ramified, is not a dominant-rooted tree ([CR2]), is obviously not (GS) as above, and hence is not restricted.

2. **Rooted products of graphs of groups**

Given rooted graphs of groups \(\mathbb{A} = (A, A, a_0), a_0 \in VA\), and \(\mathbb{B} = (B, B, b_0), b_0 \in VB\), we construct a rooted graph of groups \(\mathbb{C} = (C, C, c_0) = A \times_a a_0 = b_0 \mathbb{B}\) as follows: we set

\[
C := A \sqcup B/(a_0 = b_0 = c_0).
\]

For \(a \in VA, e \in EA\), we set

\[
C_a := A_a \times B_{b_0}, \quad C_e := A_e \times B_{b_0},
\]

and if \(\alpha_e : A_e \hookrightarrow A_{a_0 e}\), we set

\[
\gamma_e := \alpha_e \times Id_{B_{b_0}}.
\]

Similarly, for \(b \in VB, e \in EB\), we set

\[
C_b := A_{a_0} \times B_b, \quad C_e := A_{a_0} \times B_e,
\]

and if \(\beta_e : B_e \hookrightarrow B_{b_0 e}\), we set

\[
\gamma_e := Id_{A_{a_0}} \times \beta_e.
\]

If we set \((A, i^A) = I(\mathbb{A})\) and \((B, i^B) = I(\mathbb{B})\), then we have the ‘rooted union of edge-indexed graphs’:

\[
(C, i^C) := (A, i^A) \sqcup (B, i^B)/(a_0 = b_0 = c_0),
\]

for \(c_0 \in VC\), and clearly \(\mathbb{C}\) is a grouping of \((C, i^C)\).

**(2.1) Remarks.**

1. The graph of groups \(\mathbb{C}\) is faithful if and only if \(\mathbb{A}\) and \(\mathbb{B}\) are faithful. In fact, if \(N_A\) is the maximal normal subgroup of \(\mathbb{A}\), and \(N_B\) is the maximal normal subgroup of \(\mathbb{B}\), then the maximal normal subgroup of \(\mathbb{C}\) is \(N_A \times N_B\).

2. We have

\[
\pi_1(\mathbb{C}, c_0) = (\pi_1(\mathbb{A}, a_0) \times B_{b_0}) \ast ((A_{a_0} \times B_{b_0})/(A_{a_0} \times \pi_1(\mathbb{B}, b_0))).
\]

3. If \(\mathbb{A}\) and \(\mathbb{B}\) are graphs of finite groups, then so also is \(\mathbb{C}\), and

\[
\text{Vol}(\mathbb{C}) = \frac{1}{|B_{b_0}|} \text{Vol}(\mathbb{A}) + \frac{1}{|A_{a_0}|} \text{Vol}(\mathbb{B}) - \frac{1}{|A_{a_0}||B_{b_0}|}.
\]
(2.2) Functoriality.

Suppose that we have groupings $A \leq A'$ of an edge-indexed graph $(A, i^A)$ and $B \leq B'$ of an edge-indexed graph $(B, i^B)$, then we get groupings

$$C = A \times a_0 = b_0 \leq C' = A' \times a_0 = b_0$$

for $a_0 \in VA$, $b_0 \in VB$ of the edge-indexed graph

$$(C, i^C) = (A, i^A) \sqcup (B, i^B)/(a_0 = b_0 = c_0),$$

for $c_0 \in VC$. In particular, for an edge $e \in VA$ with initial vertex $a \in VA$,

$$A'_e \times B'_{b_0} \xrightarrow{\alpha'_e \times Id_{B'_{b_0}}} A'_a \times B'_{b_0} \leq \leq \leq$$

$$\leq \leq \leq$$

$$A_e \times B_{b_0} \xrightarrow{\alpha_e \times Id_{B_{b_0}}} A_a \times B_{b_0}$$

commutes, and similarly in $B$.

(2.3) Corollary. A tower $A_1 \leq A_2 \leq A_3 \leq \ldots$ yields a tower

$$A_1 \times a_0 = b_0 \leq A_2 \times a_0 = b_0 \leq A_3 \times a_0 = b_0 \leq \ldots$$

Since a unimodular edge-indexed graph with bounded denominators admits a finite faithful grouping, we can apply the above corollary repeatedly to obtain the following lemma.

(2.4) Lemma. Let $(A, i)$ be an edge-indexed graph and let $(A_0, i)$ be a core subgraph such that $(A, i)$ is obtained from $(A_0, i)$ by attaching to finitely many vertices $a_1, \ldots, a_n \in VA_0$, rooted edge-indexed graphs $(A_j, i_j, a_j)$, $j = 1, \ldots, n$ respectively. Suppose that $(A_0, i)$ admits an infinite ascending chain of finite faithful groupings of finite volume. Suppose that each of the $(A_j, i_j)$, $j = 1, \ldots, n$ are unimodular, have finite volume and bounded denominators. Then $(A, i)$ admits an infinite tower of finite faithful groupings of finite volume.

3. Infinite towers of uniform tree lattices

In [R], the second author proved the following:

(3.1) Theorem ([R]). Let $(A, i)$ be a finite permissible edge-indexed graph. Then $(A, i)$ admits an infinite tower of finite faithful groupings.

The proof of Theorem (3.1) generalizes the techniques of Bass-Kulkarni ([BK]) for constructing towers of groupings on certain fundamental examples, and uses certain constructions with edge-indexed graphs to extend to a more general setting.

Theorem (3.1) yields the following:
(3.2) Theorem ([R]). Let $X$ be a locally finite tree. The following conditions are equivalent:

(a) $X$ is uniform and not rigid.
(b) $X$ is the universal cover of a finite permissible edge-indexed graph.
(c) $\text{Aut}(X)$ contains an infinite ascending chain

$$\Gamma_1 < \Gamma_2 < \Gamma_3 < \cdots$$

of uniform $X$-lattices.
(d) The set of uniform covolumes

$$\{ \text{Vol}(\Gamma \backslash X) \mid \Gamma \text{ is a uniform } X\text{-lattice} \} \subset \mathbb{Q}_{>0}$$

is not bounded away from zero.

This generalizes Theorem 7.1(a) of [BK] which states the result for homogeneous trees.

4. Infinite towers of non-uniform $X$-lattices with quotient a tree

The techniques described in §3 extend to certain infinite edge-indexed graphs. We have the following:

(4.1) Theorem. Let $(A, i)$ be an edge-indexed graph that admits a lattice, and which is infinite, hence permissible. Assume that $(A, i)$ is a tree, but is not dominant-end-rooted. Then $(A, i)$ admits an infinite tower of finite faithful groupings.

Except for the case that $(A, i)$ is a dominant-end-rooted edge-indexed tree (similar to a dominant-rooted edge-indexed tree as defined on page 4, except that an end takes the role given to the root vertex before, that is, $i(e)=1$ for all edges directed towards that end as opposed to the root vertex (see [CR2] for more details)), the assumption that $(A, i)$ is an infinite permissible tree implies the existence of a finite permissible ‘core’ graph $(A_0, i)$ which is an edge-indexed path of length $n \geq 1$. By Theorem (3.1), $(A_0, i)$ admits an infinite tower of finite faithful groupings, and we may then apply Lemma (2.4) to extend the tower of groupings to $(A, i)$.

If $(A, i)$ is a dominant-end-rooted edge-indexed tree, then $(A, i)$ does not contain a finite permissible core. We know that in this case, the set of covolumes of non-uniform lattices in $\text{Aut}(X)$, $X = \widehat{(A, i)}$, is not bounded away from zero, however our techniques do not suffice to produce a tower of groupings on $(A, i)$.

(4.2) Theorem. Let $(A, i)$ be as in Theorem (4.1). Let $X = \widehat{(A, i)}$. Then there is an infinite ascending chain

$$\Gamma_1 < \Gamma_2 < \Gamma_3 < \cdots$$

of non-uniform $X$-lattices in $\text{Aut}(X)$. Hence $\text{Vol}(\Gamma_i \backslash X) \to 0$ as $i \to \infty$.

In Theorems (4.1), and (4.2), the covering tree $X = \widehat{(A, i)}$ may be uniform or not.
5. Infinite towers of non-uniform $X$-lattices

We have the following:

(5.1) Theorem ([R]). Let $(A, i)$ be a permissible edge-indexed graph. Suppose $(A, i)$ contains an arithmetic bridge with $n \geq 2$ edges. Then $(A, i)$ admits an infinite tower of finite faithful groupings.

Concerning existence of arithmetic bridges, we have the following:

(5.2) Theorem ([C1], [CR1]). Let $(A, i)$ be a unimodular edge-indexed graph. Let $e \in EA$ be a ramified edge such that $\Delta(e)$ is not an integer. If $e$ is not separating, then $e$ is contained in an arithmetic bridge with $n \geq 2$ edges.

Combining the results of §2, §4 and the above, we have:

(5.3) Theorem. Let $(A, i)$ be a permissible edge-indexed graph that is not a dominant-end-rooted edge-indexed tree. Then $(A, i)$ admits an infinite tower of finite faithful groupings.

A corollary of Theorem (5.3) is the following:

(5.4) Theorem. Let $X$ be a locally finite tree. If $\text{Aut}(X)$ contains a non-uniform $X$-lattice $\Gamma$, and $X$ is not the universal cover of a dominant-end-rooted edge-indexed tree, then $\text{Aut}(X)$ contains an infinite tower

$$\Gamma_1 < \Gamma_2 < \Gamma_3 < \cdots$$

of non-uniform $X$-lattices. Hence $\text{Vol}(\Gamma_i \backslash X) \to 0$ as $i \to \infty$.

6. Existence of non-uniform $X$-lattices

By Theorem (5.4), the question of existence of infinite towers of non-uniform $X$-lattices reduces to the question of existence of non-uniform $X$-lattices.

To outline the results on existence of non-uniform $X$-lattices, we make the following definition. Let $X$ be a locally finite tree, $G = \text{Aut}(X)$, and let $\mu$ be a (left) Haar measure on $G$. Suppose that $G$ is unimodular. Then $\mu(G_x)$ is constant on $G$-orbits, so we can define ([BL], (1.5)):

$$\mu(G \backslash X) := \sum_{x \in V(G \backslash X)} \frac{1}{\mu(G_x)}.$$

We have the ‘Lattice existence theorem’:

(6.1) Theorem ([BCR], (0.2)). Let $X$ be a locally finite tree, let $G = \text{Aut}(X)$, and let $\mu$ be a (left) Haar measure on $G$. The following conditions are equivalent:

(a) $G$ contains an $X$-lattice $\Gamma$.

(b) (U) $G$ is unimodular, and

(PV) $\mu(G \backslash X) < \infty$.

In particular, we have the following theorem, which together with Bass-Kulkarni’s ‘Uniform existence theorem’ ([BK], (4.10)) gives Theorem (6.1):
Theorem ([BCR], (0.5)). Let $X$ be a locally finite tree, let $G = \text{Aut}(X)$, and let $\mu$ be a (left) Haar measure on $G$. Assume that:

- (U) $G$ is unimodular;
- (FV) $\mu(G \backslash X) < \infty$, and
- (INF) $G \backslash X$ is infinite.

Then $G$ contains a (necessarily non-uniform) $X$-lattice $\Gamma$.

For uniform trees, we have the following:

Theorem ([C1], [C2]). If $X$ is uniform and not virtually rigid then $G$ contains a non-uniform $X$-lattice $\Gamma$.

References


[BT] Bass, H and Tits, J, A discreteness criterion for certain tree automorphism groups, Appendix [BT], Tree Lattices by Hyman Bass and Alex Lubotzky (2000), Progress in Mathematics 176, Birkhauser, Boston.


