THE MAP $V \rightarrow V//G$ NEED NOT BE SEPARABLE

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Abstract. We construct a vector space $V$ with a linear action of a reductive group $G$ such that the quotient map $V \rightarrow V//G$ (in the sense of geometric invariant theory) fails to be separable. This gives a counterexample to an assertion of Bardsley and Richardson.

0. Introduction

Let $G$ be a reductive algebraic group, possibly nonconnected, and let $X$ be an irreducible affine $G$–variety. Suppose that the ground field $k$ has characteristic $p > 0$. In their paper on étale slices in characteristic $p$, Bardsley and Richardson claim ([1], Section 2, 2.1.9(b)) that the canonical projection from $X$ to the quotient $X//G$ is separable. We give a counterexample to show that this map need not be separable, not even when $X$ is a vector space $V$ and $G$ acts linearly.

Bardsley and Richardson extend Luna’s important Étale Slice Theorem [2] from characteristic zero to characteristic $p$. At only one point (see [1], Section 4, 4.3) do they use the separability of the quotient map. There the group $G$ is finite and their assertion is justified: for the function field $k(X//G)$ is the whole of the field of invariants $k(X)^G$ of the function field $k(X)$, by [1], Section 4, 4.3.1, and whenever a group $\Gamma$ acts on a field $K$, the extension $K/K^\Gamma$ is separable [3], IV.1, Lemma 1.5. The main results of [1], therefore, remain valid.

Separability questions come up when one tries to generalise or strengthen the results of Bardsley and Richardson’s [1]. A major interest of the counterexample presented here is that it indicates limits on any possible Luna slice theorem in characteristic $p > 0$. The first author will explore this point further in a forthcoming paper.

Throughout this article, $k$ will be an algebraically closed field. We denote by $X//G$ the quotient of $X$ by $G$ in the sense of geometric invariant theory; see [1] for details.

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1. The counterexample

**Notation 1.1.** Let $W$ be a finite dimensional vector space over $k$, of dimension $d = \dim(W) \geq 2$. Let $G = GL(W)$ be the group of all linear automorphisms of $W$. Let $W^*$ be the dual of $W$, with the usual $G$–action. For any integer $n > 0$, we let $W^n$ stand for the direct sum $W \oplus W \oplus \cdots \oplus W$ of $n$ copies of $W$. We first wish to consider the $G$–module $W^* \oplus W^n$.

This $G$–module is naturally an affine variety. Choose a basis $(x_1, x_2, \ldots, x_d)$ for $W$, and take the natural dual basis $(y_1, y_2, \ldots, y_d)$ for $W^*$. The polynomial functions on $W^*$ form the ring $k[y_1, \ldots, y_d]$, while the polynomial functions on $W$ form the ring $k[x_1, \ldots, x_d]$. The difference is that the $G$–actions on these rings are dual. The polynomial functions on $W^* \oplus W^n$ form a ring $R = k[x_j, y^i_j]$, with $1 \leq j \leq d$ and $1 \leq i \leq n$. Note that in $y^i_j$ the $i$ is a superscript; $y^i_j$ stands for the $j$th component of the $i$th vector. We are not raising anything to the $i$th power. Our notation for raising to the $p$th power, in this article, will be $\{y^i_j\}^p$.

**Lemma 1.2.** With the notation as in Notation 1.1, let $I \subset R = k[x_j, y^i_j]$ be the ideal generated by all $\{x_j, y^i_j \mid j \geq 2\}$. Then any $G$–invariant element of $R$ that lies in the ideal $I$ must vanish. In symbols, $I \cap R^G = 0$.

*Proof.* In the $G$–orbit of any point of $W^*$ there is a point $(\lambda, 0, \cdots, 0)$. Therefore in the $G$–orbit of any point of $W^* \oplus W^n$ there is a point whose coordinates are

$$\begin{pmatrix}
\lambda \\
0 \\
\vdots \\
0
\end{pmatrix}, \begin{pmatrix}
\mu_1^1 \\
\mu^1_2 \\
\vdots \\
\mu^1_d
\end{pmatrix}, \begin{pmatrix}
\mu_1^2 \\
\mu_2^2 \\
\vdots \\
\mu^2_d
\end{pmatrix}, \cdots, \begin{pmatrix}
\mu^n_1 \\
\mu^2_2 \\
\vdots \\
\mu^n_d
\end{pmatrix}$$

Now the element of $GL(W)$ given by the diagonal matrix

$$\begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & t & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & t
\end{pmatrix}$$

takes the above to the point

$$\begin{pmatrix}
\lambda \\
0 \\
\vdots \\
0
\end{pmatrix}, \begin{pmatrix}
\mu_1^1 \\
t\mu^1_2 \\
\vdots \\
t\mu^1_d
\end{pmatrix}, \begin{pmatrix}
\mu_1^2 \\
t\mu_2^2 \\
\vdots \\
t\mu^2_d
\end{pmatrix}, \cdots, \begin{pmatrix}
\mu^n_1 \\
t\mu^2_2 \\
\vdots \\
t\mu^n_d
\end{pmatrix}$$
Taking the limit as $t \to 0$, we have that the closure of any $G$–orbit must contain a point of the form

$$
\begin{pmatrix}
\lambda \\
0 \\
\vdots \\
0
\end{pmatrix},
\begin{pmatrix}
\mu_1^1 \\
0 \\
\vdots \\
0
\end{pmatrix},
\begin{pmatrix}
\mu_1^2 \\
0 \\
\vdots \\
0
\end{pmatrix}, \ldots,
\begin{pmatrix}
\mu_1^n \\
0 \\
\vdots \\
0
\end{pmatrix}
$$

Now any polynomial in the ideal $I \subset R$ vanishes on the points above. Since $G$–invariant polynomials are constant on closures of orbits, any $G$–invariant polynomial in $I$ must vanish on all closures of all orbits; in other words, it must be identically zero.

**Definition 1.3.** Let the notation be as in Lemma 1.2. For every $1 \leq i \leq n$, we may form the polynomial

$$
Y_i = x_1 y_i^1 + x_2 y_i^2 + \cdots + x_d y_i^d.
$$

The $Y_i$’s are obviously $G$–invariant.

**Proposition 1.4.** Let the notation be as in Definition 1.3. The subring $R^G \subset R$, of all $G$–invariant polynomials in $R$, is generated by the $Y_i$’s.

**Proof.** It is easy to prove Proposition 1.4 as a consequence of the First Main Theorem of classical invariant theory. Instead, we will give an equally easy, self-contained proof.

The center $Z(G)$ of $G$, that is the set of non-zero scalar matrices, stabilises $I$. Therefore $Z(G)$ acts on $R/I$. It is very obvious that the $Z(G)$–invariant polynomials in $R/I$ are generated by the monomials $x_1 y_i^1$, and $Y_i$ is congruent mod $I$ to $x_1 y_i^1$.

Take any $G$–invariant polynomial $f \in R^G$. Then $f$ gives a $Z(G)$–invariant polynomial modulo $I$, and by the above paragraph, there exists a polynomial $P$ in $n$ variables so that $f$ is congruent to $P(Y_1, \ldots, Y_n)$ mod $I$. But then

$$
f - P(Y_1, \ldots, Y_n)
$$

is a $G$–invariant element of $I$, and by Lemma 1.2 it must vanish.

**Lemma 1.5.** With the notation as above, the $Y_i \in R^G$ are algebraically independent. Even better: any monomial in $\{x_j, y_j^i\}$ can occur in the expansion of at most one monomial $Y_1^{M_1} Y_2^{M_2} \cdots Y_n^{M_n}$.

**Proof.** By checking the degrees of the monomials in the vectors $(y_1^i, y_2^i, \ldots, y_n^i)$ for different $i$. 

**Definition 1.6.** Suppose now that $k$ is of characteristic $p > 0$. Put $X_j = \{x_j\}^P$. The ring $R = k[x_j, y_j^i]$ contains a subring $S = k[X_j, y_j^i]$. The ring $S$ is not just
a subring of $R$, it is also a $G$–submodule. In fact, $S$ can be thought of as the ring of polynomial functions on the $G$–module $\pi_* W^* \oplus W^n$. Here, $\pi_* W^*$ is the Frobenius twist of $W^*$. The vector spaces $W^*$ and $\pi_* W^*$ are identical. A matrix in $GL(d)$ acts on a vector in $\pi_* W^*$ by raising the entries of the matrix to the $p^\text{th}$ power, followed by the usual action on $W^*$.

The polynomials

$$Y_i = X_1 \left\{ y_1^i \right\}^p + X_2 \left\{ y_2^i \right\}^p + \cdots + X_d \left\{ y_d^i \right\}^p$$

$$= x_1^p \left\{ y_1^i \right\} + x_2^p \left\{ y_2^i \right\} + \cdots + x_d^p \left\{ y_d^i \right\}$$

are clearly $G$–invariant elements of the ring $S$.

**Proposition 1.7.** Let the notation be as in Definition 1.6. The subring $S^G \subset S$, of all $G$–invariant elements of $S$, is generated by the $Y_i$'s.

**Proof.** The ring $S = k[ x_j, y_j ]$ is a subring and $G$–submodule of $R = k[ x_j, y_j ]$.

By Proposition 1.4 we know that $R^G$ is generated by

$$Y_i = x_1 y_1^i + x_2 y_2^i + \cdots + x_d y_d^i.$$ 

The ring $S^G$ is nothing more than the intersection of $R^G$ with $S$.

By Lemma 1.5, the elements $Y_1^{M_1} Y_2^{M_2} \cdots Y_n^{M_n} \in R^G$ have disjoint monomial expansions. A linear combination of $Y_1^{M_1} Y_2^{M_2} \cdots Y_n^{M_n}$'s will lie in $S$ if and only if every term does. Suppose therefore that some $Y_1^{M_1} Y_2^{M_2} \cdots Y_n^{M_n}$ belongs to $S$.

In the expansion of the product, there is a term

$$x_2^{M_1} \left\{ y_2^1 \right\}^{M_1} \prod_{i=2}^n x_1^{M_i} \left\{ y_1^i \right\}^{M_i}$$

and since this lies in $S$, it follows that $p$ must divide $M_1$. By symmetry, $p$ must divide $M_i$ for every $i$. That is, our monomial is really a monomial in $Y_i^p = Y_i$.

**Theorem 1.8.** There exists a vector space $V$, and a reductive group $G$ acting on $V$, so that the geometric invariant theory map $V \rightarrow V/G$ is not separable.

**Proof.** Put $V = \pi_* W^* \oplus W^n$ as above, with $n > d = \dim(W)$. We assert that the map $V \rightarrow V/G$ is not separable. The map corresponds to the inclusion $S^G \subset S$. We know that $S^G$ is the polynomial algebra $k[ Y_1, \ldots, Y_n ]$. The derivative of the inclusion $S^G \subset S$ takes $dY_i$ to

$$\left\{ y_1^i \right\}^p dX_1 + \left\{ y_2^i \right\}^p dX_2 + \cdots + \left\{ y_d^i \right\}^p dX_d$$

which is in the linear span of $\{ dX_1, dX_2, \ldots, dX_d \}$. The image is therefore contained in a $d$–dimensional vector subspace of the 1–forms on $V$. Since the dimension of $V/G$ is $n > d$, the map $\Omega^1_{V/G} \rightarrow \Omega^1_V$ cannot be generically injective.
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